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Handout of Mathematics 1
Lessons and Examples

For First-Year LMD Students in the Science and Technology Domain.

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Introduction

Our goal in this set of lecture notes is to provide students with a strong foundation in mathematical analysis. Such a foundation is crucial for future study of deeper topics of analysis. Students should be familiar with most of the concepts presented here after completing the calculus sequence. However, these concepts will be reinforced through rigorous proofs. The lecture notes contain topics of real analysis usually covered in a 10-week course: the completeness axiom, sequences and convergence, continuity, and differentiation. The lecture notes also contain many well-selected exercises of various levels. Although these topics are written in a more abstract way compared with those available in some textbooks, teachers can choose to simplify them depending on the background of the students. For instance, rather than introducing the topology of the real line to students, related topological concepts can be replaced by more familiar concepts such as open and closed intervals. Some other topics such as lower and upper semicontinuity, differentiation of convex functions, and generalized differentiation of non-differentiable convex functions can be used as optional mathematical projects. In this way, the lecture notes are suitable for teaching students of different backgrounds. Hints and solutions to selected exercises are collected in Chapter 5. For each section, there is at least one exercise fully solved. For those exercises, in addition to the solutions, there are explanations about the process itself and examples of more general problems where the same technique may be used. Exercises with solutions are indicated by a I and those with hints are indicated by a B. Finally, to make it easier for students to navigate the text, the electronic version of these notes contains many hyperlinks that students can click on to go to a definition, theorem, example, or exercise at a different place in the notes. These hyperlinks can be easily recognized because the text or number is on a different color and the mouse pointer changes shape when going over them

Chapter 1

Logic and mathematical reasoning

At the intersection of philosophy and mathematics, logic is a fundamental branch that enables the determination of the truth value of propositions and the construction of mathematical reasoning. This document serves as an introduction to this crucial branch of mathematics. We will define the concepts of proposition and operator, construct truth tables, explain implications, reciprocal implications, and equivalence, before delving into the various types of reasoning used in mathematics.

Formal logic (symbolic logic) In mathematics, the systematic study of reasoning is called formal logic. It analyzes the structure of arguments, as well as the methods and validity of mathematical deduction and proof.

This chapter is concerned with first-order propositional logic. First-order here means that there is no quantification over propositions, and propositional means that there is no quantification over terms. We consider mainly intuitionists logic, where A ? $\overline{A}$ is not valid in general. We also study classical logic. In addition, we consider minimal logic, which is the subset of intuitionists logic with implication as only connective. In this setting we study detors and detor elimination.

1.1 Propositional Logic

Definition 1.1.1. ***(Proposition)***

A proposition is a statement which has a truth value either true or false. Notation: Variables are used to represent propositions. The most common variables used are P , Q , and R.

Example 1.1.1. ***1)*** *P : 2 is even , $Q : 2 + 2 = 5$, $R : 2 + 2 = 4$ are popositions.*
2) *$x + 2 = 2x$ is not a proposition.*

Definition 1.1.2. ***(Negation)***

The negation of a proposition P is also called not P , and is denoted by $\overline{P}$.

Example 1.1.2. ***1)*** *If P : 2 is even then $\overline{P}$: 2 is not even .*
2) *If $P : 2 + 2 = 5$ then $\overline{P} : 2 + 2 \neq 5$.*

Definition 1.1.3. ***(Truth-value)***

The truth-value is one of the two values, "true" (T) or "false" (F) , that can be taken by a given logical formula in an interpretation (model) considered . Sometimes the truth value T is denoted in the literature by 1, and F by 0 .

P	$\overline{P}$
1	0
0	1

It is important to understand how to construct a truth table as we will use it many times in this course. This connector is quite intuitive as we use it in our daily lives.

Example 1.1.3. *P : " Algiers is the capital of Algeria " (its value is V)*
$\overline{P}$: "Algiers is not the capital of Algeria" (its value is F)
Q : " π is an integer " (F)
$\overline{Q}$: " π is not an integer " (V)

1.2 Logical Connectors

1.2.1 Conjunction

Definition 1.2.1. ***(Conjunction) ; (And) : denoted*** $\wedge$

If P *and* Q *are two propositions then their conjunction is the proposition whose value is true only when both are true. A conjunction can also be written* $P \wedge Q$ *which is read* P *and* Q*. .*

P	Q	$P \wedge Q$
1	1	1
1	0	0
0	1	0
0	0	0

Example 1.2.1. ***1)*** *A triangle has three sides and a square has four sides" is a conjunction*

2) *Let* $P : 2 \leq 3$ *and* $Q : 2^2 \leq 3^2$ *, the proposition* $P \wedge Q$ *is true.*

1.2.2 Disjunction

Definition 1.2.2. ***(Disjunction);(Or) : denoted*** $\vee$

A compound statement of the form P *or* Q *is known as a disjunction and it is denoted by* $P \vee Q$ *. The disjunction of* P *and* Q *has value false only when both are false.*

P	Q	$P \vee Q$
1	1	1
1	0	1
0	1	1
0	0	0

1.2.3 Implication

Definition 1.2.3. ***(Implication) : denoted*** $(\Longrightarrow)$

*A conditional statement of the form " **if** . . . **then** . . ." is known as a conditional or an implication.*
A conditional statement has two components: If P, then Q. Statement P is called the antecedent (hypothesis, or premise) and statement Q the consequent (or conclusion) .
We write $P \Longrightarrow Q$; and read P implies Q .

A conditional statement can be written a number of different, but equivalent, ways :
- *If P then Q .*
- *P implies Q .*
- *Q if P .*
- *p only if Q .*
- *P is sufficient for Q .*
- *Q is necessary for P .*

The implication of P and Q has value false only when P is true and Q is false .

The truth table for the proposition $P \Longrightarrow Q$ is as follows :

P	Q	$P \Longrightarrow Q$
1	1	1
1	0	0
0	1	1
0	0	1

Example 1.2.2. ***1)*** *"If a polygon has three sides, then it is a triangle" is a conditional statement.*

2) *"If* $1 \leq 3$ *, then* $1+1 \leq 1+3$*" is a true implication.*

3) *"If* π *and* $2+3i$ *are real numbers then* $2+3i$ *is real number" is a true implication.*

4) *"If* $2+3=5$ *, then* $3\times 2+3\times 3=20$*" is a false implication because when* $x=5$ *, then* $3x=15$ *.*

Definition 1.2.4. ***(Converse of implication)***
The converse of $P \Longrightarrow Q$ *is the proposition* $Q \Longrightarrow P$

Example 1.2.3. *Let* P *: x is a prime number different from 2" and q* Q *: is odd". One has* $P \Longrightarrow Q$ *but we do not have*$Q \Longrightarrow P$ *.*

Theorem 1.2.1. *For all propositions* P *and* Q *, the following statements are true .*
1) $P \Longrightarrow P \vee Q$ *and* $Q \Longrightarrow P \vee Q$*.*
2) $P \wedge Q \Longrightarrow P$ *and* $P \wedge Q \Longrightarrow Q$*.*

Proof.

1) We give a truth table for $P \Longrightarrow P \vee Q$ as follows.

P	Q	$P \vee Q$	$P \Longrightarrow P \vee Q$
1	1	1	1
1	0	1	1
0	1	1	1
0	0	0	1

Then $P \Longrightarrow P \vee Q$ is always true.
The truth table for $Q \Longrightarrow P \vee Q$ is analogous to the one for $P \Longrightarrow P \vee Q$ the conclusion is the same.

2) In order to prove that $P \wedge Q \Longrightarrow P$ for all propositions P and Q , we give a truth table for $P \wedge Q \Longrightarrow P$

P	Q	$P \wedge Q$	$P \wedge Q \Longrightarrow Q$
1	1	1	1
1	0	0	1
0	1	0	1
0	0	0	1

Then $P \wedge P \Longrightarrow Q$ is always true.
The truth table for $P \wedge Q \Longrightarrow Q$ is analogous to the one for $P \wedge Q \Longrightarrow P$

□

1.2.4 Equivalence

Definition 1.2.5. ***(Equivalence) : denoted ($\Longleftrightarrow$) " P is equivalent to Q "***

The symbol for equivalence is ($\Longleftrightarrow$) a double arrow that resembles the implication arrow discussed earlier.

Two mathematical statements are equivalent if they have the same truth values. The statement of the form P if, and only if Q is called an equivalence or biconditional statement. It is often abbreviated as P iff Q and is written in symbols as $\Longleftrightarrow$. It is equivalent to the compound statement " P implies Q , and Q implies P " composed of two CONDITIONAL statements. The truth-values of P and Q must match for the biconditional statement as a whole to be true.

The truth table for the proposition $P \Longleftrightarrow Q$ is as follows :

P	Q	$P \Longleftrightarrow Q$
1	1	1
1	0	0
0	1	0
0	0	1

Example 1.2.4. ***1)*** *A triangle is equilateral if, and only if, it is equiangular" is a biconditional statement.* ***2)*** *The proposition " $(1 = 1) \Longleftrightarrow (0 = 0)$" is true, the proposition " $(1 = 0) \Longleftrightarrow (0 = 2)$ " is true, whereas the proposition " $(1 = 0) \Longleftrightarrow (0 = 0 =)$" is false.* ***3)*** *For all real x $x, x \neq 0$ and y, we have $y = x \Longleftrightarrow \frac{y}{x} = 1$ is true.* ***4)*** *The equivalence statement ($y = x \Longleftrightarrow y^2 = x^2$ is not true for all real x and y: for example $2^2 = (-2)^2 \nRightarrow 2 = -2$.*

Definition 1.2.6. ***(Contrapositive)***

The contrapositive of $P \Longrightarrow Q$ is the proposition $\overline{Q} \Longrightarrow \overline{P}$. It can be shown that these two are equivalent:

$$P \Longrightarrow Q \iff \overline{Q} \Longrightarrow \overline{P}$$

The equivalence can easily be verified using truth table:

P	Q	$\overline{P}$	$\overline{Q}$	$P \Longrightarrow Q$	$\overline{Q} \Longrightarrow \overline{P}$
1	1	0	0	1	1
1	0	0	1	0	0
0	1	1	0	1	1
0	0	1	1	1	1

Properties 1.2.2. ***1)*** *Morgan's laws :*

$$\overline{P \wedge Q} \Longleftrightarrow \overline{P} \vee \overline{Q}, \text{ and } \overline{P \wedge Q} \Longleftrightarrow \overline{P} \vee \overline{Q}$$

2) *Idempotents of* $\wedge$ *and* $\vee$ *.*

$$P \Longleftrightarrow P \wedge P, \text{ and } P \Longleftrightarrow P \vee P$$

3) *Commutativity of* $\wedge$ *and* $\vee$

$$P \wedge Q \Longleftrightarrow Q \wedge P, \text{ and } P \vee Q \Longleftrightarrow Q \vee P$$

4) *Associativity of* $\wedge$ *and* $\vee$

$$P \wedge (Q \wedge R) \Longleftrightarrow (P \wedge Q) \wedge R$$

$$P \wedge (Q \vee R) \Longleftrightarrow (P \vee Q) \vee R$$

5) *Distributivity of* $\wedge$ *over* $\vee$ *and* $\vee$ *over* $\wedge$ *respectively .*

$$P \wedge (Q \vee R) \Longleftrightarrow (P \wedge Q) \vee (P \wedge R)$$

$$P \vee (Q \wedge R) \Longleftrightarrow (P \vee Q) \wedge (P \vee R)$$

6) *Double negation law ;*

$$\overline{\overline{P}} \Longleftrightarrow P$$

7) *Absorption laws :*

$$P \vee (P \wedge Q) \Longleftrightarrow P$$

$$P \wedge (P \vee Q) \Longleftrightarrow P$$

Exercise 1.2.1. *Prove the following equivalence by drawing the truth table:*

$$P \Longrightarrow Q \Longleftrightarrow \overline{P} \vee Q.$$

Solution

P	Q	$\overline{P}$	$P \Longrightarrow Q$	$\overline{P} \wedge Q$
1	1	0	1	1
1	0	0	0	0
0	1	1	1	1
0	0	1	1	1

The truth table establishes that these corresponding pairs of compound statements are logically equivalent.

1.3 Predicates and Quantifiers

1.3.1 Predicates

Definition 1.3.1. ***(Predicate)*** *A predicate is a statement that contains variables and that may be true or false depending on the values of these variables.*

Example 1.3.1. ***1)*** *Let* $P(x) : x^2 < x$ *is a predicate . One has* $P(1) : 1^2 < 1$ *is false and* $P(2) : 2^2 < 2$ *is even false. But for* $x = \frac{1}{2}$ $P(\frac{1}{2}) : \frac{1}{4} < \frac{1}{2}$ *is true .*

A predicate can also be made a proposition by adding a quantifier. There are two quantifiers:

1.3.2 Universal quantification

Definition 1.3.2. ***(Universal quantifier)***

A universal quantifier is a quantifier meaning "for all", "for any", "for each" or "for every", denoted by $\forall$.
Here is a formal way to say that for all values that a predicate variable x can take in a domain A, the predicate is true:

$$\forall x \in A \ : \ P(x).$$

which is read as for all x belonging to A , P(x) is true .

Example 1.3.2. *All natural numbers of the form* $2n+1$ *are odd is written:* $\forall n$, $2n+1$ *is odd.*

1.3.3 Existential quantification

Definition 1.3.3. ***(Existential quantifier)***
An existential quantifier is a quantifier meaning "there exists", "there is at least one" or "for some" , denoted by $\exists$.
Here is a formal way to say that for some values that a predicate variable x can take in a domain A, the predicate is true:

$$\exists x \in A \ : \ P(x).$$

which is read some for all x belonging to A , P(x) is true .

Example 1.3.3. *There exists a natural number n satisfying ,* $n \times n = n+n$ *can be written:* $\exists n : n \times n = n+n$.

Remark 1.3.1. *A unique existential quantifier is a quantifier meaning "there is a unique", "there is exactly one" or "there exists only one" , denoted by* $\exists!$.
Here is a formal way to say that for some values that a predicate variable x can take in a domain A, the predicate is true:

$$\exists! x \in A \ : \ P(x).$$

which is read , there exists only one x belonging to A , P(x) is true .

Example 1.3.4. $\exists! x \in \mathbb{R} : x+2 = 5$ *: there is a real number x such that* $x+2=5$ *, which is true .*

Properties 1.3.1. *We observe, at least intuitively, that the negations of* $\exists$ *and* $\forall$ *are correlated in the following manner.*

$$\overline{\forall\ x, P(x)} \iff \exists\ x, \overline{P(x)}.$$

$$\overline{\exists\ x, P(x)} \iff \forall\ x, \overline{P(x)}.$$

Exercise 1.3.1. *Write the negations by interchanging* $\forall$ *and* $\exists$.

a) *There is a real number* x *such that ,* $x^2 < 0$.
b) *Every integer is even.*
c) *There is an integer* x *such that :* $x^2 + x + 3 = 0$.

Solution a) $\overline{\exists\ x\ \in \mathbb{R}\ :\ x^2 < 0.} \iff \forall\ x\ \in \mathbb{R}\ :\ x^2 \geq 0.$
. **b)** There is an integer which is not even.
c) $\overline{\exists\ x \in \mathbb{Z}\ : x^2 + x + 3 = 0} \iff \forall x \in \mathbb{Z}\ : x^2 + x + 3 \neq 0.$

1.3.4 Nested Quantifiers

Two quantifiers are nested if one is within the scope of the other. The order of existential quantifiers and universal quantifiers in a statement is important.
♠ When we have one quantifier inside another, we need to be a little careful.

Example 1.3.5. *Consider the following proposition over the integers:*

$$\forall x \in \mathbb{Z}\ , \exists y \in \mathbb{Z}\ : x + y = 0.$$

• *The proposition is true.*
• *The existence of* y *depends on* x *: if you pick any* x*, I can find a* y *that makes* $x + y = 0$ *true.*

Example 1.3.6. *Consider the following proposition over the integers:*

$$\exists\ x \in \mathbb{Z}\ , \forall\ y \in \mathbb{Z}\ : x + y = 0.$$

The proposition is false.
The existence of y *does not depend on* x *: there is no* y *that will make* $x + y = 0$ *true for every* x *.*

Example 1.3.7. *Consider the following proposition over the integers:*

$$\exists\ y \in \mathbb{Z}\ ,\forall\ x \in \mathbb{Z}\ : x + y = x.$$

The proposition is true.
There is $y = 0$ *that will make* $x + y = x$ *for every* x.

Example 1.3.8. *Suppose we claimed . " For every real number, there's a real number larger than it ".*
We'd write this as :

$$\forall x \in \mathbb{R}\ , \exists y \in \mathbb{R}\ : y > x.$$

- *The proposition is true.*

♠ We can exchange the same kind of quantifier (∃ , ∀) .
These statements are equivalent:

$$\forall x, \forall y\ , P(x,y) \iff \forall y, \forall x\ , P(x,y).$$

$$\exists x, \exists y\ , P(x,y) \iff \exists y, \exists x\ , P(x,y).$$

1.4 Methods of Proof

Now we have all the tools to carry out complete mathematical reasoning. Reasoning allows us to establish a proposition based on one or more initial propositions that are accepted (or previously proven) by following the rules of logic.

1.4.1 Direct Methods

We have already seen one way of proving a mathematical statement of the form: If P , then Q . Based of the fact that the implication $P \implies Q$ is false only when P is true and Q is false, the idea behind the method of proof that we discussed was to assume that P is true and then to proceed, through a chain of logical deductions, to conclude that Q is true. Here is the outline of the argument:

Example 1.4.1. *Prove the statement: If* n *is even, then* n^2 *is even.*
Assume that the integer n is even.

$$\exists k \in \mathbb{Z},\ n = 2k \implies n^2 = 4k^2$$
$$\implies n = 2(2k^2).$$

which shows that n^2 *is even.*

1.4.2 Proof by Contrapositive

The proposition $\overline{Q} \implies \overline{P}$ is called the contrapositive of the proposition $P \implies Q$.
A proposition and its contrapositive are equivalent, which means that one can be proven to prove the other. For example, to prove $P \implies Q$, we can use contrapositive reasoning to prove $\overline{Q} \implies \overline{P}$.

Example 1.4.2. *Prove the statement: If* n^2 *is even, then* n *is even.*
Assume that the integer n is not even . It then follows that

$$\exists k \in \mathbb{Z},\ n = 2k+1 \implies n^2 = (2k+1)^2 = 4k^2 + 4k + 1$$
$$\implies n^2 = 2(2k^2 + 2k) + 1.$$
$$\implies n^2 = 2k' + 1. \ \textit{such that} \ (k' = 2k+1).$$

which shows that n^2 *is not even.*

1.4.3 Reasoning by Contradiction (Absurd)

To prove that a proposition P is true we may assume that P is false then $\overline{P}$ is true. Therefore we show that it would lead to a contradiction or a false statement

Example 1.4.3. *Prove that* $\sqrt{2}$ *is irrational .*
Let $P:$ $\sqrt{2}$ *is irrational. Now assume that P is false then* $\overline{P}$ *is true, that is* $\sqrt{2}$ *is rational. Then there are some integers* a *and* b *with no common factors:*

$$\exists a \in \mathbb{Z}, \exists b \in \mathbb{Z}^\star, \sqrt{2} = \frac{a}{b} \implies a^2 = 2b^2$$

$$\implies a^2 \ \ iseven$$

$$\implies a^2 = 2c \ , \ c \in \mathbb{Z}$$

$$\implies 4c^2 = 2b^2 \ , \ by\ substituting$$

$$\implies 2c^2 = b^2$$

$$\implies b^2 \ is\ even$$

$$\implies b \ is\ even$$

This means that a and b have a common factor 2 which is a contradiction, and so $\overline{P}$ *must be false and P is true*

1.4.4 Proof by Counter-Example

To show that a proposition of the form $\forall\ x : P(x)$ is false ", we show that its negation $\exists\ x : \overline{P(x)}$ is true ". This is providing a counter-example.

Example 1.4.4. *Let P be the proposition* $\forall n \in \mathbb{N}$, $n^2 + 1$ *is a prime number ".*
Prove that P is false . To prove that P is false, we will show that its negation $\overline{P}$ *is true.*
$\overline{P}$ *is the proposition* $\exists n \in \mathbb{N}$ *suchthat* $n^2 + 1$ *is not a prime number*
Let $n = 3$ *. Then* $n^2 + 1 = 10$ *. 10 is not a prime number. n is a counter-example of proposition P .*

1.4.5 Proof by Mathematical Induction

To prove a proposition in the form $\forall n\ :\ P(n)$ where n is a natural number, it suffices to prove it in two steps:

1) $P(n_0)$ is true for a certain base step n_0 : Usually the base case is $n = 1$ or $n = 0$.
2) $P(n) \Longrightarrow P(n+1)$. That is, if $P(n)$ is true, then $P(n+1)$ is true.

Example 1.4.5. *Prove that* 3^n-1 *a multiple of 2 for all natural numbers n .*

1) *Show it is true for* $n = 1$ $3^1 - 1 = 3 - 1 = 2$: *One has 2 is a multiple of 2. That was easy.* $3^1 - 1$ *is true*

2) *Assume it is true for n and prove that* $3^n - 1$ *is a multiple of 2 ?*

$$\begin{aligned} 3^n - 1 = 2k &\implies 3^n \times 3 - 1 \times 3 = 2k \times 3 \\ &\implies 3^{n+1} \times 3 - 1 = 2 + 2k \times 3 \\ &\implies 3^{n+1} \times 3 - 1 = 2(1 + 3k) \\ &\implies 3^{n+1} \times 3 - 1 = 2k' \ , \ suchthat \ \ k' = 1 + 3k. \end{aligned}$$

Therefore $\forall n \in \mathbb{N}$, $3^n - 1$ *a multiple of 2*

Exercise 1.4.1. *Rewrite the following sentences using quantifiers:*

1. *f is a constant function on* $\mathbb{R}$ *.*

2. *The graph of the function f intersects the line* $y = x$*.*

3. *The equation sin* $x = x$ *has one and only one solution in* $\mathbb{R}$ *.*

4. *For every integer, there exists an integer that is strictly greater.*

Exercise 1.4.2. *Negate the following formulas:*

- $0 \leq x \leq 25 \implies \sqrt{x} \leq 5.$
- $0 < x \leq 1$ *or* $2 \leq y \leq 3.$
- $\exists x \in \mathbb{R} \ : \ cos(x) = 0 \ \ and \ \ \exists x \in \mathbb{R} \ : \ sin(x) = 0.$
- $\forall \epsilon > 0, \exists \eta > 0, \forall x \in D \ ; \ \mid x - x_0 \mid < \eta \implies \mid f(x) * f(x_0) \mid < \epsilon.$

Exercise 1.4.3. *Prove the following formulas:*

1.4 Methods of Proof

1. $\mid x \mid < 0.1 \implies \mid 2x^2 - x \mid < 0.2$ *(Direct proof).*

2. *For any integer* n, $n^2 + 3n$ *is even (Proof by cases).*

3. $\forall n \in \mathbb{N} \ : \ n^2$ *is even* $\implies$ n *is even(Contrapositive)..*

4. $\sqrt{2}$ *is irrational (Proof by contradiction).*

5. $\forall \ a, b \in \mathbb{R}^+ \ : \ \dfrac{a}{1+b} = \dfrac{b}{1+a} \implies a = b$ *(Proof by contradiction).*

6. $\forall n \in \mathbb{N} \ : \ 2^n > n$ *(Proof by induction).*

7. *For real numbers a, b, c, and d such that* $a \leq b$ *and* $c \leq d$ *, is it always true that* $ac \leq bd$ *? (Counterexample)*

Chapter 2

Sets, Relations and Applications

This course aims to cover some basic of set theory and it's properties. Throughout this chapter, we will learn about sets, relations and functions. we will infer that sets and relations are interconnected with each other (relations define the connection between the two given sets). After that, we will delve further in relations where we define another kind that can be considered a function.

2.1 Basic concepts of set theory

2.1.1 Sets and Elements

Definition 2.1.1. *Intuitively, a set is a collection of objects with certain properties. The objects in a set are called the elements or members of the set. We usually use uppercase letters to denote sets and lowercase letters to denote elements of sets. If a is an element of a set A, we write* $a \in A$ *. If a is not an element of a set A, we write* $a \in A$ *. To specify a set, we can list all of its elements, if possible, or we can use a defining rule. For instance, to specify the fact that a set A contains four elements* a, b, c, d *we write :*

$$A = \{a, b, c, d\}.$$

Although sets can contain many different types of elements, numbers are probably the most common for mathematics. For this reason particular im-

portant sets of numbers have been given their own symbols.
$\mathbb{N}$: *The set of natural numbers .*
$\mathbb{Z}$: *The set of integers .*
$\mathbb{Q}$: *The set of rational numbers.*
$\mathbb{R}$: *The set of real numbers .*
$\mathbb{C}$: *The set of complex numbers .*

An empty set, denoted by $\emptyset$ *, is a set that does not contain any elements. A set* $E = \{a\}$ *, consisting of a single element, is called a singleton.*

Example 2.1.1. ***1)*** *The set A given by* $A = \{a1, 2, 3, 4\}$ *is an explicit description.*
2) *The set* $\{x$ *, is a prime number*$\}$ *is implicit.*
3) $\{x \ : \ x \in \{1, 3, 5\}$ *and* $x \leq 1\}$ *is an empty set.*

Definition 2.1.2. *If a set A contains exactly n elements where n is a non-negative integer, then A is a finite set, and n is called the cardinality of A. We write Card(A) = n .*
If Card(A) is finite, A is a finite set; otherwise, A is infinite .

Example 2.1.2. ***1)*** $A = \{1, 5, 9, 4, \sqrt{3}\}$, *Card (A) = 5 .*
2) *The set* $B = \{x \ x \in \mathbb{Z} \ : -3 < x < 7\}$ *Card (B) = 8.*
3) *Card* $(\emptyset = 0)$.
4) *The set of positive integers is an infinite set.*

Definition 2.1.3. *(Equality)*
Two sets A and B are equal if each element of A is an element of B and vice versa. This is denoted, $A = B$ *. Formally ;*

$$A = B \iff \forall\, x \ : \ x \in A \iff x \in B. \tag{2.1.1}$$

Example 2.1.3. *Let* $A = \} - 2, -1, 0, 1, 2\}$ *and* $B = \{n \ : n \in \mathbb{Z} \ , \ -2 \leq n \leq 2\}$. *Then A=B.*

Definition 2.1.4. *(Equivalent sets) The two sets are equivalent, if the number of elements is the same for two different sets. For example,* $A = \{1, 2, 3, 4\}$ *and* $B = \{7, a, 3, z\}$ *are equivalent.*

Definition 2.1.5. *(Subset) The set A is a subset of B denoted by* $A \subseteq B$ *if and only if every element of A is olso an element of B.*

$$A \subseteq B \iff \forall\, x \ : \ x \in A \implies x \in B. \tag{2.1.2}$$

Remark 2.1.1.
• *If* $A \subseteq B$ *, and* $A \neq B$ *, then A is said to be a proper subset of B and it is denoted by* $A \subset B$ *.*
• $A \nsubseteq B$ *is the negation of* $A \subseteq B$ *More formally,*

$$A \nsubseteq B \iff \overline{A \subseteq B}$$

$$\iff \overline{\forall\, x \;:\; x \in A \implies x \in B}$$

$$\implies \exists x \;:\; x \in A \;\wedge x \notin B$$

• *The empty set is a subset of every set,including the empty set itself .*

Example 2.1.4. ***1.*** $\{14, 22, 55\} \subseteq \{14, 22, 55\}$.
2. $\{a, x\} \subset \{a, y, y, t\}$.
3. $\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R}$.
4. $\{\sqrt{2}, i\} \nsubseteq \mathbb{R}$.

Definition 2.1.6. *Given a set E, the power set of E is the set of all subsets of E. The power set is denoted by P(E). Formally*

$$P(E) = \{A, A \subseteq E\}. \tag{2.1.3}$$

Remark 2.1.2. • *The number of elements in the power set of E is* 2^n *, where n is the number of elements in set E . That is card(P(E))* $= 2^n$ *, where n = card(E).*
• $A \in P(E)$ *, means* $A \subseteq E$*.*

Example 2.1.5. *If* $E = \{1, 2, 3\}$ *, then card(P(E)) = 8 and*

$$P(E) = \{\emptyset, \{1\}, , \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}.$$

2.1.2 Set Operations

Definition 2.1.7. *(Set Intersection)*
The intersection of A and B denoted by $A \cap B$ *is the set of all elements that are in both A and B. That is,*

$$A \cap B = \{x \;:\; x \in A \;\; and \;\; x \in B\}. \tag{2.1.4}$$

We say that A and B are disjoint sets if $A \cap B = \emptyset$*.*

Example 2.1.6. ***1.*** $\{1,2,3,4,5\} \cap \{6,8,4,5,3\} = \{3,4,5\}$.
2. $\{x \,; x \geq 0\} \cap \{x; x \geq 2\} = \{x; x \geq 2\}$.
3. $\mathbb{N} \cap \mathbb{Z} \cap \mathbb{R} = \mathbb{N}$.

Definition 2.1.8. *(Set Union)*
The union of A and B denoted by $A \cup B$ *is the set of all elements that are in A or in B or in both A and B. That is,*

$$A \cup B = \{x \ : \ x \in A \ \ or \ \ x \in B\}. \tag{2.1.5}$$

Example 2.1.7. ***1.*** $\{1,2,3,4,5\} \cup \{6,8,4,5,3\} = \{1,2,3,4,5,6,8\}$.
2. $\{x \,; x \geq 0\} \cup \{x; x \geq 2\} = \{x; x \geq 0\}$.
3. $\mathbb{N} \cup \mathbb{Z} \cup \mathbb{R} = \mathbb{R}$.

Lemma 2.1.1. *For any two sets A and B, we have*

$$card(A \cup B) = card(A) + card(B) - card(A \cap B). \tag{2.1.6}$$

Theorem 2.1.2. *If A and B are any sets, then :*
1. $A \cap B \subseteq A$ *and* $A \cap B \subseteq B$.
2. $A \subseteq A \cup B$ *and* $B \subseteq A \cup B$.

Properties 2.1.3. ***1.*** *Commutative laws:*

$$A \cap B = B \cap A \ \ ; \ \ A \cup B = B \cup A.$$

2. *Associative laws:*

$$(A \cap B) \cap C = A \cap (B \cap C) \ \ ; \ \ (A \cup B) \cup C = A \cup (B \cup C).$$

3. *Distributive laws:*

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C) \ \ ; \ \ A \cup (B \cap C) = (A \cup B) \cap (A \cup C).$$

4. *Domination laws:*

$$A \cap \emptyset = \emptyset \ ; \ A \cup E = E.$$

5. *Identity laws:*

$$A \cup \emptyset = A \ ; \ A \cap E = A.$$

6. *Idempotent laws:*

$$A \cup A = A \ ; \ A \cap A = A.$$

Definition 2.1.9. *(Set complement)*
Let A subset of the universal set E. The complement of A relative to E denoted by $C_E(A)$ *or* $\overline{A}$ *is the set of elements that are in E and not in A. That is,*

$$C_E(A) = \overline{A} = \{x \in E \ ; \ x \notin A\}. \tag{2.1.7}$$

Example 2.1.8. *Let the universe be* $\mathbb{R}$ *:*
1. $\overline{\{0\}} = \{x; \ x \neq 0\} = \mathbb{R}^*$
2. $\overline{\{-1, 1\}} =]-\infty, -1[\cup]1, +\infty[$.

Properties 2.1.4. *Let A and B subsets of the universal set E :*
▶ $\overline{E} = C_E^E = \emptyset$..
▶ $C_E(C_E A) = \overline{\overline{A}} = A$.
▶ $A \cup \overline{A} = E$ *and* $A \cap \overline{A} = \emptyset$.
▶ $A \subset B \implies \overline{B} \subset \overline{A}$.
▶ $\overline{A \cap B} = \overline{A} \cup \overline{B}$ *and* $\overline{A \cup B} = \overline{A} \cap \overline{B}$.

Definition 2.1.10. *(Set difference)*
The difference of A and B is the set of elements that are in A but not in B , denoted by $A \backslash B$ *. That is,*

$$A \backslash B = \{x; x \in A \wedge x \notin B\}. \tag{2.1.8}$$

Example 2.1.9. ***1.*** $\{1, 2, 3, 4\} \setminus \{3, 4, 5\} = \{1, 2\}$.
2. $\mathbb{R} \backslash \{0\} = \{x; x \in \mathbb{R} \ \wedge x \neq 0\}$.

Properties 2.1.5. *Let A and B subsets of the universal set E:*
• $A \backslash B = A \cap \overline{B}$.
• $A \backslash B \subset A$.
• $A \backslash A = \emptyset$.
• $A \subset B \iff A \backslash B = \emptyset$.

Definition 2.1.11. *(Set Symmetric Difference)*
The symmetric difference of set A and set B, denoted by $A \triangle B$*, is the set containing those elements in exactly one of A and B. Formally :*

$$A \triangle B = (A \backslash B) \cup (B \backslash A) = (A \cap \overline{B}) \cup (B \cap \overline{A}). \tag{2.1.9}$$

Example 2.1.10. *If* $A = \{1, 2, 3, 4, 5, 10\}$ *and* $B = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ *, then* $A \backslash B = \{10\}$ *and* $B \backslash A = \{0, 7, 8, 9\}$ *. Hence* $A \triangle B = \{0, 7, 8, 9, 10\}$.

Properties 2.1.6. • *$A \triangle B = B \triangle A$.*
• $A \triangle \emptyset = A$.
• $A \triangle A = \emptyset$.
• $A \triangle B = \overline{A} \triangle \overline{B}$.
• $A \triangle B = (A \cup B) \backslash (A \cap B)$.

Definition 2.1.12. *(Set Cartesian product)*
The Cartesian product of A and B denoted by $A \times B$ is the set of all ordered pairs. That is :

$$A \times B = \{(a,b) : a \in A \wedge b \in B\}. \tag{2.1.10}$$

The equality in $A \times B$ is defined by:

$$(x_1, y_1) = (x_2, y_2) \iff x_1 = x_2 \wedge y_1 = y_2.$$

Example 2.1.11. *Let $A = \{1,2,3\}$ and $B = \{4,5\}$, then :*
$A \times B = \{(1,4),(1,5),(2,4),(2,5),(3,4),(3,5)\}$.
$B \times A = \{(4,1),(5,1),(4,2),(5,2),(4,3),(5,3)\}$.

Remark 2.1.3. *In general,*
• $A \times B \neq B \times A$.
• *card* $(A \times B) =$ *card* $(A) \times$ *card* (B).

Exercise 2.1.1. *Let A, B and C be three subsets of the set E. Show that:*

1. $A = B \iff A \cap B = A \cup B$.
2. $A \cup B = A \cap C \iff B \subset A \subset C$.
3. $A \cap B = \phi \iff \overline{A} \cup \overline{B} = E$.
4. $A \Delta B = \phi \iff A = B$.
5. $(A \cap B) \backslash C = (A \backslash C) \cap (B \backslash C) = (A \backslash C) \cap B = (B \backslash C) \cap A$.

2.2 Binary Relations on a Set

A relation is an association between objects. A book on a table is an example of the relation of one object being on another. It is especially common to speak of relations among people. For example, one person could be the niece of another. In mathematics, there are many relations such as equals and less-than that describe associations between numbers. To formalize this idea, we make the next definition.

2.2.1 Basic Defnitions

Definition 2.2.1. *(Binary Relation)*

Let E be a set. A binary relation $\Re$ on E is a property that applies to pairs of elements from E . We denote $x\Re y$ to indicate that the property is true for the pair $(x;y) \in E \times E$.

Example 2.2.1. ***1.*** *The inequality $\leq$ is a relation on $\mathbb{N}$; $\mathbb{Z}$, and $\mathbb{R}$.*

2. *The inclusion relation in the power set of E : $A\Re B \iff A \subset B$.*

3. *The divisibility relation on the integers: $m\Re n \iff m$ divides n.*

Remark 2.2.1. • *In other words, a relation from A to B is a set $\Re$ of ordered pairs where the first element of each ordered pair comes from A and the second element comes from B.*

- *If $(a, b) \in \Re$, then we say that a is related to b by $\Re$.*

- *$a\Re b$ write to express that $(a, b) \in \Re$ and $a\Re b$ to express that $(a, b) \notin \Re$*
- *A relation on a set A is a subset of $A \times A$.*

Definition 2.2.2. *(Properties of Relations)*
There are several properties that are used to classify relations on a set. We will introduce the most important of these here :
Let $\Re$ be a binary relation in E :

- *$\Re$ is reflexive Iff $\forall x \in E : x\Re x$.*

- *$\Re$ is symmetric Iff $\forall x, y \in E : x\Re y \implies y\Re x$.*

- $\Re$ *is anti-symmetric Iff* $\forall x, y \in E : (x\Re) \wedge (y\Re x) \implies x = y.$
- $\Re$ *is transitive Iff* $\forall x, y, z \in E : (x\Re) \wedge (y\Re z) \implies x\Re z.$

2.2.2 Equivalence Relation

Definition 2.2.3. *(Equivalence Relation)*

A relation $\Re$ is said to be an equivalence relation if it is simultaneously reflexive, symmetric, and transitive on E .

Example 2.2.2. *The relation $\Re$ of " being parallel " is an equivalence relation for the set E of all lines in the plane :*

1. *Reflexivity : A line is parallel to itself .*
2. *Symmetry : If line D is parallel to D', then D' is parallel to D .*
3. *Transitivity : If line D is parallel to D' and D' is parallel to D", then D is parallel to D" .*

Definition 2.2.4. *(Equivalence Classes)*
Let $\Re$ be an equivalence relation on E . Let $x \in E$, the equivalence class of x denoted by $C(x)$ or $\dot{x}$ is defined as the set of all those point of E which are related to x under the relation $\Re$.

$$C(x) = \dot{x} = \{y \in E \ : \ y\Re x\}.$$

Let $\Re$ be an equivalence relation on E and $a, b \in E$, where $a\Re b$, then a and b have the same equivalence class.

Definition 2.2.5. *(Quotient set of relation)*

Let $\Re$ be an equivalence relation on a set E. The quotient set of E by $\Re$ is the set of equivalence classes of $\Re$, denoted by $E/\Re$:

$$E/\Re \ = \ \{\dot{x} \ : \ x \in E\}.$$

Example 2.2.3. *Consider the following relation on $\mathbb{Z}$:*

$$x\Re y \iff \exists k \in \mathbb{Z} \ : \ x - y = 2k.$$

1. *Reflexivity :* $\Re$ *is reflexive because* $\exists k = 0 \;:\; x - x = 2\times 0 = 0,$ *thus* $x\Re x$.
2. *Symmetry : Suppose* $x\Re y$ *, then :*

$$\exists k \in \mathbb{Z} \;:\; x - y = 2k \implies y - x = 2k',\ \textit{with}\ k' = -k \in \mathbb{Z}.$$

Therefore, $y\Re x$ *. Hence,* $\Re$ *is symmetric.*
3. *Transitivity : Suppose* $x\Re y$ *and* $y\Re z$ *. Then,*

$$\exists k \in \mathbb{Z} \;:\; x - y = 2k\ ,\ \textit{and}\ \exists k' \in \mathbb{Z} \;:\; y - z = 2k'$$

by adding these equations, we obtain $x - z = 2k'',$ *with* $k'' = k + -k' \in \mathbb{Z}.$ *Thus,* $x\Re z$ *. Therefore,* $\Re$ *is transitive. Consequently,* $\Re$ *is an equivalence relation.*
The equivalence class of an element x *:*

$$\begin{aligned}
\dot{x} &= \{y \in E \;:\; y\Re x\} \\
&= \{y \in E \;:\; x - y = 2k\} \\
&= \{x - 2k\ ,\ k \in \mathbb{Z}\} \\
\dot{0} &= \{y \in E \;:\; y\Re 0\} \\
&= \{y \in E \;:\; 0 - y = 2k\} \\
&= \{-2k\ ,\ k \in \mathbb{Z}\} \\
&= \{..., -4, -2, 0, 2, 4, ...\} \\
\dot{1} &= \{y \in E \;:\; y\Re 1\} \\
&= \{y \in E \;:\; 1 - y = 2k\} \\
&= \{1 - 2k\ ,\ k \in \mathbb{Z}\} \\
&= \{..., -3, -1, 0, 1, 3, ...\}
\end{aligned}$$

and $\dot{0} = \dot{2}$, $\dot{1} = \dot{3}$.
Therefore,

$$E/\Re = \{\dot{0}, \dot{1}\}.$$

2.2.3 Order Relation

Definition 2.2.6. *(Order Relation)*
A binary relation $\Re$ on E is an order relation if and only if it is reflexive, anti-symmetric, and transitive. We then say that $(E;\Re)$ is an ordered set.

Two elements x and y of E are said to be comparable if $x\Re y$ or $y\Re x$.

Definition 2.2.7. *(Total Order and Partial Order)*
Let $\Re$ be an order relation on E. If any two elements x and y are always comparable, we say that $\Re$ is a total order relation and the set E is called totally ordered. Otherwise (i.e., if there exist at least two non-comparable elements x and y), we say that $\Re$ is a partial order relation and the set E is called partially ordered.

Total Order :

$$\forall x \in E, \forall y \in E \ : \ (x\Re y) \vee (y\Re x).$$

Partial Order :

$$\exists x \in E, \exists y \in E \ : \ \overline{(x\Re y)} \wedge \overline{(y\Re x)}.$$

Example 2.2.4. • $\leq$ *is a total order on* $\mathbb{N}$; $\mathbb{Z}$, *and* $\mathbb{R}$.
• *The divisibility relation in* N^* *is a partial order.*

Exercise 2.2.1. *In* $\mathbb{R}$, *the binary relation* $\Re$ *is defined as follows :*

$$\forall x, y \in \mathbb{R} \ : \ x^2 - 1 = y^2 - 1.$$

1. *Show that* $\Re$ *is an equivalence relation on* $\mathbb{R}$.

2. *Determine the quotient set* $\Re/\mathbb{R}$.

Exercise 2.2.2. *Let* $\Re$ *be a binary relation on* $\mathbb{R}^3$ *defined by*

$$(x, y, z)\Re(a, b, c) \iff (\mid x - a \mid \leq b - y \text{ and } z = c).$$

1. *Show that* $\Re$ *is a partial order relation on* $\mathbb{R}^3$.

2. *Is the order total on* $\mathbb{R}^3$?

2.3 Applications

In this section we formally define what we mean by a" function ". This is done using only the set-theoretic concepts developed up to now. We introduce notation to simplify the discussion and definition of these concepts. Examples of simple functions such as the identity function, the characteristic function and constant functions are presented. Given a function, f, on a set A, we define the restriction of this function on a subset, D, of A. We state what we mean by " equal functions ". The expressions " one-to-one ", " injective ", " onto ", " surjective " and " bijective functions " are also defined.

2.3.1 Defnitions and Examples

Definition 2.3.1. *Let E and F be two sets. An application from E to F is any correspondence f associating each element x of E a single element y of F. E is the starting set. An element of E, usually x, is an antecedent or a pre-image. F is the arrival set. An element of F, usually y, which is associated with x by the application f is the image of x by f. All this is denoted by :*

$$f \ : \ E \longrightarrow F$$

$$x \longmapsto y = f(x)$$

We say that:

- *y is the image of x (under f).*
- *x is the pre-image of y (under f).*
- *f maps x onto y, and symbolize this statement by :* $f \ : \ x \longmapsto y$
- *E is the starting set of f*
- *F is the arrival set or codomain of f .*

Example 2.3.1.

- *The application : :*

$$f \ : \ E \longrightarrow E$$

$$x \longmapsto y = f(x) = x.$$

is called the identity application in E and is denoted : Id_E. • *Let* $A \subset E$ *. The application :*

$$f \ : \quad E \longrightarrow \ \{0,1\}$$

$$x \ \longmapsto \ y = f(x) = \begin{cases} 1, \ if \ x \in A. \\ 0, \ if \ x \in \overline{A}. \end{cases} .$$

is called the indicator or characteristic application of part A and is denoted by : 1_A or χ_A.

Remark 2.3.1. ***1.*** *An application is a function whose domain of dentition is the entire starting set.*
2. *The graph of an application $f: \ E \longrightarrow \ E$ is the set :*

$$\Gamma_f = \{(x, f(x)) \in E \times F \ ; x \in E\}.$$

Definition 2.3.2. *An application f is a function of E in F whose domain definition D_f is equal to E.*

Definition 2.3.3. *An application from a set E to a another set F is a relation which to every element $x \in E$ assigns a unique element $y \in F$. Formally, using predicate logic:*

$$f \ Application \iff \forall \ x \in E \ , \ \exists! \ y \in F \ : \ y = f(x). \tag{2.3.1}$$

Definition 2.3.4. *(The equality of applications)*
Two applications $f \ : \ E \longrightarrow \ F$ and $g \ : \ E' \longrightarrow \ F'$ are called equal if and only if they have the same domain $E = E'$, the same codomain $F = F'$ and $\forall \ x \in E \ : \ f(x) = g(x)$.

$$f = g \iff \begin{cases} E = E' \ and \ \ F = F' \\ \forall \ x \in E \ : \ f(x) = g(x). \end{cases} \tag{2.3.2}$$

Definition 2.3.5. *(Restrictions and Extensions)*
Let f be a function from E to F.
1. *The restriction of f to a subset $A \subset E$ is the function denoted $f_{|A} : A \ \longrightarrow F$ defined by :*

$$f_{|A} = f(x), \ \forall \in A.$$

2. *The extension of f to a set E' containing E is any function g from E' to F whose restriction is f.*

Example 2.3.2. *If f is the identity function from $\mathbb{R}^+$ to itself, it has infinitely many extensions to $\mathbb{R}$, among which:*

1. *The identity function on $\mathbb{R}$.*

2. *The absolute value function from $\mathbb{R}$ to itself .*

3. *The function h defined by $h(x) = \dfrac{1}{2}(x + \mid x \mid)$, which is identically zero on $\mathbb{R}^-$.*

2.3.2 Direct Image and Inverse Image

Often in mathematics, particularly in analysis and topology, one is interested in finding the set of image points or inverse image of an application acting on a given set, which brings us to the two following definitions that are waiting to be understood .

Definition 2.3.6. *(Image of a Subset)*
Let E and F be two sets and f an application from E to F .

For any part A of E , the direct image of A by f, denoted $f(A)$, is defined by :

$$f(A) = \{f(x)\ ,\ x \in A\}.$$
$$y \in f(A) \iff \exists x \in A, y = f(x).$$

i.e.: the images, of all elements x of A, which belong to F .
We have $f(A) \subset F$.

Definition 2.3.7. *(Inverse Image of a Subset)*
Let , $f : E \longrightarrow F$ nd consider the subset $B \subset F$. The inverse image of the subset B under f , which we write $f^{-1}(B)$ is the subset of E that consists of the pre-images of elements in B .

$$f^{-1}(B) = \{x \in E\ :\ f(x) \in B\}.$$

i.e.: The elements of E (not necessarily all of them) whose images $f(x)$ belong to B. We have, for $x \in E$, the equivalence :

$$x \in f^{-1}(B) \iff f(x) \in B..$$

Exercise 2.3.1. *Let* $f : \mathbb{R} \longrightarrow \mathbb{R}$ *be an application defined by :*

$$\forall x \in \mathbb{R}, , f(x) = x + 1.$$

1. *Find the direct image by f of the set :* $A = \{-2, -1, 1, 2, 3, 4\}$.

2. *. Find the reciprocal image by f of the set* $B = [-3, 2]$

Solution $f : \mathbb{R} \longrightarrow \mathbb{R}$, $f(x) = x + 1$

1. *et's find the direct image by f of the set* $A = \{-2, -1, 1, 2, 3, 4\}$:

$f(-3) = -2, f(-2) = -1, f(-1) = 0, f(0) = 1, f(1) = 2, f(2) = 3, f(3) = 4, f(4) = 5.$
Hence $f(A) = \{-2, -1, 0, 1, 2, 3, 4, 5\}$.

2. *Now , we have to find* $f^{-1}(B)$ *; the reciprocal image , by f , of the interval* $B = [-3, 2]$:

$$f^{-1}(B) = f^{-1}([-3, 2]) = \{x \in \mathbb{R}, f(x) \in [-3, 2]\}.$$

Let $x \in \mathbb{R}$:

$$\begin{aligned} x \in f^{-1}(B) \quad &\Longleftrightarrow \quad f(x) \in [-3, 2] \\ &\Longleftrightarrow \quad -3 \leq f(x) \leq 2 \\ &\Longleftrightarrow \quad -3 \leq x + 1 \leq 2 \\ &\Longleftrightarrow \quad -4 \leq x \leq 1 \\ &\Longleftrightarrow \quad x \in [-4, 1]. \end{aligned}$$

Hence , $f^{-1}(B) = [-4, 1]$.

Properties 2.3.1. *Let* $f : E \longrightarrow F$ *be a application , A ; B be two subsets of E, and C ; D be two subsets of F , we have the following properties. Notice how the inverse image always preserves unions and intersections, although not always true for the image of an application. Then the images of intersections and unions satisfy :*

- $f(A \cap B) \subset f(A) \cap f(B)$.

- $f(A \cup B) = f(A) \cup f(B)$.
- $f^{-1}(C \cap D) \ = \ f^{-1}(C) \cap f^{-1}(D)$.
- $f^{-1}(C \cup D) = f^{-1}(C) \cup f^{-1}(D)$.
- $A \subset B \implies f(A) \subset f(B)$.
- $C \subset D \implies f^{-1}(C) \subset f^{-1}(D)$.
- $f^{-1}(\overline{C}) = \overline{f^{-1}(C}$.
- $A \subset \ f^{-1}(f(A))$.
- $f^{-1}(f(C)) \subset C$.

Example 2.3.3. *Let the application :* $f : \mathbb{R} \longrightarrow \mathbb{R}$ *be defined by :* $\forall x \in \mathbb{R}, f(x) = x^2$ *and the subsets of* $\mathbb{R}$ *,* $A = [0,1]$ *and* $B = [-1,0]$ *. We then have :*
$f(A) = f(B) = [0,1]$ *hence* $f(A) \cap f(B) = [0,1]$ *, but* $A \cap B = \{0\}$ *, hence* $f(A \cap B) = f(0) = \{0\}$ *so* $f(A \cap B) \subset f(A) \cap f(B)$ *only without equality.*

2.3.3 Injective, Surjective and Bijective Applications

Definition 2.3.8. *(Injective)*
Let E ; F be two sets and $f : E \longrightarrow F$ *be a application . f is injective (one-to-one) if every element in E is mapped to a unique element in F. More formally :*

$$f \text{ Injective} \iff \forall x_1, x_2 \in E \ : x_1 \neq x_2 \implies f(x_1) \neq f(x_2).$$

or

$$f \text{ Injective} \iff \forall x_1, x_2 \in E \ \ f(x_1) = f(x_2) \implies x_1 = x_2.$$

Example 2.3.4. *Prove that* $f : [0, +\infty[\longrightarrow \mathbb{R}$ *defined by* $f(x) = x^2$ *is injective.*
Suppose that $f(x_1) = f(x_2)$ *for some* $x_1, x_2 \in [0, +\infty[$*, then*

$$f(x_1) = f(x_2) \iff x_1^2 = x_2^2$$

$$\iff | x_1 |=| x_2 |$$

$$\iff x_1 = x_2.$$

Thus, f is injective.

Definition 2.3.9. *(Surjective)*
An application $f : E \longrightarrow F$ is surjective (or onto) if and only if for every element $y \in F$, there is an element $x \in E$ with $y = f(x)$.

$$\forall\ y \in F, \exists\ x \in E\ :\ y = f(x).$$

Another formulation: f is surjective if and only if $f(E) = F$.

Example 2.3.5. *Prove that $f :\ \mathbb{R} \longrightarrow [0, +\infty[$ defined by $f(x) = x^2$ is surjective.*
proof that : $\forall\ y \in [0, +\infty[, \exists\ x \in \mathbb{R}\ :\ y = f(x)$.

$$y = f(x) \iff y = x^2$$

$$\iff | x |= \sqrt{y}$$

$$\iff (x = \sqrt{y})\ \vee\ (x = -\sqrt{y}).$$

Hence, $\forall\ y \in [0, +\infty[, \exists\ x = \sqrt{y} \in \mathbb{R}\ :\ y = f(x).$. *Consequently , f is surjective*

Definition 2.3.10. *(Bijective)*
An application $f : E \longrightarrow F$ is bijective if and only if for every element $y \in F$, there is a unique element $x \in E$ with $y = f(x)$:

$$\forall\ y \in F, \exists!\ x \in E\ :\ y = f(x).$$

$$f \text{ bijective} \iff f \text{ injective} \wedge f \text{ surjective}.$$

Example 2.3.6. *Prove that $f :\ [0, +\infty[\longrightarrow [0, +\infty[$ defined by $f(x) = x^2$ is bijective.*
proof that : $\forall\ y \in [0, +\infty[, \exists!\ x \in [0, +\infty[\ :\ y = f(x)$.

$$y = f(x) \iff y = x^2$$

$$\iff x = \sqrt{y}$$

Hence, $\forall\ y \in [0,+\infty[, \exists!\ x = \sqrt{y} \in \mathbb{R}\ :\ y = f(x).$. *Consequently* , f *is bijective*

Definition 2.3.11. *(Inverse Application)*
Let f be bijection from the set E to the set F. The inverse function of f is the function that assigns to an element y belonging to F the unique element x in E such that $f(x) = y$*. The inverse function of f is denoted by :* $f^{-1} : F \longrightarrow E$. *Hence,* $f^{-1}(y) = x$ *when* $f(x) = y$.

Properties 2.3.2. *Let* $f : E \longrightarrow F$ *be a bijective application , then* $f^{-1} : F \longrightarrow E$ *is a bijective application.*

Example 2.3.7. *Let* $f : \mathbb{R} \longrightarrow \mathbb{R}$ *defined by* $f(x) = \dfrac{4x+2}{5}$ *suppose f is invertible so , find the inverse of f .*
proof that : $\forall\ y \in [0,+\infty[, \exists!\ x \in [0,+\infty[\ :\ y = f(x).$

$$f(x) = \frac{4x+2}{5} \iff y = \frac{4x+2}{5}$$

$$\implies x = \frac{5y-2}{4}$$

Then : $f^{-1}(x) = \dfrac{5x-2}{4}$.

Definition 2.3.12. *(Composite Applications)*

Given functions $f : E \longrightarrow F$ *and* $g : F \longrightarrow G$ *, we define the composition function* $g \circ f$ *of f and g as the function* $g \circ f : E \longrightarrow G$ *given by*

$$(g \circ f)(x) = g(f(x)),\ \forall x \in E.$$

Example 2.3.8. $f : \mathbb{R} \longrightarrow R$ *and* $g : \mathbb{R} \longrightarrow \mathbb{R}$ *with* $f(x) = -3x + 1$ *and* $g(x) = 2x - 5$ *:*
find $(f \circ g)(x)$ *and* $(g \circ f)(x)$.

$$\begin{aligned}(f \circ g)(x) &= f[g(x)] \\ &= f(2x-5) \\ &= -3(2x-5)+1. \\ &= -6x-15.\end{aligned}$$

$$\begin{aligned}(g\circ f)(x) &= g[f(x)]\\ &= g(-3x+1)\\ &= 2(-3x+1)-5.\\ &= -6x-3.\end{aligned}$$

We remark that $(f\circ g)(x)$ *and* $(g\circ f)(x)$ *produced different answers.*

Definition 2.3.13. *If* $f : E \longrightarrow F$ *and* $g : F \longrightarrow E$ *, then f and g are inverse applications of one another relative to composition iff .*

$$g\circ f = I_E \text{ and } f\circ g = I_F.$$

▶ If f has an inverse, then it is unique

Theorem 2.3.3. *Let* $f : E \longrightarrow F$ *and* $g : F \longrightarrow G$ *be two functions and let* $B \subset G$ *. The following hold:*

1. $(g\circ f)^{-1}(B) = f^{-1}(g^{-1}(B))$.
2. *If f and g are injective, then* $g\circ f$ *is injective .*
3. *If f and g are surjective, then* $g\circ f$ *is surjective.*
4. *If* $g\circ f$ *is injective, then f is injective .*
5. *If* $g\circ f$ *is surjective, then g is surjective.*

Exercise 2.3.2. *Let* $f : E \longrightarrow F$ *be a function. Let A; B be two subsets of the set E and C; D be two subsets of the set F. Show that:*

1. $f(A\cap B) \subset f(A)\cap f(B)$, $f(A\cup B) = f(A)\cup f(B)$.
2. *f is injective* $\Longleftrightarrow$ $f(A\cap B) = f(A)\cap f(B)$.
3. $f^{-1}(C\cap D) = f^{-1}(C)\cap f^{-1}(D)$, $f^{-1}(C\cup D) = f^{-1}(C)\cup f^{-1}(D)$.
4. $f(f^{-1}(C)) \subset C$.

5. *f is surjective* $\iff$ $f(f^{-1}(C)) = C$.

6. $f^{-1}(\overline{C}) = \overline{f^{-1}(C)}$.

7. $f^{-1}(C\Delta D) = f^{-1}(C)\Delta f^{-1}(D)$.

Exercise 2.3.3. *Consider the function f defined by :*

$$\begin{array}{rcl} f & : & \mathbb{R} \longrightarrow \mathbb{R} \\ & & x \longmapsto f(x) = \dfrac{2x}{1+x^2}. \end{array}$$

1. *Is f injective ? Surjective ?*

2. *Show that $f(\mathbb{R}) = [-1, 1]$.*

3. *Show that the function g defined by :*

$$\begin{array}{rcl} g & : & [-1,1] \longrightarrow [-1,1] \\ & & x \longmapsto g(x) = f(x). \end{array}$$

is a bijection and find its inverse function g^{-1} .

Chapter 3

Real Functions of One Real Variable

This chapter is devoted to the functions of a real variable which are often modeled for the study of curves and mechanical calculations. In this regard, we present the foundations of the functions of a real variable, where the objective is to know and interpret the notion of the limit, continuity and differentiability of a function, and to present some of their properties

3.1 Notions of function

3.1.1 General definitions

Definition 3.1.1. *We call digital function on a set D any process which, at all element x of D , allows to associate at most one element of the set $\mathbb{R}$, then called image of ? and denoted $f(x)$. The elements of $\mathbb{R}$ which have an image by f form the definition set of f , noted D .*

Definition 3.1.2. *We call a graph, or representative curve, of a function f defined on an interval $D \subseteq \mathbb{R}$, the set*

$$\Gamma_f = \{(x, f(x)) \ , \ x \in D\}.$$

formed from the points $(x, f(x)) \in \mathbb{R}^2$ of the plan provided with an orthonormal coordinate system (o, I, J).

3.1.2 Bounded functions, monotonic function

Definition 3.1.3. *Let $f : D \longrightarrow \mathbb{R}$ be a function. We say that:*

a) *f is bounded from above if there is a number M such that for all x from D , $f(x) \leq M$, we write:*

$$\exists M \in \mathbb{R} , \forall x \in D : \; f(x) \preceq M.$$

b) *f is bounded from below if there is a number m such that for all x from D : $f(x) \geq m$. we write: such that for all x from D , $f(x) \leq M$, we write:*

$$\exists m \in \mathbb{R} , \forall x \in D : \; f(x) \geq m.$$

c) *f is bounded if it is bounded both from above and below. It is to say: such that for all x from D , $f(x) \leq M$, we write:*

$$\exists M \in \mathbb{R} , \forall x \in D : \mid f(x) \mid \leq m.$$

d) *Function that is not bounded is called unbounded function .*

Definition 3.1.4. *Let $f : D \longrightarrow \mathbb{R}$ be a function . We say that:*

►f *is increasing on* $D \iff \forall \, x, y \in D \; : \; x < y \implies f(x) \leq f(y).$

►f *is strictly increasing on* $D \iff \forall \, x, y \in D \; : \; x < y \implies f(x) < f(y).$

►f *is decreasing on* $D \iff \forall \, x, y \in D \; : \; x < y \implies f(x) \geq f(y).$

►f *is strictly decreasing on* $D \iff \forall \, x, y \in D \; : \; x < y \implies f(x) > f(y).$

► *f is monotonic (strictly monotonic, respectively) on D if f is increasing or decreasing (strictly increasing or strictly decreasing, resp) on D .*

Example 3.1.1.

a) *Exponential functions $exp : \mathbb{R} \longrightarrow \mathbb{R}$ is strictly increasing.*

b) *The absolute value function $x \longrightarrow |x|$ defined on $\mathbb{R}$ is not monotonic*

3.1.3 Even , odd , periodic function

Definition 3.1.5. *Let I be an interval of $\mathbb{R}$ symmetric with respect to 0 . Let $f : I \longrightarrow \mathbb{R}$ be a function. We say that:*

$$\blacktriangleright\ f \text{ is even on } I \iff \forall\, x \in I\ :\ f(-x) = f(x).$$

$$\blacktriangleright\ f \text{ is odd on } I \iff \forall\, x \in I\ :\ f(-x) = -f(x).$$

Example 3.1.2.

a) *The function defined on $\mathbb{R}$ by $x \longrightarrow x^m$, ($m \in \mathbb{N}$) is even.*

b) *The function defined on R by $x \longrightarrow x^{2m+1}$, ($m \in \mathbb{N}$) is odd.*

Definition 3.1.6. *Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be a function and T a real number , $T > 0$. The function f is called periodic of period T if:*

$$\forall x \in \mathbb{R}\ :\ f(x+T) = f(x).$$

Example 3.1.3. *The functions* cos *and* sin *are* 2π *periodic . The tangent function is* π *periodic.*

3.2 Limit of a Function

The concept of a function is the fundamental concept of calculus and analysis. Real function f of one real variable is a mapping from the set $D \subseteq \mathbb{R}$, a subset in real numbers $\mathbb{R}$, to the set of all real numbers $\mathbb{R}$.

$$f\ :\ D \longrightarrow \mathbb{R}$$

$$x \longmapsto f(x).$$

D is the domain of the function f , where $D = \{x \in \mathbb{R}, f(x)\ makes\ sense\}$.

3.2.1 Limits

Definition 3.2.1. *(Limits)*
Limits are used to analyze the local behavior of functions near points of interest. A function f is said to have a limit l at x_0 if it is possible to make

the function arbitrarily close to l by choosing values closer and closer to x_0. Note that the actual value at x_0 is irrelevant to the value of the limit.

The notation is as follows:

$$\lim_{x \longrightarrow x_0} f(x) = l.$$

which is read as : (the limit of $f(x)$ as x approaches x_0 is l) We can write :

$$\lim_{x \longrightarrow x_0} f(x) = l \iff \forall \epsilon > 0, \exists \eta > 0, \forall x \in D : \mid x - x_0 \mid < \eta \implies \mid f(x) - l \mid < \epsilon.$$

3.2.2 Left-hand Limits

Definition 3.2.2. *(The left-hand limit)*
The left-hand limit of a function f as it approaches x_0 is the limit :

$$\lim_{x \longrightarrow x_0^-} f(x) = l.$$

We can write :

$$\lim_{x \longrightarrow x_0^-} f(x) = l \iff \forall \epsilon > 0, \exists \eta > 0, \forall x \in D : x \in]x_0 - \eta, x_0[\implies \mid f(x) - l \mid < \epsilon.$$

3.2.3 Right-hand Limits

Definition 3.2.3. *(The right-hand limit)*
The right-hand limit of a function f as it approaches x_0 is the limit :

$$\lim_{x \longrightarrow x_0^+} f(x) = l.$$

We can write :

$$\lim_{x \longrightarrow x_0^+} f(x) = l \iff \forall \epsilon > 0, \exists \eta > 0, \forall x \in D : x \in]x_0, x_0 + \eta[\implies \mid f(x) - l \mid < \epsilon.$$

3.2.4 Existence of the limit

Theorem 3.2.1. *(Existence of the limit)*
Let f be a real valued function defined on a set $D \subseteq \mathbb{R}$ *, then* $\lim\limits_{x \longrightarrow x_0} f(x) = l.$
exists if and only if:

$$\lim_{x \longrightarrow x_0^-} f(x) = \lim_{x \longrightarrow x_0^+} f(x) = \lim_{x \longrightarrow x_0} f(x).$$

Example 3.2.1. *Let f be a real valued function defined by:*

$$f \ : \ [-1;1] \ \longrightarrow \ \mathbb{R}$$

$$f(x) = \begin{cases} 0, & , \ -1 \leq x \leq 0. \\ 1, & , \ 0 \leq x \leq 1. \end{cases} .$$

$\lim\limits_{x \longrightarrow x_0} f(x)$ *does not exist because* $\lim\limits_{x \longrightarrow 0^-} f(x) = 0$ *and* $\lim\limits_{x \longrightarrow 0^+} f(x) = 1.$

3.2.5 Limit of a function at infinity

Definition 3.2.4. *Limits at infinity are used to describe the behavior of functions as the independent variable increases or decreases without bound. we write :*

$$\lim_{x \longrightarrow \pm\infty} f(x) = l.$$

The graph of the function will have a horizontal asymptote at $y = l$.

a) $\lim\limits_{x \longrightarrow +\infty} f(x) = l.$ if

$$\forall \ \varepsilon > 0 \ , \exists \ A > 0 \ : \ x > A \ \Longrightarrow | \ f(x) - l \ | < \varepsilon.$$

We say that f has a (finite) limit l at $+\infty$ if when x becomes very large $f(x)$ becomes very close to l .

b) $\lim\limits_{x \longrightarrow -\infty} f(x) = l.$ if

$$\forall \ \varepsilon > 0 \ , \exists \ A > 0 \ : \ x < -A \ \Longrightarrow | \ f(x) - l \ | < \varepsilon.$$

we say that f has a (finite) limit at $-\infty$ if, when x becomes very large in negative value , $f(x)$ becomes very close to l.

3.2.6 Properties of the limit

Properties 3.2.2.

The following properties remain true if one replaces each limit by a one-sided limit, or a limit for $x \longrightarrow \infty$.
Let f and g be two given functions whose limits for $x \longrightarrow x_0$ *we know,*

$$\lim_{x \longrightarrow x_0} f(x) = l_1 \ , \ \lim_{x \longrightarrow x_0} g(x) = l_2.$$

1) $$\lim_{x \longrightarrow x_0} [f(x) + g(x)] = \lim_{x \longrightarrow x_0} f(x) + \lim_{x \longrightarrow x_0} g(x) = l_1 + l_2.$$

2) $$\lim_{x \longrightarrow x_0} [f(x) \times g(x)] = \lim_{x \longrightarrow x_0} f(x) \times \lim_{x \longrightarrow x_0} g(x) = l_1 \times l_2.$$

3) $$\lim_{x \longrightarrow x_0} [\lambda f(x)] = \lambda \lim_{x \longrightarrow x_0} f(x) = \lambda l_1.$$

4) $$\lim_{x \longrightarrow x_0} \frac{f(x)}{g(x)} = \frac{l_1}{l_2}. \ \ if \ \ l_2 \neq 0$$

Theorem 3.2.3. *Suppose that :*

$$f(x) \leq h(x) \leq g(x).$$

(for all x) and that

$$\lim_{x \longrightarrow x_0} f(x) = \lim_{x \longrightarrow x_0} g(x).$$

Then

$$\lim_{x \longrightarrow x_0} f(x) = \lim_{x \longrightarrow x_0} g(x) = \lim_{x \longrightarrow x_0} h(x).$$

Corollary 3.2.4. *If* $\lim_{x \longrightarrow x_0} f(x) = 0$ *and g is a bounded function. Then*

$$\lim_{x \longrightarrow x_0} (f \times g)(x) = 0.$$

Indeterminate form Some forms of limits are called indeterminate. An indeterminate form is an expression whose limit cannot be determined solely from the limits of the individual functions. Example of indeterminate forms:

$$\frac{0}{0} \ , \ \frac{\infty}{\infty} \ , \ +\infty - \infty \ , \ 0 \times \infty \ , \ \infty^0.$$

Exercise 3.2.1. *Use the definition of limit to prove that :*

$$\textbf{\textit{1)}}\quad \lim_{x\ \longrightarrow 2}(3x-7)=-1.$$

$$\textbf{\textit{2)}}\quad \lim_{x\ \longrightarrow 3}(x^2+1)=10.$$

$$\textbf{\textit{2)}}\quad \lim_{x\ \longrightarrow 1}\frac{x+3}{x+1}=2.$$

Exercise 3.2.2. *Let f be the function given by :*

$$f\ :\ \mathbb{R}\ \longrightarrow\ \mathbb{R}$$

$$f(x)=\begin{cases} x, & if\ x\in\mathbb{Q}\cap[0,1]. \\ 1-x, & if\ x\in\overline{\mathbb{Q}}\cap[0,1]. \end{cases}$$

Determine which of the following limits exist. For those that exist find their values.

1) $\lim\limits_{x\longrightarrow\frac{1}{2}} f(x).$

2) $\lim\limits_{x\longrightarrow 0} f(x).$

3) $\lim\limits_{x\longrightarrow 1} f(x).$

Exercise 3.2.3. *Let $a\in\mathbb{R}$, let f be the function given by :*

$$f\ :\ \mathbb{R}\ \longrightarrow\ \mathbb{R}$$

$$f(x)=\begin{cases} x^2, & if\ x>1. \\ ax-1, & if\ x\leq 1. \end{cases}$$

Find the value of a such that $\lim\limits_{x\longrightarrow 1} f(x)$ exists.

3.3 Continuous Functions

Continuous functions are functions that take nearby values at nearby points.

3.3.1 Continuity of a function at a point

Definition 3.3.1. *Let $I \subseteq \mathbb{R}$ and $f : \mathbb{R} \longrightarrow \mathbb{R}$ be a function. we say that f is continuous at a point $x_0 \in I$ if :*

$$\lim_{x \longrightarrow x_0} f(x) = f(x_0).$$

Otherwise, f is said to be discontinues at x_0
We can write :

$$f \text{ is continuous at } x_0 \iff \forall \epsilon > 0, \exists \eta > 0, \forall x \in D : \mid x - x_0 \mid < \eta \implies \mid f(x) - f(x_0) \mid < \epsilon.$$

Remark 3.3.1. *A function f is continuous at $x = x_0$ if the following three conditions hold:*

1) *$f(x0)$ is defined (that is, x_0 belongs to the domain of f)*

2) *$\lim_{x \longrightarrow x_0} f(x)$ exists (that is, left-hand limit = right-hand limit)*

3) *$\lim_{x \longrightarrow x_0} f(x) = f(x_0)$.*

Example 3.3.1. *Consider the function*

$$f : \mathbb{R} \longrightarrow \mathbb{R}$$

$$f(x) = \mid x \mid = \begin{cases} x, & if\ x \geq 0. \\ -x, & if\ x \leq 0. \end{cases}$$

This function is continuous at all x_0.

Example 3.3.2. *Consider the function*

$$f : \mathbb{R} \longrightarrow \mathbb{R}$$

$$f(x) = \begin{cases} \dfrac{\sin x}{x}, & if\ x \neq 0. \\ 1, & if\ x = 0. \end{cases}.$$

This function is continuous at all 0 , because :

$$\lim_{x \longrightarrow 0} f(x) = \lim_{x \longrightarrow 0} \frac{\sin x}{x} = f(0) = 1.$$

3.3.2 Continuity of a function in an interval

Definition 3.3.2. ***1.*** *f is said to be continuous in an open interval $]a,b[$ if it is continuous at every point x_0 in this interval.*

2. *f is said to be continuous in the closed interval $[a;b]$ if :*

- *f is continuous in $]a;b[$:*
- *f is right continuous at a point a , i.e.* $\lim\limits_{x\longrightarrow a^+} f(x) = f(a)$.
- *f is left continuous at a point b , i.e.* $\lim\limits_{x\longrightarrow b^-} f(x) = f(b)$.

Example 3.3.3.
- *Every polynomial function is continuous on $\mathbb{R}$.*
- *Every rational function is continuous on its domain.*
- $\sin(\cdot)$ *and* $\cos(\cdot)$ *are continuous everywhere on* $\mathbb{R}$.
- *The square root is continuous on $\mathbb{R}^+$.*

3.3.3 Continuity on the left on the right

Definition 3.3.3. *Consider a function $f : D \longrightarrow \mathbb{R}$, I being an interval of $\mathbb{R}$.*
1. *The function is continuous to the right at x_0 if :*

$$\lim_{x\longrightarrow x_0^+} f(x) = f(x_0) \iff \forall\epsilon > 0, \exists\eta > 0, \forall x \in D : x_0 < x < x_0+\eta \implies |\ f(x)-f(x_0)\ | < \epsilon.$$

2. *The function is continuous to the left at x_0 if :*

$$\lim_{x\longrightarrow x_0^-} f(x) = f(x_0) \iff \forall\epsilon > 0, \exists\eta > 0, \forall x \in D : x_0-\eta < x < x_0 \implies |\ f(x)-f(x_0)\ | < \epsilon.$$

3. *f is continuous at x_0 if and only if these two limits exist and are equal:*

$$\lim_{x\longrightarrow x_0^+} f(x) = f(x_0) = \lim_{x\longrightarrow x_0^-} f(x).$$

Exercise 3.3.1. *Consider the function :*

$$f \;:\; \mathbb{R} \longrightarrow \mathbb{R}$$

$$f(x) = \begin{cases} x^2 + a, & if\ x > 2. \\ ax - 1, & if\ x \leq 2. \end{cases} .$$

Find the value of a such that f is continuous.

3.3.4 Operations on continuous functions

Theorem 3.3.1. *The basic properties of continuous functions follow from those of limits: If* $f : D \longrightarrow \mathbb{R}$ *and* $g : D \longrightarrow \mathbb{R}$ *are continuous at* x_0 *of* D *, and* α *is a constant, then :*

1. $f + g$ *is continuous at* x_0
2. αf *is continuous at* x_0
3. $f \times g$ *is continuous at* x_0
4. *If* $f(x_0) \neq 0$ *, then* $\dfrac{1}{f}$ *is continuous at* x_0.
5. *If* $g(x_0) \neq 0$ *, then* $\dfrac{f}{g}$ *is continuous at* x_0.

Theorem 3.3.2. *Let* $f : I \longrightarrow \mathbb{R}$ *and* $g : J \longrightarrow \mathbb{R}$ *two functions such that* $f(I) \subseteq J$ *. If f is continuous at* x_0 *of I and if g is continuous at* $f(x_0)$ *, then* $g \circ f$ *is continuous at* x_0

Example 3.3.4. *Determine whether* $h(x) = cos(x^2 - 5x + 2)$ *is continuous. Note that ,* $h(x) = f(g(x))$ *, where* $f(x) = cos(x)$ *and* $g(x) = x^2 - 5x + 2$ *Since both f and g are continuous for all x , then h is continuous for all x.*

3.3.5 Operations on continuous functions

We can redefine functions with removable discontinuities to obtain continuous functions.

Proposition 3.3.3. *When we can remove a discontinuity by redefining the function at that point, we call the discontinuity removable. (Not all discontinuities are removable, however.)*

If $\lim_{x \longrightarrow x_0} f(x) = l$ *, but* $f(x_0)$ *is not defined, we define a new function*

$$\widetilde{f}(x) = \begin{cases} f(x) \ , \ for \ x \neq x_0 \\ l \ , \quad for \ x = x_0. \end{cases}$$

which is continuous at x_0 *. It is called the continuous extension of* $f(x)$ *to* x_0*.*

Example 3.3.5. *Find a continuous extension of the function* $f(x) = \dfrac{\sin x}{x}$.
The domain of f is $D = \mathbb{R}^*$ *, then f is discontinuous at* $x = 0$ *because* $f(0)$ *is not defined.*
Since $\lim_{x \longrightarrow 0} f(x)$ *exists, the discontinuity is removable. We know that* $\lim_{x \longrightarrow 0} \dfrac{\sin x}{x} = 1$*. For the function to be continuous at zero we need to define* $f(0)$ *we make*

$$f(0) = \lim_{x \longrightarrow 0} \widetilde{f}(x) = \frac{\sin x}{x} = 1.$$

and redefine the function :

$$\widetilde{f}(x) = \begin{cases} f(x) \ , \ for x \neq 0 \\ 1 \ , \quad for \ x = 0. \end{cases}$$

We say $\widetilde{f}$ *is the continuous extension of f to* $x = 0$.

3.3.6 Intermediate Value Theorem (IVT)

he intermediate value theorem describes a key property of continuous functions. It states that a continuous function on an interval takes on all values between any two of its values.

Theorem 3.3.4. *Let* $f : [a, b] \longrightarrow \mathbb{R}$ *such that :*
- *f is continuous on the closed interval* $[a, b]$
- *k be any number between* $f(a)$ *and* $f(b)$.

Then, there exists at least $c \in]a, b[$ *such that* $f(c) = k$.

The most used version of the intermediate value theorem given as :

Theorem 3.3.5. *Let $f : [a, b] \longrightarrow \mathbb{R}$ such that :*
- *f is continuous on the closed interval $[a, b]$*
- *$f(a)f(b) < 0$*

Then, there exists at least $c \in]a, b[$ such that $f(c) = 0$.

Example 3.3.6. *Show that the equation $4x^3 - 6x^2 + 3x - 2 = 0$ has a solution in the interval $[1, 2]$.*

Consider the function $f(x) = 4x^3 - 6x^2 + 3x - 2$ over the closed interval $[1, 2]$.
The function f is a polynomial, therefore it is continuous over $[1, 2]$.
We have $f(1) = -1$ and $f(2) = 12$, hence $f(1)f(2) < 0$ by the Mean-Value-Theorem there exists a value c in the interval $]1, 2[$ such that $f(c) = 0$, i.e. there is a solution for the equation $f(x) = 0$, in the interval $]1, 2[$.

3.4 Differentiability of a function

3.4.1 Differentiability of a function at a point

Below, we note I a non-empty interval of $\mathbb{R}$.

Definition 3.4.1. *Let $f : D \longrightarrow \mathbb{R}$ be a function, and let $x_0 \in D$. we say that f is differentiable at x_0 if the limit*

$$\lim_{x \longrightarrow x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

exists, and finite. This limit is called the derivative of f at x_0, we note $f'(x_0)$.

Alternative formula for the derivative:

$$\lim_{h \longrightarrow 0} \frac{f(x_0 + h) - f(x_0)}{h}.$$

Properties 3.4.1.

If f is differentiable at x_0 , then the curve representing the function f have a tangent to the point $(x_0, f(x_0))$, with the slope $f'(x_0)$.

3.4.2 Left Differentiability and Right Differentiability

Definition 3.4.2.

- *The left-hand derivative of a function f at x_0.*

$$\lim_{x \longrightarrow x_0^-} \frac{f(x) - f(x_0)}{x - x_0}$$

- *The right-hand derivative of a function f at x_0.*

$$\lim_{x \longrightarrow x_0^+} \frac{f(x) - f(x_0)}{x - x_0}$$

f is differentiable at x_0 if and only if these two limits exist and are equal.

Example 3.4.1. *Show that $f(x) = \mid x - 1 \mid$ is not differentiable at $x = 0$.*

- *The right-hand derivative at x = 0 :*

$$\lim_{x \longrightarrow 1^+} \frac{\mid x - 1 \mid - 0}{x - 1} = \lim_{x \longrightarrow 1^+} \frac{x - 1}{x - 1} = 1$$

- *The left-hand derivative at x = 0 :*

$$\lim_{x \longrightarrow 1^-} \frac{\mid x - 1 \mid - 0}{x - 1} = \lim_{x \longrightarrow 1^-} \frac{-(x - 1)}{x - 1} = -1$$

So the right-hand and left-hand derivatives differ.

Remark 3.4.1. *We say that a function f is differentiable on an interval D when f is differentiable in any point of D .*

3.4.3 Differentiability of a function in an interval

Definition 3.4.3.

1. *f is said to be differentiable in an open interval $]a, b[$ if it is differentiable at every point x0 in this interval.*

2. *f is said to be differentiable in the closed interval $[a, b]$ if :*

- *f is differentiable in $]a, b[$:*

- *f is right differentiable at a point a , i.e.*

$$\lim_{x\longrightarrow a^+}\frac{f(x)-f(x_0)}{x-x_0}=f'(a)$$

- *f is left continuous at a point b , i.e.*

$$\lim_{x\longrightarrow b^-}\frac{f(x)-f(x_0)}{x-x_0}=f'(b)$$

3.4.4 Operations on derivative

Let $f,g:D\longrightarrow\mathbb{R}$ two functions. We assume that f and g are differentiable of x. Therefore,

1. $f+g$ is differentiable , and

$$(f+g)'(x)=f'(x)+g'(x).$$

2. $f\times g$ is differentiable , and

$$(f\times g)'(x)=f'(x)\times g(x)+f(x)\times g'(x).$$

3. λf is differentiable , and

$$(\lambda f)'(x)=\lambda f'(x).$$

4. If $g(x)\neq 0$, then $\dfrac{f}{g}$ is differentiable , and

$$(\frac{f}{g})'(x)=\frac{f'(x)\times g(x)-f(x)\times g'(x)}{(g(x))^2}.$$

Theorem 3.4.2. *(Derivatives of composite functions)*
Let $f:I\longrightarrow\mathbb{R}$ and $g:J\longrightarrow\mathbb{R}$ two functions such that $f(I)\subseteq J$. If f is differentiable of x, and g is differentiable of $f(x)$, then $g\circ f$ is differentiable of x and

$$(g\circ f)'(x)=g'((f(x))f'(x).$$

Example 3.4.2. *The function $f(x)=\sin(2x)$ is the composition of two simpler functions, namely: $f(x)=g(h(x))$ where $g(u)=\sin u$ and $h(x)=2x$. Since g and h are differentiable then $g'(u)=\cos u$ and $h'(x)=2$:*

Therefore the derivative of the composite functions rule implies that :

$$f'(x)=(g(h(x)))'=h'(x)\cdot g'(h(x))=2\cos(2x).$$

Definition 3.4.4. *(Derivative of inverse function)*

Let f be a function that is differentiable on an interval I . If f has an inverse function f^{-1} , then f^{-1} is differentiable at any x for which $f'(f^{-1}(x))$. Moreover,

$$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}.$$

3.4.5 Applications of Derivatives

Derivatives have various applications in Mathematics, We'll learn about these two applications of derivatives:

I. Monotonicity of functions
Derivatives can be used to determine whether a function is increasing, decreasing or constant on an interval.

Theorem 3.4.3. *Let f be a differentiable function on an interval I :*

1. *f is increasing on I* $\iff \forall x \in I: \ f'(x) \geq 0.$

2. *f is decreasing on I* $\iff \forall x \in I: \ f'(x) \leq 0.$

3. *f is constant on I* $\iff \forall x \in I: \ f'(x) = 0.$

II. Extremum of Functions
An extremum of a function is the point where we get the maximum or minimum value of the function in some interval.

Proposition 3.4.4. *Let $f : I \longrightarrow \mathbb{R}$ be a function, and let $c \in I$. We say that c is a critical point of f if $f'(c) = 0$ or $f'(c)$ is undefined.*

Let $f : I \longrightarrow \mathbb{R}$ is differentiable, and $c \in I$ be a critical point of f . Then

1. *If $f'(x) > 0$ for all $x < c$ and $f'(x) < 0$ for all $x > c$, then $f(c)$ is the maximum value of f .*

2. *If* $f'(x) < 0$ *for all* $x < c$ *and* $f'(x) > 0$ *for all* $x > c$ *, then* $f(c)$ *is the minimum value of f .*

Example 3.4.3. *Find the extremum of* $f(x) = 3x^2 - 18x + 5$ *on* $[0, 7]$.
First, we find all possible critical points :
$f'(x) = 0 \iff 6x - 18 = 0 \iff x = 3.$

for $x \in [0, 3[$ *, we have* $f'(x) < 0$ *and for* $x \in]3, 7]$ *, we have* $f'(x) > 0$
Then $f(3) = -22$ *is the minimum value of f on* $[0, 7]$.

III. Rolle's Theorem

In analysis, special case of the mean-value theorem of differential calculus is Rolle's theorem.

Theorem 3.4.5. *Let* $f : [a, b] \longrightarrow \mathbb{R}$ *such that*

- *f is continuous on the closed interval [a, b],*
- *f is differentiable on the open interval]a, b[,*
- *f (a) = f (b).*

Then, there exists $c \in]a, b[$ *such that f'(c) = 0.*

IV. Mean Value Theorem
Let $a, b \in \mathbb{R}$ with $a < b$ and

$f : [a, b] \longrightarrow \mathbb{R}$. Suppose f is continuous on $[a, b]$ and differentiable on

]a , b [. Then there exists $c \in]a, b[$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

V . Indeterminate Forms and L'Hospital's Rule

In this section, we will learn how to evaluate functions whose values cannot be found at certain points.

Theorem 3.4.6. *Consider f and g are continuous functions on [a , b] which are differentiable at every point in]a , b[, except possibly at $x_0 \in [a, b]$. Assume that:*

1. $g(x) \neq 0$ *and* $g'(x) \neq 0$ *at every point in [a , b]:*

$$\textbf{\textit{2.}} \lim_{x \longrightarrow x_0} \frac{f(x)}{g(x)} = \frac{0}{0} \quad or \quad \lim_{x \longrightarrow x_0} \frac{f(x)}{g(x)} = \frac{\infty}{\infty}.$$

$$\textbf{\textit{3.}} \lim_{x \longrightarrow x_0} \frac{f'(x)}{g'(x)} \ exists. \ Then \ :$$

$$\lim_{x \longrightarrow x_0} \frac{f(x)}{g(x)} \ exists \ and \ \lim_{x \longrightarrow x_0} \frac{f(x)}{g(x)} = \lim_{x \longrightarrow x_0} \frac{f'(x)}{g'(x)}.$$

Remark 3.4.2. *Note that the rule is also valid for one-sided limits and for limits at infinity or negative infinity.*

Example 3.4.4. *Find* $\lim_{x \longrightarrow 1} \frac{\ln x}{x - 1}$

$\lim_{x \longrightarrow 1} \frac{\ln x}{x - 1} = \frac{0}{0}.$

Thus, we can apply l'Hospital's Rule:

$$\lim_{x \longrightarrow 1} \frac{\ln x}{x - 1} = \lim_{x \longrightarrow 1} \frac{(\ln x)'}{(x - 1)'} = \lim_{x \longrightarrow 1} \frac{\frac{1}{x}}{1} = 1$$

Exercise 3.4.1. *Determine the values of x at which each function is continuous. The domain of all the functions is* $\mathbb{R}$.

$$\textbf{\textit{(1)}} \ f(x) = \begin{cases} \mid \dfrac{sinx}{x} \mid , \ if \ x \neq 0. \\ 0 \ , \ \ if \ \ x = 0. \end{cases}$$

$$\textbf{\textit{(2)}} \ f(x) = \begin{cases} \dfrac{sinx}{\mid x \mid} \ , \ if \ x \neq 0. \\ 0 \ , \ \ if \ \ x = 0. \end{cases}$$

$$\textbf{\textit{(3)}}\ f(x) = \begin{cases} x sin\frac{1}{x}\ ,\ if\ x \neq 0. \\ 0\ ,\ \ if\ \ x = 0. \end{cases}$$

$$\textbf{\textit{(4)}}\ f(x) = \begin{cases} cos\frac{\pi x}{2}\ ,\ if\ \ |\ x\ | \leq 1. \\ |\ x - 1\ |\ ,\ \ if\ \ \ |\ x\ | > 1. \end{cases}$$

Exercise 3.4.2. *Prove that the equation* $x^2 - 2 = \cos(x+1)$ *has at least two real solutions. (Assume known that the function* $\cos x$ *is continuous.)*

Exercise 3.4.3. *Prove that the equation* $cosx = x$ *has at least on solution in R. (Assume known that the function* $\cos x$ *is continuous.)*

Exercise 3.4.4. *Determine the values of x at which each function is differentiable.*

$$\textbf{\textit{1.)}}\ f(x) = \begin{cases} x sin\frac{1}{x}\ ,\ if\ x \neq 0. \\ 0\ ,\ \ if\ \ x = 0. \end{cases}$$

$$\textbf{\textit{2.)}}\ f(x) = \begin{cases} x^2 sin\frac{1}{x}\ ,\ if\ x \neq 0. \\ 0\ ,\ \ if\ \ x = 0. \end{cases}$$

Exercise 3.4.5. *Use L'Hospital's Rule to find the following limits (you may assume known all the relevant derivatives from calculus)*

a. $\displaystyle\lim_{x \longrightarrow 2} \frac{x^3 - 4x}{3x^2 + 5x - 2}.$

b. $\displaystyle\lim_{x \longrightarrow 0} \frac{e^x - e^{-x}}{sin(x)cos(x)}.$

c. $\displaystyle\lim_{x \longrightarrow 1} \frac{x - 1}{\sqrt{x+1} - \sqrt{2}}.$

d. $\displaystyle\lim_{x \longrightarrow 0} \frac{e^x - e^{-x}}{\ln(1+x)}.$

Exercise 3.4.6. *Consider the function :*

$$f(x) = \begin{cases} e^{\frac{-1}{x^2}}\ ,\ if\ x \neq 0. \\ 0\ ,\ \ if\ \ x = 0. \end{cases}$$

Prove that $f \in C^n(\mathbb{R})$ *for every* $n \in \mathbb{N}$.

Chapter 4

Usual functions

In this chapter, we review some fundamental functions frequently encountered in calculus and analysis. These include the logarithmic and exponential functions, as well as the trigonometric and hyperbolic functions. Understanding their definitions, properties is essential for solving a wide range of mathematical problems.

4.1 Logarithmic Functions

Definition 4.1.1. *(The Natural Logarithm function)*

The function that satisfies the following two conditions is called the natural logarithm function and is denoted by ln*:*

1. $\forall\, x \in \mathbb{R}_+^* \ : \ (\ln(x))' = \dfrac{1}{x}$.

2. $\ln(1) = 0$.

Properties 4.1.1. *According to the previous dentition, the function* $\ln(x)$ *is differentiable on* $\mathbb{R}_+^*$ *and* $\forall\, x \in \mathbb{R}_+^* \ ; \ (\ln(x))' = \dfrac{1}{x}$.

- *Let* g *be a positive function differentiable and non-zero on* I *then the function* $\ln(g(x))$ *is differentiable on* I *and its derivative:*

$$(\ln(g(x)))' = \frac{g'(x)}{g(x)}.$$

Properties 4.1.2. *(Limits and classical inequalities)*

- $\lim_{x \longrightarrow +\infty} ln(x) = +\infty.$
- $\lim_{x \longrightarrow 0^+} ln(x) = -\infty.$
- $\lim_{x \longrightarrow +\infty} \frac{ln(x)}{x} = 0.$
- $\lim_{x \longrightarrow +\infty} \frac{ln(x)}{x^\alpha} = 0.$ $\alpha \in \mathbb{R}_+^*.$
- $\lim_{x \longrightarrow 0^+} xln(x) = 0.$
- $\lim_{x \longrightarrow 0} \frac{ln(x+1)}{x} = 1.$

Properties 4.1.3. *(Algebraic properties of the function ln(x))*

For all $x, y \in \mathbb{R}_+^*$ *and* $\alpha \in \mathbb{Q}$, *we have the following properties:*

- $ln(x \times y) = ln(x) + ln(y).$
- $ln(\frac{x}{y}) = ln(x) - ln(y).$
- $ln(\frac{1}{x}) = -ln(x).$
- $ln(x^\alpha) = \alpha ln(x).$

4.2 Exponential Functions

Definition 4.2.1. *The inverse function of the function* $ln(\cdot)$ *is called the exponential function and is denoted by:* $exp(\cdot)$ *or* $e^{\cdot}$ *, and satisfies the following properties:*

1) $\forall\ x \in]0, +\infty[\ :\ x = e^{ln(x)}.$

1) $\forall\ y \in \mathbb{R}\ :\ y = ln(e^y).$

Proposition 4.2.1.

1) *The function* $\exp(\cdot)$ *is continuous and strictly increasing on* $\mathbb{R}$.

2) *The function* e^x *is differentiable on* $\mathbb{R}$ *and we have:*

$$\forall\ x \in \mathbb{R}\ :\ (e^x)' = e^x.$$

3) *If* u *is differentiable on* I *then: the function* $e^{u(x)}$ *is differentiable on* I *and its derivative defined by:*

$$\forall\ x \in I\ :\ (e^{u(x)})' = u'(x)e^{u(x)}.$$

Proposition 4.2.2. *(Limits and inequalities)*

- $\lim\limits_{x \longrightarrow -\infty} e^x = 0.$
- $\lim\limits_{x \longrightarrow +\infty} e^x = +\infty.$
- $\lim\limits_{x \longrightarrow +\infty} xe^{-x} = 0.$
- $\lim\limits_{x \longrightarrow +\infty} \dfrac{x^\alpha}{e^x} = 0,\ \alpha \in \mathbb{R}$
- $\lim\limits_{x \longrightarrow +\infty} \dfrac{e^x}{x^\alpha} = +\infty\ ,\ \alpha \in \mathbb{R}$
- $\lim\limits_{x \longrightarrow 0} \dfrac{e^x - 1}{x} = 1.$

Properties 4.2.3. *(Algebraic properties of the exponential function)*

For all $x, y \in \mathbb{R}_+^*$ *and* $\alpha \in \mathbb{Q}$, *we have the following properties:*

- $e^{x+y} = e^x \cdot e^y.$
- $e^{-x} = \frac{1}{e^x}.$
- $e^{x-y} = \dfrac{e^x}{e^y}.$
- $e^{\alpha x} = (e^x)^\alpha.$

4.3 Hyperbolic cosine, sine and tangent functions

Any function f defined on $\mathbb{R}$ can be uniquely decomposed into a sum of two functions fev and fod where fev is an even function and odd is an odd function. This means for every $x \in \mathbb{R}$ we can write :

$$f(x) = \frac{f(x) + f(-x)}{2} + \frac{f(x) - f(-x)}{2}.$$

and we choose :

$$f_{ev}(x) = \frac{f(x) + f(-x)}{2} \quad \textit{and} \quad f_{od}(x) = \frac{f(x) - f(-x)}{2}.$$

We can easily check that this decomposition is unique, and f_{ev} is an even function and f_{od} is an odd function.

4.3.1 Hyperbolic cosine

Definition 4.3.1. *We call the hyperbolic cosine function and denoted ch or* cosh *, the even part of the exponential function defined by*

$$\begin{aligned} \cosh \ : \ & \mathbb{R} \longrightarrow \mathbb{R} \\ x \ \longmapsto & \ \cosh(x) = \frac{e^x + e^{-x}}{2}. \end{aligned}$$

4.3.2 Hyperbolic sine

Definition 4.3.2. *We call the hyperbolic cosine function and denoted sh or* sinh *, the even part of the exponential function defined by*

$$\begin{aligned} \sinh \ : \ & \mathbb{R} \longrightarrow \mathbb{R} \\ x \ \longmapsto & \ \sinh(x) = \frac{e^x - e^{-x}}{2}. \end{aligned}$$

4.3.3 Hyperbolic tangent

Definition 4.3.3. *We call the hyperbolic cosine function and denoted th or* tanh *, the even part of the exponential function defined by*

$$\begin{aligned} \tanh \ : \quad & \mathbb{R} \longrightarrow \mathbb{R} \\ x \ \longmapsto \ & \tanh(x) = \frac{\sinh(x)}{\cosh(x)} = \frac{e^x - e^{-x}}{e^x + e^{-x}}. \end{aligned}$$

Proposition 4.3.1. *The functions* $\cosh(x)$, $\sinh(x)$ *and* $th(x)$ *have the following properties:*

- *The function* $\cosh(x)$ *is a function defined on* $\mathbb{R}$ *, continuous and even .*

- *The function* $\sinh(x)$ *is a function defined on* $\mathbb{R}$*, continuous and odd.*

- *The function* $\tanh(x)$ *is a function defined on* $\mathbb{R}$*, continuous and odd.*

- *The functions* $\cosh(x)$, $\sinh(x)$ *and* $th(x)$ *are differentiable on* $\mathbb{R}$ *and their derivatives are defined by:*

$$\forall\ x \in \mathbb{R}, \begin{cases} (\cosh(x))' = \sinh(x). \\ (\sinh(x))' = \cosh(x). \\ (\tanh(x))' = \frac{1}{\cosh^2(x)} = 1 + \tanh^2(x) \end{cases}$$

- $\cosh(0) = 1$, $\sinh(0) = 0$ *and* $\tanh(0) = 0$.

- $\lim\limits_{x \longrightarrow -\infty} \sinh(x) = -\infty$, $\lim\limits_{x \longrightarrow -\infty} \cosh(x) = +\infty$, *and* $\lim\limits_{x \longrightarrow -\infty} \tanh(x) = -1$.

- $\lim\limits_{x \longrightarrow +\infty} \sinh(x) = +\infty$, $\lim\limits_{x \longrightarrow +\infty} \cosh(x) = +\infty$, *and* $\lim\limits_{x \longrightarrow +\infty} \tanh(x) = 1$.

Proposition 4.3.2. *For every real* x*, we have:*

- $\cosh(x) + \sinh(x) = e^x$.

- $\cosh(x) - \sinh(x) = e^{-x}$.

- $\cosh^2(x) - \sinh^2(x) = 1.$

For all $(x, y) \in \mathbb{R}^2$ *, we have the following formulas:*

- $\cosh(x + y) = \cosh(x)\cosh(y) + \sinh(x)\sinh(y).$
- $\cosh(x - y) = \cosh(x)\cosh(y) - \sinh(x)\sinh(y).$
- $\sinh(x + y) = \sinh(x)\cosh(y) + \cosh(x)\sinh(y).$
- $\sinh(x - y) = \sinh(x)\cosh(y) - \cosh(x)\sinh(y).$
- $\tanh(x + y) = \dfrac{\tanh(x) + \tanh(y)}{1 + \tanh(x)\tanh(y)}.$
- $\tanh(x - y) = \dfrac{\tanh(x) - \tanh(y)}{1 - \tanh(x)\tanh(y)}.$

4.4 The inverse hyperbolic functions

4.4.1 The inverse of hyperbolic Sine function

From the above table of variation of $\sinh(\cdot)$ we have: $\sinh(\cdot)$ is continuous and strictly increasing on $\mathbb{R}$. Hence, it realizes a bijection from $\mathbb{R}$ into $\mathbb{R}$.

Definition 4.4.1. *The inverse function of the hyperbolic sine function on* $\mathbb{R}$ *is denoted* $argsh(x)$ *or* $sh^{-1}(x)$.

$$argsh \;:\; \mathbb{R} \longrightarrow \mathbb{R}$$

$$x \longmapsto argsh(x).$$

Proposition 4.4.1. *The function* $argsh(x)$ *has the following properties:*

1. *The function* $argsh(x)$ *is defined on* $\mathbb{R}$*, it is continuous and strictly increasing on* $\mathbb{R}$*.*

2. $\forall\, x \in \mathbb{R}\ ;\ argsh(\sinh(x)) = x.$

3. $\forall\, y \in \mathbb{R}\ ;\ \sinh(argsh(y)) = y.$

4. $\forall\ x, y \in \mathbb{R}\ ;\ y = \sinh(x) \iff x = argsh(y).$

5. *$argsh(x)$ is odd function.*

Proposition 4.4.2. *The function $argsh(x)$ is differentiable on $\mathbb{R}$ and verifies:*

$$\forall x \in \mathbb{R}\ ;\ (argsh(x))' = \frac{1}{\sqrt{1+x^2}}.$$

Proposition 4.4.3.

$$\forall x \in \mathbb{R}\ ;\ argsh(x) = \ln(x + \sqrt{1+x^2}).$$

4.4.2 The inverse of hyperbolic Cosine function

From the table of variation of the function $\cosh(x)$ above we have: $\cosh(x)$ is continuous and strictly increasing on $[0, +\infty[$. So it forms a bijection from $[0, +\infty[$ into $[1, +\infty[$

Definition 4.4.2. *The inverse function of the restriction of $\cosh(x)$ on $[0, +\infty[$ is denoted by $argch(x)$ or $ch^{-1}(x)$.*

$$argch\ :\ [1, +\infty[\longrightarrow [0, +\infty[$$

$$x \longmapsto argch(x).$$

Proposition 4.4.4. *The $argch(x)$ function has the following properties:*

► *The function $argch(x)$ is defined on $[1, +\infty[$, it is continuous and strictly increasing on $[1, +\infty[$.*

► $\forall\ x \in [0, +\infty[\ ;\ argch(\cosh(x)) = x.$

► $\forall\ y \in [1, +\infty[\ ;\ \cosh(argch(y)) = y.$

► $\forall\ x \in [0, +\infty[\ ;\ \forall\ y \in [1, +\infty[\ ;\ y = \cosh(x) \iff x = argch(y).$

Proposition 4.4.5. *The inverse hyperbolic cosine function is differentiable on $]1, +\infty[$ and verifies:*

$$\forall\ x \in]1, +\infty[\ :\ (argch(x))' = \frac{1}{\sqrt{x^2-1}}.$$

Proposition 4.4.6.

$$\forall x \in]1, +\infty[\ ;\ argch(x) = \ln(x + \sqrt{x^2 - 1}).$$

4.4.3 The inverse hyperbolic tangent function

From the table of variation of the function $\tanh(x)$ above we have: $\tanh(x)$ is continuous and strictly increasing on $\mathbb{R}$. So it makes is a bijection from $\mathbb{R}$ into $]-1, 1[$.

Definition 4.4.3. *The inverse function of the function* $\tanh(x)$ *on* R *is denoted by* $argth(x)$ *or* $th^{-1}(x)$

$$argth \ :\ ]-1, 1[\longrightarrow \mathbb{R}$$

$$x \longmapsto argth(x).$$

Proposition 4.4.7. *The function* $argth(x)$ *has the following properties:*

- *The function* $argth(x)$ *is defined on* $]-1, 1[$ *, it is continuous and strictly increasing on* $]-1, 1[$ *.*
- $\forall\ x \in \mathbb{R}\ ;\ \ argth(\tanh(x)) = x.$
- $\forall\ y \in]-1, 1[\ ;\ \ \tanh(argth(y)) = y.$
- $\forall\ x \in \mathbb{R}\ ;\ \forall\ y \in]-1, 1[\ ;\ y = \tanh(x) \iff x = argth(y).$
- *The* $argth(x)$ *function is odd.*

Proposition 4.4.8. *The function* $argth(x)$ *is differentiable on* $]-1, 1[$ *and verifies:*

$$\forall\ x \in]-1, 1[\ :\ (argth(x))' = \frac{1}{1 - x^2}.$$

Proposition 4.4.9.

$$\forall x \in]-1, 1[\ ;\ argth(x) = \frac{1}{2}\ln(\frac{1+x}{1-x}).$$

4.4 The inverse hyperbolic functions

Exercise 4.4.1. *solve the equation :*

$$4\cosh(2x) + 10\sinh(2x) = 5 \text{ .,} \quad \cosh(x) = \frac{13}{5}$$

Exercise 4.4.2.

1. *Calculate :*

$$S = \sum_{k=0}^{n} \sinh(kx)... \quad , \; C = \sum_{k=0}^{n} \cosh(kx).$$

2. *Linearize* $\sinh x \cdot \cosh(2x)$, $\cosh x \cdot \cosh(2x)$.

3. *Verify that* $\sinh(2x) = 2\sinh x \cdot \cosh x$ *and then calculate.*

Exercise 4.4.3.

Consider the following hyperbolic equation, given in terms of a constant k .

$$2\cosh^2(x) = 2\sinh(x) + k.$$

a. *Find the range of values of k for which the above equation has no real solutions.*

b. *Given further that k =1 , find in exact logarithmic form, the solutions of the above equation.*

Exercise 4.4.4. *Let* $f : \mathbb{R} \longrightarrow \mathbb{R}$ *be the function defined by*

$$f(x) = argch\sqrt{1 + x^2}$$

1. *Determine the domain of definition of f*

2. *Calculate* $argch(\cosh(t))$*, for all* $t \in \mathbb{R}$

3. *Show that* $\forall\, x \in \mathbb{R} : f(x) = argsh \mid x \mid$.

4. *Calculate* $f'(x)$, *for all* $x \in \mathbb{R}^*$.

5. *Is f differentiable at 0 ?*

4.5 Trigonometric Functions

4.5.1 Recalls on the functions $\cos(x)$ and $\sin(x)$

Proposition 4.5.1. *The functions* $x \longmapsto \cos(x)$ *and* $x \longmapsto \sin(x)$ *are defined on* $\mathbb{R}$ *and satisfy the following properties:*

1. $\forall\, x \in \mathbb{R} \;\; ; \mid \cos(x) \mid \leq 1 \wedge \mid \sin(x) \mid \leq 1.$

2. $\cos(x)$ *and* $\sin(x)$ *are* 2π*-periodic i.e.:*

$$\forall\, x \in \mathbb{R}\;;\; \cos(x+2\pi) = \cos(x) \text{ and } \sin(x+2\pi) = \sin(x).$$

3. *The function* $\cos(x)$ *is even and the function* $\sin(x)$ *is odd, i.e.:*

$$\forall\, x \in \mathbb{R}\;;\; \cos(-x) = \cos(x) \text{ and } \sin(-x) = -\sin(x).$$

4. $\forall\, x \in \mathbb{R} \;:\; \cos^2 x) + \sin^2(x) = 1.$

5. *The functions* $\cos(x)$ *and* $\sin(x)$ *belong to* $C^\infty(\mathbb{R})$ *and we have: is odd, i.e.:*

$$\forall\, x \in \mathbb{R}\;;\; (\cos(x))' = -\sin(x) \text{ and } (\sin(x))' = \cos(x).$$

Proposition 4.5.2. *(Trigonometric addition formulas)*
For all $(x, y) \in \mathbb{R}^2$ *, we have the following formulas:*

- $\cos(x+y) = \cos(x)\cos(y) - \sin(x)\sin(y).$
- $\cos(x-y) = \cos(x)\cos(y) + \sin(x)\sin(y).$
- $\sin(x+y) = \sin(x)\cos(y) + \cos(x)\sin(y).$
- $\sin(x-y) = \sin(x)\cos(y) - \cos(x)\sin(y).$
- $\cos(2x) = \cos^2(x) - \sin^2(x) = 2\cos^2(x) - 1 = 1 - 2\sin^2(x).$
- $\sin(2x) = 2\sin(x)\cos(x).$

- $\sin(x) + \sin(y) = 2\sin(\frac{x+y}{2})\cos(\frac{x-y}{2})$.
- $\sin(x) - \sin(y) = 2\cos(\frac{x+y}{2})\sin(\frac{x-y}{2})$.
- $\cos(x) + \cos(y) = 2\cos(\frac{x+y}{2})\cos(\frac{x-y}{2})$.
- $\cos(x) - \cos(y) = 2\sin(\frac{x+y}{2})\sin(\frac{x-y}{2})$.

4.5.2 Recalls on the Tangent function.

Definition 4.5.1. *The tangent function is one of the main trigonometric functions and defined by:*

$$\tan \ : \ \mathbb{R}\backslash\{\tfrac{\pi}{2} + k\pi \ , k \in \mathbb{Z}\} \ \longrightarrow \mathbb{R}$$

$$x \ \longmapsto \ \tan(x) = \tfrac{\sin(x)}{\cos(x)}.$$

Proposition 4.5.3. *The function* $\tan(x)$ *is differentiable on* $\mathbb{R}\backslash\{\frac{\pi}{2} + k\pi \ , k \in \mathbb{Z}\}$ *and we have:*

$$\forall \ x \in \mathbb{R}\backslash\{\frac{\pi}{2} + k\pi \ , k \in \mathbb{Z}\} \ : \ (\tan(x))' = \frac{1}{\cos^2(x)} = 1 + \tan^2(x).$$

Proposition 4.5.4. *The function* $\tan(x)$ *checks the following properties:*

1. *The function tan(x) is* π*-periodic i.e :*

$$\forall \ x \in \mathbb{R}\backslash\{\frac{\pi}{2} + k\pi \ , k \in \mathbb{Z}\} \ : \ \tan(x+\pi) = \tan(x).$$

2. *For any* $x, y \in \mathbb{R}\backslash\{\frac{\pi}{2} + k\pi \ , k \in \mathbb{Z}\}$ *we have:*

$$\tan(x+y) = \frac{\tan(x) + \tan(y)}{1 - \tan(x)\tan(y)} \quad \textit{and} \quad \tan(x-y) = \frac{\tan(x) - \tan(y)}{1 + \tan(x)\tan(y)}.$$

3. $\forall \ x \in \mathbb{R}\backslash\{\frac{\pi}{2} + k\pi \ , k \in \mathbb{Z}\} \ , \ \tan(2x) = \dfrac{2\tan(x)}{1 - \tan^2(x)}$

4.6 Inverse Trigonometric Functions

In this section, we concern ourselves with finding inverses of the (circular) trigonometric functions.

4.6.1 Inverse Sine Function ($\arcsin(x) = \sin^{-1}(x)$)

The function $\sin(x)$ is continuous and strictly increasing on $[-\frac{\pi}{2}, \frac{\pi}{2}]$, then the function $\sin(x)$ represents a bijection from $[-\frac{\pi}{2}, \frac{\pi}{2}]$ to $[-1, 1]$.

Definition 4.6.1. *We first consider* $f(x) = \sin(x)$ *in a similar manner, although the interval of choice is* $[-\frac{\pi}{2}, \frac{\pi}{2}]$.

It should be no surprise that we call $f^{-1}(x) = \arcsin(x)$ *, which is read " arc-sin of x ".*

$$f^{-1} \quad : \quad [-1, 1] \longrightarrow [-\tfrac{\pi}{2}, \tfrac{\pi}{2}]$$

$$x \longmapsto f^{-1}(x) = \arcsin(x).$$

The trigonometric function $\sin(x)$ *is not one-to-one functions, hence in order to create an inverse, we must restrict its domain. The restricted sine function is given by*

$$f^{-1}(x) = y \iff y = f(x).$$

We have

$$arcsin(x) = \sin^{-1} x = y \ , \ y \in [-\frac{\pi}{2}, \frac{\pi}{2}] \iff \sin(y) = x \ , \ x \in [-1, 1].$$

Properties 4.6.1. *The function* $arcsin(x)$ *has the following properties:*

- $\sin(\arcsin(x)) = x$ *provided* $x \in [-1, 1]$.
- $\arcsin(\sin(x)) = x$ *provided* $x \in [-\frac{\pi}{2}, \frac{\pi}{2}]$.
- *The function* $\arcsin(x)$ *is continuous and strictly increasing on* $[-1, 1]$. *(According to the inverse function theorem).*
- *The function* $\arcsin(x)$ *is odd.*

Example 4.6.1. *Find the exact values of the following :*

$$\textbf{1.}\ \arcsin(\frac{\sqrt{2}}{2})\ ,\ \textbf{2.}\ \arcsin(\frac{-1}{2}).$$

1. *The value of* $\arcsin(\frac{\sqrt{2}}{2})$ *is a real number* $y \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ *. with* $\sin(y) = \frac{\sqrt{2}}{2}$ *The number we seek is* $y = \frac{\pi}{4}$ *. Hence ,* $\arcsin(\frac{\sqrt{2}}{2}) = \frac{\pi}{4}$.
2. *To find* $\arcsin(\frac{-1}{2})$ *, we seek the number* $y \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ *with* $\sin(y) = \frac{-1}{2}$ *. The answer is* $y = -\frac{\pi}{6}$ *so that* $\arcsin(\frac{-1}{2}) = -\frac{\pi}{6}$.

Proposition 4.6.2. *The arcsine function is differentiable on* $]-1,1[$ *and verifies:*

$$\forall\ x \in]-1,1[\ ,\ (\arcsin(x))' = \frac{1}{\sqrt{1-x^2}}.$$

4.6.2 Inverse Cosine Function ($\arccos(x) = \cos^{-1}(x)$)

The function $f(x) = \cos(x)$ is continuous and strictly decreasing on $[0,\pi]$, so the function $cos(x)$ makes a bijection from $[0,\pi]$ into $[-1,1]$.

Definition 4.6.2. *The inverse function of the restriction of* $\cos(x)$ *on* $[0,\pi]$ *is called the arccosine function and is denoted by* $\arccos(x)$ *or* $\cos^{-1}(x)$ *:*

$$\arccos\ :\ [-1,1] \longrightarrow [0,\pi]$$

$$x \longmapsto \arccos(x).$$

We use the notation $f^{-1}(x) = \arccos(x)$ *, read " arc-cosine of* x *". Formally:*

$$f^{-1}\ :\ [-1,1] \longrightarrow [0,\pi]$$

$$x \longmapsto f^{-1}(x) = \arccos(x).$$

We have

$$\arccos(x) = \cos^{-1} x = y\ ,\ y \in [0,\pi] \iff \cos(y) = x\ ,\ x \in [-1,1].$$

Properties 4.6.3. *The function* $\arccos(x)$ *has the following properties:*

- *The function* $\arccos(x)$ *is continuous and strictly decreasing on* $[-1,1]$. *(From the inverse function theorem)*

- *The function* $\arccos(x)$ *is neither even nor odd.*

- $\forall\ x \in [0,\pi]\ ;\ \arccos(\cos(x)) = x.$

- $\forall\ y \in [-1,1]\ ;\ \cos(\arccos(y)) = y.$

Example 4.6.2. *Find the exact values of the following :*

$$\textbf{\textit{1.}}\ \arccos(\frac{1}{2})\ ,\ \textbf{\textit{2.}}\ \arcsin(\frac{-\sqrt{2}}{2}).$$

1. *To find* $\arccos(\frac{1}{2})$ *, we need to find the real number y (or, equivalently, an angle measuring y radians) which verifies* $y \in [0,\pi]$ *and with* $\cos(y) = \frac{1}{2}$ *. We know* $y = \frac{\pi}{3}$ *meets these criteria, so* $\arccos(\frac{1}{2}) = \frac{\pi}{3}$.

2. *The number* $y = \arccos(\frac{-\sqrt{2}}{2}) \in [0,\pi]$ *with* $\cos(y) = \frac{-\sqrt{2}}{2}$ *. Our answer is* $y = \frac{3\pi}{4}$ *.*

Proposition 4.6.4. *The arccosine function is differentiable on* $]-1,1[$ *and verifies:*

$$\forall\ x \in]-1,1[\ ,\ (\arccos(x))' = -\frac{1}{\sqrt{1-x^2}}.$$

4.6.3 Inverse Tangent Function ($\arctan(x) = \tan^{-1}(x)$)

The function $\tan(x) = \dfrac{\sin x}{\cos x}$ is defined on $D = \{\frac{\pi}{2} + k\pi;\ k \in \mathbb{Z}\}$. It is continuous and differentiable on its domain of dentition and for all $x \in D$ we have:

$$(\tan(x))' = \frac{1}{\cos^2 x} = 1 + \tan^2(x).$$

The function $\tan(x)$ is continuous and strictly increasing on $[-\frac{\pi}{2}, \frac{\pi}{2}]$, then the function $\tan(x)$ makes a bijection from $[-\frac{\pi}{2}, \frac{\pi}{2}]$ *into* $\mathbb{R}$.

Definition 4.6.3. *We call the arctangent function* $\arctan(x)$ *or* $\tan^{-1}(x)$ *the inverse of the tangent function on* $]-\frac{\pi}{2},\frac{\pi}{2}[$ *defend by*

$$\arctan \;:\;]-\infty,+\infty[\longrightarrow]-\tfrac{\pi}{2},\tfrac{\pi}{2}[$$

$$x \longmapsto \arctan(x).$$

Proposition 4.6.5. *The function* $\arctan(x)$ *has the following properties:*

1. *The function* $\arctan(x)$ *is continuous and strictly increasing on R, with values in* $]-\frac{\pi}{2},\frac{\pi}{2}[$.

2. *The function* $\arctan(x)$ *is odd.*

3.

$$\forall\; x \in]-\frac{\pi}{2},\frac{\pi}{2}[\;;\; \arctan(\tan(x)) = x.$$

4.

$$\forall\; y \in \mathbb{R}\;;\; \tan(\arctan(y)) = y.$$

5.

$$\forall\; x \in]-\frac{\pi}{2},\frac{\pi}{2}[\;,\; \forall\; y \in \mathbb{R}\;:\; \tan(x) = y \iff x = \arctan(y).$$

Example 4.6.3. *Find the exact values of the following :* $\arctan(\sqrt{3})$.

We know $\arctan(\sqrt{3})$ *is the real number* $y \in]-\frac{\pi}{2},\frac{\pi}{2}[$ *with* $\tan(y) = \sqrt{3}$. *We find* $y = \frac{\pi}{3}$ *, so* $\arctan(\sqrt{3}) = \frac{\pi}{3}$:

Proposition 4.6.6. *The function* $\arctan(x)$ *is differentiable on* $\mathbb{R}$ *and verifies:*

$$\forall\; x \in \mathbb{R}\;;\; (\arctan(x))' = \frac{1}{1+x^2}.$$

Proposition 4.6.7. *(Some properties)*

▶ *For any* $x \in [-1,1]$ *we have:*

$$\arccos(x) + \arcsin(x) = \frac{\pi}{2}.$$

- ► $x \in [-1,1]$ *we have:*

$$\sin(\arccos(x)) = \cos(\arcsin(x)) = \sqrt{1-x^2}.$$

- ► *For all* $x \in \mathbb{R}^*$ *we have:*

$$\arctan(x) + \arctan(\frac{1}{x}) = \frac{\pi}{2}.$$

Exercise 4.6.1. *Show that for all* $x \in [-1;1]$ *, we have*

$$\sin(\arccos(x)) = \cos(\arcsin(x)) = \sqrt{1-x^2}.$$

Exercise 4.6.2. ***1.*** *Calculate :* $\arcsin(\sin\frac{\pi}{3})$, $\arccos\cos(\frac{\pi}{3})$, $\arccos(\sin\frac{\pi}{3})$.

2. *Calculate :* $\arccos(\cos\frac{4\pi}{3})$, $\arccos\cos(\frac{7\pi}{3})$, $\arcsin(\sin\frac{2\pi}{3})$, $\arcsin(\sin\frac{7\pi}{3})$.

Exercise 4.6.3. ***1.*** *Show that*

$$\arctan a + \arctan b = \arctan\frac{a+b}{1-ab} \text{ , with } ab < 1.$$

2. *Calculate :* $\arctan(\frac{1}{2}) + \arctan(\frac{1}{3})$.

Exercise 4.6.4. *Let* $f : \mathbb{R} \longrightarrow \mathbb{R}$ *be a continuous function defined by*

$$f(x) = \begin{cases} \arctan(\frac{1}{x^2}) \text{ , } if\ x \neq 0. \\ l \text{ , } \ if\ \ x = 0. \end{cases}$$

1. *Determine* l *.*

2. *Show that* f *is differentiable on* $\mathbb{R}^*$ *and calculate* f'*.*

3. *Show that* f *is differentiable at 0 and calculate* $f'(0)$ *(Apply MVT between 0 and x).*

4. *Deduce that* f *is* C^∞ *.*

5. *Calculate* g' *where* g *is the function defined on* $\mathbb{R}$ *by* $g(x) = \arctan x^2$.

6. *Calculate :*

$\arctan x^2 + \arctan(\frac{1}{x^2})$, $\forall x \in \mathbb{R}^*$ *and deduce* $arctan x + arctan(\frac{1}{x})$, $\forall x \in \mathbb{R}^*$.

7. *Show that* $g : [0,+\infty[\longrightarrow [0,\frac{\pi}{2}[$ *is bijective and calculate* g^{-1}.

8. *Calculate* $(g^{-1})'$ *in two ways.*

Chapter 5

Finite Expansions

5.1 Finite expansions at zero

Definition 5.1.1. *Let f be a real valued function. We said that the function f is represented by a finite expansion at zero if there exist real numbers $a_0, a_1, ..., a_n$ and a real valued function ϵ such that*

$$f(x) = a_0 + a_1 x + a_2 x^2 + ... + a_n x^n + x^n \epsilon(x) \ , \ \lim_{x \longrightarrow 0} \epsilon(x) = 0.$$

Then the function f is represented by the polynomial approximation of degree n, denoted by $P_n(x)$, for x near to zero, which is called the main part of finite expansions at zero, such that:

$$P_n(x) = a_0 + a_1 x + a_2 x^2 + ... + a_n x^n.$$

Remark 5.1.1. *Note that $x^n \epsilon(x) = O(x)$.*

Example 5.1.1. *Using the Euclidean division by increasing power order, one has the finite expansion at zero of $f(x) = \frac{1}{1-x}$.*

$$\frac{1}{1-x} = 1 + x + x^2 + ... + x^n + \frac{x^{n+1}}{1-x} = 1 + x + x^2 + ... + x^n + x^n(\frac{x}{1-x}).$$

in this case $\epsilon(x) = \dfrac{x}{1-x}$. We generally do not try to determine the function $\epsilon(x)$.

Properties 5.1.1. ***1.*** *If the function f can be expanded at zero, then this expansion is unique.*

2. *If the function f can be expanded at zero, then $\lim_{x \longrightarrow 0} f(x)$ exists and equal to a_0 . This criterion is generally used to demonstrate that a function does not admits an expansion.*

Example 5.1.2. *The function $f(x) = lnx$ does not have an expansion at zero, because $\lim_{x \longrightarrow 0^+} f(x) = -\infty$.*

5.2 Algebraic combinations of finite expansions

Definition 5.2.1. *If f and g can both be expanded at zero and ? is any constant, then each of the following functions is also can be expanded at zero: The sum $f + g$, the difference $f - g$, the constant multiple λf , the product $f \times g$, the quotient $\frac{f}{g}$, if $g(x) \neq 0$.*
Consider the finite expansions at zero of f and g:

$$f(x) = a_0 + a_1 x + a_2 x^2 + ... + a_n x^n + O(x).$$

$$g(x) = b_0 + b_1 x + b_2 x^2 + ... + b_n x^n + O(x).$$

- *The finite expansion at zero of the sum $f + g$ is:*

$$(f + g)(x) = (a_0 + b_0) + (a_1 + b_1)x + (a_2 + b_2)x^2 + ... + (a_n + b_n)x^n + O(x).$$

- *The finite expansion at zero of the $f \cdot g$ is obtained by the product and keeping only the monomials of degree less than n in the product*

$$(a_0 + a_1 x + a_2 x^2 + ... + a_n x^n) + (b_0 + b_1 x + b_2 x^2 + ... + b_n x^n).$$

- *The finite expansion at zero of the quotient f/g is obtained by the Euclidean division of*

$$(a_0 + a_1 x + a_2 x^2 + ... + a_n x^n) + (b_0 + b_1 x + b_2 x^2 + ... + b_n x^n).$$

by increasing power order.

Example 5.2.1. *Find the finite expansion at zero of* $f(x) = \sinh x$ *of the degree 4.*

$$\begin{aligned}\sinh x &= \frac{e^x - e^{-x}}{2}\\ &= \frac{1}{2}[(1+\frac{x}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}) - (1-\frac{x}{1!}+\frac{x^2}{2!}-\frac{x^3}{3!}+\frac{x^4}{4!})] + O(x^4)\\ &= \frac{1}{2}(2x + 2\frac{x^3}{3!} + O(x^4).\\ &= x + \frac{x^3}{3!} + O(x^4).\end{aligned}$$

5.3 Composite of finite expansions

Definition 5.3.1. *If* g *can be expanded at zero of degree* n *and if* f *can be expanded at* $g(0)$ *of degree* n *such that* $g(0) = 0$ *. Then the composite function* $(f \circ g)$ *can be expanded at zero of degree* n *by replacing the finite expansion of* g *in the finite expansion of* f *and by keeping only the monomials of degree* $\leq n$*.*

Example 5.3.1. *Prove that the finite expansion at zero of* $f(x) = \exp(\sin x)$ *is given by*

$$f(x) = \exp(\sin x) = 1 + x + \frac{x^2}{2} + O(x^3).$$

5.4 Finite expansions at a point

Definition 5.4.1. *We said that the function* $f : x \longrightarrow f(x)$ *can be represented by a finite expansion at point* x_0 *if the function* $F : X \longrightarrow F(X)$ *can be represented by finite expansion at zero* $X_0 = 0$ *such that* $F(X) = f(x_0 + X)$ *and*

$$F(X) = a_0 + a_1 X + a_2 X^2 + ... + a_n X^n + X^n \epsilon(X)\ ,\ \lim_{X \longrightarrow 0} O(X) = 0.$$

$f(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + ... + a_n(x - x_0)^n + O((x - x_0))$, $\lim_{x \longrightarrow 0} O((x - x_0)^n) = 0.$

Example 5.4.1. *Find the finite expansion at a point* $x_0 = 1$ *of* $f(x) = \exp(x)$ *of the degree 3 :*

$$\begin{aligned} F(X) &= f(x_0 + X) \\ &= \exp(1 + X) \\ &= \exp(1) \times \exp X. \\ &= \exp(1) \times [1 + \frac{X}{1!} + \frac{X^2}{2!} + \frac{X^3}{3!} + O(X^3). \\ &= \exp(1) \times (1 + \frac{(x-1)}{1!} + \frac{(x-1)^2}{2!} + \frac{(x-1)^3}{3!} + O((x-1)^3).. \end{aligned}$$

5.5 Finite expansions at Infinity

Definition 5.5.1.
We said that the function $f : x \longrightarrow f(x)$ *can be represented by a finite expansion at infinity if the function* $F : X \longrightarrow F(X)$ *can be represented by finite expansion at zero* $X_0 = 0$ *such that* $F(X) = f(\frac{1}{x})$ *and*

$$F(X) = a_0 + a_1X + a_2X^2 + ... + a_nX^n + X^n\epsilon(X) \ , \ \lim_{X \longrightarrow 0} O(X) = 0.$$

$$f(x) = a_0 + \frac{a_1}{x} + \frac{a_2}{x^2} + ... + \frac{a_n}{x^n} + O(\frac{1}{x^n}).$$

Example 5.5.1. *Find the finite expansion at infinity of* $f(x) = \cos(\frac{1}{x})$ *: Let* $X = \frac{1}{x}$ *and thus:*

$$\begin{aligned} \cos(\tfrac{1}{x}) = \cos(X) &= 1 - \frac{X^2}{2!} + \frac{X^4}{4!} + ... + \frac{(-1)^n X^{2n}}{2n!} + O(X^{2n}) \\ &= 1 - \frac{1}{2!x^2} + \frac{1}{4!x^4} + ... + \frac{(-1)^n}{2n!x^{2n}} + O(\frac{1}{x^{2n}}). \end{aligned}$$

5.6 Using finite expansions to evaluate limits

The finite expansions provide a good way to understand the behavior of a function near a specified point and so are useful for solving some indeterminate forms. When taking a limit as $x \longrightarrow 0$, we can often simplify the statement by substituting in finite expansions that we know.

Chapter 6

Linear Algebra

In this chapter we introduce the notion of a vector space, which is fundamental for the approximation methods that we will later develop, in particular in the form of an orthogonal projection onto a subspace representing the best possible approximation in that subspace. Any vector in an vector space can be expressed in terms of a set of basis vectors, and we here introduce the process of constructing an orthonormal basis from an arbitrary basis, which provides the foundation for a range of matrix factorization methods we will use to solve systems of linear equations and eigenvalue problems. We use the Euclidian space $\mathbb{R}^n$ as an illustrative example, but the concept of a vector space is much more general than that, forming the basis for the theory of function approximation and partial differential equations.

6.1 Internal Composition Laws and Their Properties

Definition 6.1.1. *(Internal Composition Laws)*

Let E be a set . An internal composition law $\star$ on E is a mapping from $E \times E$ to E, associating every pair (x, y) in $E \times E$ with an element of E :

$$\star \; : \; E \times E \longrightarrow E$$

$$(x, y) \longrightarrow x \star y$$

Remark 6.1.1. ***1.*** *The internal composition law can be noted by :*

($\star$ $\triangle$, T , $\perp$, . . .) , or other symbols.
2. *$(E;\star)$ is often used to denote a set E equipped with an internal operation $\star$.*

Properties 6.1.1. • ***Associativity :*** *we say that $\star$ is associative if and only if :*

$$\forall\ x, y, z \in E\ :\ x \star (y \star z) = (x \star y) \star z. \tag{6.1.1}$$

Example 6.1.1. *Let $\star$ be an internal composition law defined on $\mathbb{R}$ by :*

$$x \star y = x + y - 1.$$

$$\begin{aligned} (x \star y) \star z &= [(x+y) - 1] \star z \\ &= [(x+y) - 1] + z - 1 \\ &= x + y + z - 2 \end{aligned}$$

and

$$\begin{aligned} x \star (y \star z) &= x \star [(y+z) - 1] \\ &= x + [(y+z) - 1] - 1 \\ &= x + y + z - 2. \end{aligned}$$

Then $\star$ is associative.

- ***Commutativity :*** *we say that $\star$ is commutative if and only if :*

$$\forall\ x, y \in E\ :\ x \star y = y \star x. \tag{6.1.2}$$

Example 6.1.2. *Let $\star$ be an internal composition law defined on $\mathbb{R}$ by :*

$$x \star y = x + y - 1.$$

$$\begin{aligned} x \star y &= (x+y) - 1 \\ &= (y+x) - 1 \\ &= y \star x. \end{aligned}$$

Then $\star$ is commutative. .

• ***Neutral element:*** *The law of internal composition $\star$ admits a neutral element e on set E if and only if :*

$$\exists\ e \in E, \forall\ x \in E\ :\ x \star e = e \star x = x. \tag{6.1.3}$$

Example 6.1.3. *Let $\star$ be an internal composition law defined on $\mathbb{R}$ by :*

$$x \star y = x + y - 1.$$

$$x \star e = x \implies (x + e) - 1 = x$$

$$\implies e = 1.$$

Then $e = 1$ is a neutral element .

• ***Symmetric element:*** *We assume that E has a neutral element e for $\star$. Let x and x' be two elements of E. We say that x' is symmetric to x (for the law $\star$) if:*

$$\forall\ x \in E\ , \exists\ x' \in E, :\ x \star x' = x' \star x = e. \tag{6.1.4}$$

Example 6.1.4. *Let $\star$ be an internal composition law defined on $\mathbb{R}$ by :*

$$x \star y = x + y - 1.$$

$$x \star x' = e \implies (x + x') - 1 = 1$$

$$\implies x' = (2 - x) \in \mathbb{R}.$$

Then $x' = 2 - x$ is a symmetric element.

• ***Distributivity :***

Given two laws of internal composition $\star$ and T defined on E.

- *We say that the law T is left distributive with respect to the law $\star$ if:*

$$\forall\ x, y, z \in E\ :\ x\ T\ (y \star z) = (x\ T\ y) \star (x\ T\ z). \tag{6.1.5}$$

We say that the law T is right distributive with respect to the law $\star$ if:

$$\forall\ x, y, z \in E\ :\ (y \star z)\ T\ x = (y\ T\ x) \star (z\ T\ x). \tag{6.1.6}$$

The law T is said to be distributive with respect to the law $\star$ if it is both left and right distributive with respect to $\star$.

Example 6.1.5. *Let $\star$ be an internal composition law defined on $\mathbb{R}$ by :*

$$x \star y = x + y - 1.$$

and Let T be an internal composition law defined on R by :

$$xTy = x + y - xy$$

Then the law T is said to be distributive with respect to the law $\star$. When T is commutative, it is then demonstrated that T is left distributive with respect to the law $\star$.

$$\begin{aligned} xT(y \star z) &= xT(y + z - 1) \\ &= x + (y + z - 1) - x(y + z - 1) \\ &= 2x + y + z - xy - xz \end{aligned}$$

and

$$\begin{aligned} (x\ T\ y) \star (x\ T\ z) &= (x + y - xy) \star (x + z - xz) \\ &= (x + y - xy) + (x + z - xz) - 1 \\ &= 2x + y + z - xy - xz. \end{aligned}$$

Then the law T is distributive with respect to the law $\star$.

6.2 Algebraic Structures

Definition 6.2.1. *(Group)*

A group is a non-empty set equipped with an internal composition law $(G; \star)$ such that:

- *$\star$ is associative .*
- *$\star$ has a neutral element e.*
- *every element in x is invertible (has an inverse) for $\star$.*

Remark 6.2.1. *If $\star$ is commutative, we say that $(G;\star)$ is commutative or abelian.*

Example 6.2.1. ***1.*** $(\mathbb{Z};+),(\mathbb{Q};+),(\mathbb{R};+)$, *and* $(\mathbb{C};+)$ *are abelian groups;*
2. $(\mathbb{N};+),(\ R;\times)$ *, are not groups.*

Definition 6.2.2. *(Subgroup)*

Let $(G;\star)$ be a group and let H be a non-empty subset of G . We say that H is a subgroup of G if:

1. *H is closed under $\star$:*

$$\forall\ x,y\in H\ :\ x\star y\in H.$$

2. *H is closed under taking inverses:*

$$\forall\forall\ x\in H\ :\ x'\in H\ \ (x' \textit{ the inverse of } x).$$

Definition 6.2.3. *(Ring)*
Let A be a set equipped with two binary operations , $\star$ and $\perp$.
$(A\ ;\ \star\ ;\perp\ )$ is called a ring if :

1. *$(A,\star)$ is a commutative group;*

2. *$\perp$ is associative;*

3. *$\perp$ is distributive over $\star$.*

Remark 6.2.2.
1. *If $\perp$ is commutative, then $(A\ ;\ \star\ ;\perp\ )$ is called a commutative ring.*

2. *If $\perp$ has a neutral element, then $(A\ ;\ \star\ ;\perp\ )$ is called a unitary ring.*

Example 6.2.2. $(\mathbb{Z};+,\times),(\mathbb{Q};+,\times),(\mathbb{R};+,\times)$, *and* $(\mathbb{C};+,\times)$ *are commutative rings;*

Definition 6.2.4. *(Field)*
A field is a commutative ring in which every non-zero element is invertible for the second operation.

Remark 6.2.3. *If the second operation is also commutative, the field* $(K\ ;\ \star\ ;\perp\ )$ *is called a commutative field.*

Example 6.2.3. $(\mathbb{Q};+,\times),(\mathbb{R};+,\times)$, *and* $(\mathbb{C};+,\times)$ *are fields , but* $(\mathbb{Z};+,\times)$ *is not . (2 is not invertible).*

Exercise 6.2.1. *Let* $\star$ *the internal composition law defined in* $\mathbb{R}$ *by*

$$\forall x,y\in\mathbb{R}\ :\ x\star y=x+y+x^2y^2.$$

1. *Verify* $\star$ *are commutative laws.*

2. *The law* $\star$ *is it associative ?*

3. *Show that* $\mathbb{R}$ *have a neutral element for the law* $\star$ *then calculate this neutral.*

4. *Solve the following equations :* $1\star x=1$, $2\star x=7$.

Exercise 6.2.2. *We provide* $A=\mathbb{R}\times\mathbb{R}$ *of two laws defined by :*

$$(x,y)+(\grave{x},\grave{y})=(x+\grave{x},y+\grave{y})\ \text{and}\ (x,y)\times(\grave{x},\grave{y})=(x\grave{x},x\grave{y}+\grave{x}y).$$

1. *Show that* $(A,+)$ *is an commutative group.*

2. ***(a)*** *Show that the law* $\times$ *is commutative.*

(b) *Show that* $\times$ *is associative .*

(c) *Find the neutral element of* A *for the law* $\times$.

(d) *Show that* $(A,+,\times)$ *is a commutative ring.*

6.3 Vector spaces

Underlying every vector space (to be defined shortly) is a scalar field $\mathbb{K}$.

Definition 6.3.1. *(Vector spaces)*
A vector space is a nonempty set V , whose objects are called vectors, equipped with two operations, called addition and scalar multiplication: For any two vectors u, v in V and a scalar λ , there are unique vectors $u+v$ and λu in V such that the following properties are satisfied :

a. *An internal operation (vector addition) and*

$$+ \ : \ V \times V \longrightarrow V$$

$$(x, y) \longrightarrow x + \mathbf{y}.$$

b. *An external scalar (scalar multiplication)*

$$\cdot \ : \ K \times V \longrightarrow V$$

$$(\lambda, x) \longrightarrow \lambda x$$

such that the following properties are satisfied: $\forall\ u, v \in V$, $\forall \lambda$, $\gamma \ \in K$

1. $u + v = v + u$ *;*

2. $(u + v) + w = u + (v + w)$ *;*

3. *There is a vector* 0 *, called the zero vector, such that* $u + 0 = u$ *;*

4. *For any vector u there is a vector $-u$ such that* $u + (-u) = 0$;

5. $\lambda(u + v) = \lambda u + \lambda v$;

6. $(\lambda + \gamma)u = \lambda u + \gamma u$;

7. $\lambda(\gamma u) = (\lambda\gamma)u$;

8. $1u = u$.

Example 6.3.1.
a. *The Euclidean space $\mathbb{R}^n$ is a vector space under the ordinary addition and scalar multiplication.*

b. *The set P_n of all polynomials of degree less than or equal to n is a vector space under the ordinary addition and scalar multiplication of polynomials.*

c. *The set $C[a;b]$ of all continuous functions on the closed interval $[a;b]$ is a vector space under the ordinary addition and scalar multiplication of functions.*

6.3.1 Subspaces of a vector space

Definition 6.3.2. *(Subspace)*
A subspace of a vector space V is a nonempty subset F of V that has three properties:

1. $0_V \in F$.

2. *F is closed under vector addition. That is, $\forall\ u, v \in F\ :\ u + v \in F$.*

3. *F is closed under multiplication by scalars. That is,*
$\forall\ u \in F, \forall\ \lambda \in K\ : \lambda u \in F$.

Lemma 6.3.1. *A subset F of E is a vector subspace of E if :*

1. *$(F, +)$ is a subgroup of $(E, +)$.*

2. $\forall \in F, \forall \lambda \in \mathbb{K}\ :\ \lambda x \in F$.

The following proposition presents a characterization of a vector subspace of E.

Proposition 6.3.2. *A subspace of a vector space V is a a nonempty subset F of V if and only if:*

$$\forall\ u, v \in F;\ \forall\ \lambda, \gamma \in K\ :\ \lambda u + \gamma v \in F.$$

Example 6.3.2.

- *$\mathbb{R}^{n-1}$ is a subspace of $\mathbb{R}^n$.*
- *0_V and V are vector subspace of V :*

- *Show that* $F = \{(0, y, z)\ ;\ y, z \in R\}$ *is a subspace of real vector space* $\mathbb{R}^3$:

- $0_{\mathbb{R}^3} \in F$, *then* F *is a nonempty subset of* $\mathbb{R}^3$: .

- *Let* , $u = (0, x_1, y_1),\ v = ((0, x_2, y_2) \in F$ *and* $\lambda, \gamma \in \mathbb{R}$. *Then :*

$$\begin{aligned} \lambda u + \gamma v &= \lambda(0, x_1, y_1) + \gamma(0, x_2, y_2) \\ &= (0, \lambda x_1, \lambda y_1) + (0, \gamma x_2, \gamma y_2) \\ &= (0, \lambda x_1 + \gamma x_2, \lambda y_1 + \gamma y_2)\ \in F \end{aligned}$$

Hence, F *is a subspace of* $\mathbb{R}^3$.

6.3.2 Operations on Vector Spaces

Definition 6.3.3.

- *The addition of two subsets* F *and* G *of a vector space is defined by :*

$$F + G = \{u + v\ :\ u \in F,\ v \in G\}.$$

- *The intersection* $\cap$ *of two subsets* F *and* G *of a vector space is defined by :*

$$F \cap G = \{u;\ u \in F\ \wedge\ u \in G\}.$$

- *A vector space* E *is called the direct sum of* F *and* G , *denoted* $F \bigoplus G$ *if* F *and* G *are subspaces of* W *with* $F \cap G = \{0_E\}$ *and* $F + G = E$.

Proposition 6.3.3. *Let* $\mathbb{K}$ *be a field,* E *a* $\mathbb{K}-$*vector space,* F *and* G *two subspaces of* E*; then :*

1. $F \cap G$ *is a subspace of* E .

2. $F \sqcup G$ *is a subspace of* E *if and only if ,* $F \subset G$ *or* $G \subset F$.

Remark 6.3.1. *We generalize the property* (1) *to any family of vector subspaces, i.e. If* $(F_i)_{i \in I, I \subset \mathbb{N}}$, *is a family of vector subspaces, then* $\cap_{i \in I} F_i$ *is a subspace.*

Theorem 6.3.4. *Let* $\mathbb{K}$ *be a field,* E *a vector space on* $\mathbb{K}$, F *and* G *two subspaces of* E*: The set* $F+G$ *defined by :*

$$F+G=\{u+v \ : \ u\in F, v\in G\} \ \subset E \tag{6.3.1}$$

is a subspace of E *called sum of the subspaces* F *and* G*. If in addition* $F\cap G=\{0_E\}$ *, we say that the sum* $F+G$ *is a direct sum and we write* $F\bigoplus G$ *.*

Example 6.3.3. *Let* $E=\mathbb{R}^3$ *be the vector space on* $\mathbb{R}$ *. Consider the following subspaces* F *and* G *:*

$F=\{(x,y,z)\in\mathbb{R}^2 \ : \ x+y+z=0\}$ *and* $G=\{(x,y)\in\mathbb{R}^2 \ : \ x=y=0\}$.

We have : $F+G=F\bigoplus G$ *. Indeed :*

Let $(x,y,z)\in F\cap G$ *so* $(x,y,z)\in F$ *i.e* $x+y+z=0$ *and* $(x,y,z)\in G$ *i,e* $x=y=0$ *, so* $x=y=z=0$ *, therefore* $F\cap G=\{0_{\mathbb{R}^3}\}$.

6.3.3 Linear combinations, generating families, linearly independent families, bases, dimension.

Definition 6.3.4. *(Linear combinations)*
Let E *be a vector space on* $\mathbb{K}$ *. We say that the vector* u *is a linear combination of the vectors* $v_1, v_2, ..., v_n$ *of* E *if :*

$$\exists\lambda_1,\lambda_2,...,\lambda_n\in\mathbb{K} \ : \ u=\lambda_1v_1+\lambda_2v_2+...+\lambda_nv_n=\sum_{i=1}^{n}\lambda_iv_i.$$

The scalars $\lambda_1,\lambda_2,...,\lambda_n$ *are called the coefficients of the linear combination.*

Definition 6.3.5. *(Spanning sets)*
Let E *be a vector space over*$\mathbb{K}$ *and* $H=\{v_1,v_2,...,v_n\}$ *be a subset of* E*. We say that* H *is a spanning set of* E *if every vector* v *of* E *can be written as a liner combination of vectors in* H *. In such cases, we say that* H *spans* E*.*

Definition 6.3.6. *The span of $H = \{v_1, v_2, ..., v_n\}$ is the set of all linear combinations of $v_1, v_2, ...\ v_n$. is denoted by span(H):*

$$spanH = span\{v_1, v_2, ...\ v_n\} = \{\sum_{i=1}^{n} \lambda_i v_i = \lambda_1 v_1 + \lambda_2 v_2 + ... + \lambda_n v_n\ :\ \lambda_1, \lambda_2, ..., \lambda_n \in \mathbb{K}.$$

The span of a set of vectors in E is a subspace of E.

Theorem 6.3.5. *Let E be a vector space over $\mathbb{R}$ and $H = \{v_1, v_2, ...\ v_n\}$ be a subset of E. Then span(H) is a subspace of E.*

Further, span(H) is the smallest subspace of E that contains H . This means, if L is a subspace of E and L contains H , then span(H) is contained in L.

Definition 6.3.7. *(Generating families)*
The family $H = \{v_1, v_2, ...\ v_n\}$ is a generating family of the vector space E if every vector v of E is a linear combination of the vectors $v_1, v_2, ..., v_n$. This can also be written :

$$\forall\ v \in E,\ \exists \lambda_1, \lambda_2, ..., \lambda_n \in \mathbb{K}\ :\ v = \lambda_1 v_1 + \lambda_2 v_2 + ... + \lambda_n v_n = \sum_{i=1}^{n} \lambda_i v_i.$$

We also say that the family $\{v_1, v_2, ...\ v_n\}$ generates the vector space(E) and we write $E = \{v_1, v_2, ...\ v_n\}$

Example 6.3.4. ***1.*** *Let $E = \mathbb{R}_n[X]$ be the vector space of polynomials of degree $\leq n$. Then the polynomials $\{1, X, X^2, ..., X^n\}$ form a generating family of E.*

2. *The set $F = \{(1,1,1); (2,2,0); (3,0,0)\}$ is a system of generators of $\mathbb{R}^3$.*
Let $u = (x, y, z)$ be a vector, we check the scalars $\alpha, \beta, \gamma \in \mathbb{R}$ such that

$$u = (x, y, z) \quad = \quad \alpha(1,1,1) + \beta(2,2,0) + \gamma(3,0,0)$$

$$\Longrightarrow \begin{cases} x = \alpha + 2\beta + 3\gamma. \\ y = \alpha + 2\beta. \\ z = \alpha \end{cases}$$

then , $\alpha = z$, $\beta = \dfrac{y-z}{2},\ \gamma = \dfrac{x-y}{3}$.

Definition 6.3.8. *(Linearly independent families)*
A family $H = \{v_1, v_2, ...\ v_n\}$ of vectors of a vector space E is linearly independent if the only linear combination of these vectors equal to the zero vector is the one whose coefficients are all zero. We also say that vectors $\{v_1, v_2, ...\ v_n\}$ are linearly independent.
This can be expressed as : $\{v_1, v_2, ...,\ v_n\}$ is a linearly independent family is equivalent to :

$$\lambda_1, \lambda_2, ..., \lambda_n \in \mathbb{K},\ and\ \lambda_1 v_1 + \lambda_2 v_2 + ... + \lambda_n v_n = 0 \implies \lambda_1 = \lambda_2 = ... = \lambda_n = 0.$$

Definition 6.3.9. *(Linearly dependent families)*
A non linearly independent family is called a linearly dependent family. We also say that vectors $\{v_1, v_2, ...\ v_n\}$ are linearly dependents. This can be expressed as : $\{v_1, v_2, ...,\ v_n\}$ is a linearly dependent family is equivalent to :

$$\exists\ \lambda_1, \lambda_2, ..., \lambda_n \in \mathbb{K}\backslash\{0\},\ :\ \lambda_1 v_1 + \lambda_2 v_2 + ... + \lambda_n v_n = 0.$$

Example 6.3.5. *In the vector space $\mathbb{R}^4$ defined over the field $\mathbb{R}$, consider the following vectors :*

$$v_1 = (1, 0, -1, 1), v_2 = (0, 1, 1, 0),\ v_3 = (1, 0, 0, 1),\ v_4 = (0, 0, 0, 1),\ v_5 = (1, 1, 0, 1),$$

The set $\{v_1, v_2, v_3, v_4\}$ is linearly independent (to be verified). The set $\{v_1, v_2, v_5\}$ is linearly dependent ($v_5 = v_1 + v_2$).

Theorem 6.3.6. *Let E be a vector space over the field $\mathbb{K}$. A set $F = \{v_1, v_2, ...,\ v_n\}$ of n vectors of E, $(n > 2)$ is linearly dependent if and only if at least one of the vectors of F is a linear combination of the other vectors of F.*

Remark 6.3.2. ***1.*** *Any family containing a linearly dependent family is linearly dependent .*

2. *Any family included in a linearly independent family is linearly independent .*

3. *$\{v\}$ is linearly independent if and only if $v \neq 0_E$.*

4. *Any set containing the null vector is linearly dependent.*

Definition 6.3.10. *(Basis)*

A basis of a vector space is linearly independent generating family. If $B = (x_i)_{i\in I}$, $I \subset \mathbb{N}$ is a basis of E, then any $x \in E$ is uniquely written as a linear combination of elements of B :

$$x = \sum_{i=1}^{n} \lambda_i x_i.$$

The scalars $(\lambda_i)_{i\in I}$; are called the coordinates of x in the basis B.

Definition 6.3.11. *Let E be a vector space and $H = \{v_1, v_2, ..., v_n\}$ be a set of elements (vectors)in E . We say that H is a basis of E if :*
1. *H spans E . (H is a generator of E).*
2. *H is linearly independent.*

Example 6.3.6.
1. *In $\mathbb{R}^3$, the set $\{e_1 = (1,0,0),\ e_2 = (0,1,0),\ e_3 = (0,0,1)$, forms a basis of $\mathbb{R}^3$, called canonical basis of $\mathbb{R}^3$.*

2. *the set $\{v_1 = (1,0,1),\ v_2 = (1,-1,1),\ v_3 = (0,1,1)$, forms a basis of $\mathbb{R}^3$:*

a. *The family is linearly independent.*

Let $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}$ such that : $\lambda_1 v_1 + \lambda_2 v_2 + \lambda_3 v_3 = 0_{\mathbb{R}^3}$. Then

$$\lambda_1(1,0,1) + \lambda_2(1,-1,1) + = \lambda_3(0,1,1) = (0,0,0)$$

$$\Longrightarrow \begin{cases} \lambda_1 + \lambda_2 = 0 \\ \lambda_2 + \lambda_3 = 0. \\ \lambda_1 + \lambda_2 + \lambda_3 = 0. \end{cases}$$

which leads $\lambda_1 = \lambda_1 = \lambda_1 = 0$

b. *The set is generating of $\mathbb{R}^3$. Let $u = (x,y,z) \in \mathbb{R}^3$: We are looking for $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}$ such that :*

$$u = (x, y, z) \;=\; \lambda_1 v_1 + \lambda_2 v_2 + = \lambda_3 v_3$$

$$\Longrightarrow \begin{cases} \lambda_1 + \lambda_2 = x \\ \lambda_2 + \lambda_3 = y. \\ \lambda_1 + \lambda_2 + \lambda_3 = z. \end{cases}$$

and we find :

$$\lambda_1 = 2x + y - z \; ; \; \lambda_2 = x - y + z \; : \; \lambda_3 = -x + z.$$

So span $\{v_1, v_2, v_3\} = \mathbb{R}^3$ *. Then* $\{v_1, v_2, v_3\}$ *a basis of* $\mathbb{R}^3$*.*

6.3.4 Finite dimensional vector spaces

Definition 6.3.12. *If a vector space is spanned by a finite number of vectors, it is said to be finite-dimensional. Otherwise it is infinite-dimensional.*

The number of vectors in a basis for a finite-dimensional vector space E is called the dimension of E and denoted dim(E). By convention, we say that $\{0_E\}$ *is a finite-dimensional space.*

Definition 6.3.13. *A family* $\{v_1, v_2, ..., v_n\}$ *of vectors of E is said to be a basis of E if and only if, we have :*

1. $\{v_1, v_2, ..., v_n\}$ *is a linearly independent family of E and*

2 $\{v_1, v_2, ..., v_n\}$ *is a generating family of E.*

Theorem 6.3.7. *(Theorem of the extracted basis)*
From any finite generating family of E, we can extract a basis of E. In particular, a finite-dimensional space admits a basis.

Theorem 6.3.8. *(Incomplete basis theorem)*
If E is finite-dimensional, then any linearly independent family of E can be completed into a basis of E. To complete it, simply consider certain vectors of a generating family of E.

Theorem 6.3.9. *(Dimension)*
If E is finite-dimensional, then all bases of E have the same number of vectors (dimension of E).

Corollary 6.3.10. *If E is a finite-dimensional vector space $(dim E = n)$ and if $B = \{v_1, v_2, ..., \ v_n\}$ is a family of n vectors of E, then the following conditions are equivalent:*

1. *B is linearly independent.*

2. *B is a generating set of E.*

3. *B is a basis of E.*

Definition 6.3.14. *(Rank of a finite family of vectors)*
Let E be a $\mathbb{K}$-vector space and $H = \{v_1, v_2, ..., \ v_n\}$ a family of m vectors of E. The rank of the family H noted rank(H) is the dimension of the vector subspace $F = Vect\{v_1, v_2, ..., \ v_n\}$ generated by the vectors $\{v_1, v_2, ..., \ v_n\}$:

$$rank(H) = \ dim(F).$$

Properties 6.3.11. *Let E be a$\mathbb{K}$-vector space and $H = \{v_1, v_2, ..., \ v_n\}$ a family of vectors of E. So we have :*

- *If $dim(F) = n$ (finite) , then $rank(G) = n$.*
- *If $dim(F) = n$ (finite) , then $rank(H\ ) = n$.*
- *$rank(H) = m$ if and only if H is free. (linearly independent).*

Example 6.3.7.

Let $H = \{v_1 = (2, 3), v_2 = (4, 2), v_3 = (-3, 4)\}$ be a family of the vector space $\mathbb{R}^2$. Determine the rank of H . It is clear that v_2 and v_3 are linearly independent. On the other hand, by solving the linear system :

$$\lambda_1 v_1 + \lambda_2 v_2 + \lambda_3 v_3 = 0_{\mathbb{R}^2}.$$

we get $2v_1 - v_2 - v_3 = 0_{\mathbb{R}^2}$. The family H is therefore dependent. We deduce that $Vect(v_1, v_2, v_3) = Vect(v_2, v_3)$. So rank$(H\ ) = 2$.

6.3.5 Existence of additional subspaces in finite dimension:

The incomplete basis theorem says that in a finite dimensional vector space, any free family can be completed into a basis of the space. We immediately deduce the existence of supplementary ones .

Proposition 6.3.12. *Let E be a finite dimensional vector space and F_1 a vector subspace of E. There exists a vector subspace F_2 such that*

$$E = F_1 \bigoplus F_2 \quad and \quad dim(E) = dim(F_1) + dim(F_2).$$

Theorem 6.3.13. *(Grassmann formula). If F_1 and F_2 are vector subspaces of E and $F_1 + F_2$ is of finite type,then :*

$$dim(F_1 + F_2) = dim(F_1) + dim(F_2) - dim(F_1 \cap F_1).$$

Theorem 6.3.14. *(Characterization of supplementary). If E is of finite type, then the following conditions are equivalent.*

- $E = F_1 \bigoplus F_2$.
- $F_1 \cap F_1 = \{0_E\}$ *and* $dim(E) = dim(F_1) + dim(F_2)$.
- $E = F_1 + F_2$ *and* $dim(E) = dim(F_1) + dim(F_2)$.

Exercise 6.3.1. *Let $\mathbb{R}^3$ be the vector space on the field $\mathbb{R}$,*

$$G = \{(1, 1, 0), (0, 0, 1), (1, 1, 1)\}.$$

subspace of $\mathbb{R}^3$ and let the set F be defined as:

$$F = \{(x, y, z) \in \mathbb{R}^3 \ : \ 2x + y - z = 0\}.$$

- *Show that F is a vector subspace of $\mathbb{R}^3$.*
- *Find a basis for each of : $F \cap G$, $F + G$, G , F (if any), and give their dimensions.*
- *Is $\mathbb{R}^3 = F \bigoplus G$?*

Exercise 6.3.2. *Let F and G two vector subspaces of $\mathbb{R}^3$ defined by :*

$$F = \{(x, y, z) \in \mathbb{R}^3 \; ; \; x-2y+z = 0\} \;\; , \; G = \{(x, y, z) \in \mathbb{R}^3 \; ; \; 2x-y+2z = 0\}.$$

1. *Give a basis for F, a basis for G, and deduce their respective dimensions.*

2. *Give a basis for $F \cap G$, and give its dimension.*

2. *Do we have $F \bigoplus G = \mathbb{R}^3$?*

6.4 Linear Maps

In this chapter we use $\mathbb{K}$ which represents etch $er\mathbb{R}$ or $\mathbb{C}$.

6.4.1 Generalities

Given two vector spaces E and F , both over the same field $\mathbb{K}$, a linear map is a function from E to F that is compatible with scalar multiplication and vector addition. More precisely, we have the following :

Definition 6.4.1. *(Linear map)*

A linear map f from a vector space E into a vector space F is a rule that assigns to each vector x in E a unique vector $f(x)$ in F, such that:

$$f \;:\; E \longrightarrow F$$

$$x \longmapsto y = f(x)$$

1. $\forall \; x, y \in E \;:\; f(x + y) = f(x) + f(y).$

1. $\forall x \in E \;,\; \forall \lambda \in \mathbb{K} \;:\; f(\lambda x) = \lambda f(x)$

This definition is equivalent to :

$$\forall \; x, y \in E, \; \forall \; \lambda, \gamma \in \mathbb{K} \;:\; f(\lambda x + \gamma y) = \lambda f(x) + \gamma f(y)$$

Example 6.4.1. *The map :*

$$f \ : \ \mathbb{R}^2 \longrightarrow \mathbb{R}^3$$

$$f(x,y) = (3x - y, 0, 2y)$$

is linear map:
1. *Let* $X = (x,y)$; $Y = (x', y') \in \mathbb{R}^2$:

$$\begin{aligned} f(X+Y) = f(x+x', y+y') &= [3(x+x') - (y+y'), 0, 2(y+y')]. \\ &= [(3x-y) + (3x'-y'), 0+0, 3x+3y']. \\ &= (3x-y,0,2y) + (3x'-y', 0, 2y'). \\ &= f(X) + f(Y). \end{aligned}$$

2. *Let* $X = (x,y) \in \mathbb{R}^2, \lambda \in \mathbb{R}$*:*

$$\begin{aligned} f(\lambda X) = f[\lambda(x,y)] = f(\lambda x, \lambda y) &= (3\lambda x - \lambda y, 0, 2\lambda y) \\ &= \lambda(3x-y,0,2y) = \lambda f(x,y) = \lambda f(X). \end{aligned}$$

Properties 6.4.1. *Here are some simple properties of linear maps*

$$f : E \longrightarrow F$$

- $f(0_E) = 0_F$.
- $f(-u) = -f(u)$, $u \in E$.

6.4.2 Operations on linear maps

Definition 6.4.2. *The set of linear mappings from* E *into* F *is denoted* $L(E;F)$.

Theorem 6.4.2. *Let* f , g *be two linear maps from* E *into* F *and* $k \in \mathbb{K}$. *Then* $f+g$ *and* kf *are linear maps from* E *into* F.

Proposition 6.4.3. $L(E;F)$ *with addition and multiplication has a vector space structure on* $\mathbb{R}$.

Theorem 6.4.4. *(Composition) Let* E , F *and* G *be vector spaces over a common field* $\mathbb{K}$, *and suppose* f *be a linear map from* E *to* F *and* g *a linear map from* F *to* G. *Then* $g \circ f$ *is a linear map from* E *to* G.

6.4.3 Endomorphisms , Isomorphisms and automorphisms

Observe that the definition of a linear map is suited to reflect the structure of vector spaces, since it preserves vector spaces two main operations, addition and scalar multiplication. In algebraic terms, a linear map is said to be a homomorphism of vector spaces. An invertible homomorphism where the inverse is also a homomorphism is called an isomorphism. If there exists an isomorphism from E to F, then E and F are said to be isomorphic, and we write $E \cong F$. Isomorphic vector spaces are essentially "the same" in terms of their algebraic structure. It is an interesting fact that finite-dimensional vector spaces of the same dimension over the same field are always isomorphic.

Definition 6.4.3. *(Endomorphisms)*
Let E be a vector space over $\mathbb{K}$ *. An endomorphism of E is a linear map from E to itself. We denote by $L(E)$ the set of endomorphisms of E.*

Remark 6.4.1. *For endomorphisms, we use this notation : $f \circ f \circ f = f^3$.*

Example 6.4.2. *Why f has no meaning if f is the linear map from $\mathbb{R}^2$ to $\mathbb{R}$ defined by $f(x, y) = x$?*

Definition 6.4.4. *(Isomorphisms and automorphisms)*
Let f be a linear map from E to F two vector spaces over $\mathbb{K}$.

1. *f is an isomorphism if and only if f is bijective.*

2. *f is an automorphism if and only if f is an endomorphism and is bijective, so is both an endomorphism and an isomorphism.*

Theorem 6.4.5. *Let f be an isomorphism from E to F. Then f^{-1} is an isomorphism from F to E.*

Proposition 6.4.6. *Let f be an automorphism of E (isomorphism from E to E). Then f^{-1} is an automorphism of E. Let f and g be two automorphisms of E, then $g \circ f$ is an automorphism of E and we have $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$.*

Definition 6.4.5. *(The kernel of a linear map)*

The kernel (or null space) of such a f *, denoted by* $ker(f)$ *, is the set of all* u *in* E *such that* $f(u) = 0_F$ *(the zero vector in* F*):*

$$ker(f) = \{u \in E \ : \ f(u) = 0_F\} = f^{-1}(\{0_F\}).$$

Example 6.4.3. ***1.*** *Let's consider* f *:*

$$f \ : \ \mathbb{R}^3 \longrightarrow \mathbb{R}^2$$

$$(x, y, z) \longmapsto f(x, y, z) = (y \ , x + y + z).$$

Find the kernel of f*.*

2. *Let's consider* g *:*

$$g \ : \ \mathbb{R}^2 \longrightarrow \mathbb{R}^3$$

$$(x, y) \longmapsto g(2x - y \ , \ x + 2y \ , \ x + y).$$

Find the kernel of g*.*

Proposition 6.4.7. *If* $f : E \longrightarrow F$ *is a linear map, then* $ker f$ *is a subspace of* E*.*

Proposition 6.4.8. *A linear map* $f : E \longrightarrow F$ *is injective if and only if* $ker f = \{0_E\}$ *:*

$$f \text{ is injective} \iff ker f = \{0_E\}.$$

6.4.4 Linear maps and dimension

Definition 6.4.6. *(The kernel of a linear map)*

The image of f *, denoted by* $Im(f)$ *, is the set of all vectors in* F *of the form* $f(x)$ *for some* x *in* E *.*

$$Im(f) = \{f(x) \ : \ x \in E\} = f(E).$$

Proposition 6.4.9. *If* $f : E \longrightarrow F$ *is a linear map, then* $Im(f)$ *is a subspace of* F*.*

Proposition 6.4.10. *A linear map* $f : E \longrightarrow F$ *is surjective if and only if* $Im(f) = f(E)$ *:*

$$f \text{ is surjective} \iff Im(f) = f(E).$$

Example 6.4.4. ***1.*** *Let's consider the map* f *defined as :*

$$f \ : \ \mathbb{R}^3 \longrightarrow \mathbb{R}^2$$

$$(x,y,z) \longmapsto f(x,y,z) = (x+y\ ,\ z).$$

$$f \text{ is injective} \iff ker(f) = \{0_{\mathbb{R}^3}\}.$$

$$ker(f) = \{X = (x,y,z) \in \mathbb{R}^3 \ : \ f(X) = 0_{\mathbb{R}^2}\}.$$

$$\begin{aligned} ker(f) &= \{(x,y,z) \in \mathbb{R}^3 \ : \ f(x,y,z) = (0,0)\} \\ &= \{(x,y,z) \in \mathbb{R}^3 \ : \ (x+y\ ,\ z) = (0,0). \\ &= \{(x,y,z) \in \mathbb{R}^3 \ : \ x+y = 0 \text{ and } z = 0\}. \\ &= \{(x,y,z) \in \mathbb{R}^3 \ : \ x = -y \text{ and } z = 0\}. \\ &= \{y(-1,1,0) \in \mathbb{R}\}. \end{aligned}$$

$ker(f) = span\{(-1,1,0)\}$.
$ker(f)$ *is generated by the vector* $(-1,1,0)$ *, then* $ker(f) \neq \{0_{\mathbb{R}^3}\}$.
Hence f *is not injective.*

$$Im(f) = \{f(X) \ : \ X \in \mathbb{R}^3\} = \mathbb{R}^2.$$

$$\begin{aligned} Im(f) &= \{f(x,y,z) \ : \ (x,y,z) \in \mathbb{R}^3\} \\ &= \{(x+y\ ,\ z) \ : \ (x,y,z) \in \mathbb{R}^3\} \\ &= \{x(1,0) + y(1,0) + z(0,1) \ : \ (x,y,z) \in \mathbb{R}^3\}. \end{aligned}$$

$Im(f) = span\{(1,0),(0,1)\}$.
Hence, $Im(f)$ *is generated by two vectors* $(1,0),(1,0)$ *which are the canonical basis of* $\mathbb{R}^2$ *. Then* $Im(f) = \mathbb{R}^2$ *and* f *is surjective.*

Theorem 6.4.11.

Let E and F be two vector spaces over $\mathbb{K}$ and $f : E \longrightarrow F$ a linear map. If $V = (e_1, e_2, ..., e_p)$ is a spanning set of E, which means $E = Vect(e_1, e_2, ..., e_p)$ then $V' = (f(e_1), f(e_2), ..., f(e_p))$ is a spanning set of $Im(f)$.

This theorem allows to find $Im(f)$ using only a spanning set of E.

Proposition 6.4.12. *Let $f : E \longrightarrow F$ be a linear map, with V finite-dimensional. Then:*

$$dimE = dimker(f) + dimIm(f).$$

Definition 6.4.7. *(The rank of a linear map)*
The rank of a linear map f is the dimension of its image, written $rankf$:

$$rankf = dim(Im(f)).$$

and we have :

$$rankf = dimE - dim(ker(f)).$$

Theorem 6.4.13. *Let E and F be two $\mathbb{K}$ vector spaces of finite dimension and f a linear mapping of E into F then we have the following equivalences :*

- *f is injective $\iff rank(f) = dimE$.*
- *f is surjective $\iff rank(f) = dimF$.*
- *f is bijective $\iff dimE = rank(f) = dimF$.*

Theorem 6.4.14.

Let E and F be two finite dimensional vector spaces over $\mathbb{K}$ with the same dimension and f a linear map from E to F . The following sentences are equivalent :

1. *f is injective.*

2. *f is surjective.*

3. *f is bijective.*

Exercise 6.4.1. *Let f be the map defined by :*

$$f \ : \ \mathbb{R}^2 \longrightarrow \mathbb{R}^2$$

$$(x,y) \longmapsto f(x,y) = (4x+6y \ , \ 2x-3y).$$

1. *Show that f is linear .*

2. *Show that f is a projection ie $f \circ f = f$.*

3. *Determine $Ker(f)$ et $Im(f)$.*

4. *Is f injective, surjective?*

Exercise 6.4.2. *Let f be a linear mapping from $\mathbb{R}^2$ into $\mathbb{R}^5$, defined by (x,y) of $\mathbb{R}^2$:*

$$f(x,y) = (x+2r, -2x+3y, x+y, 3x+5y, -x+2y).$$

1. *Show that f is linear .*

2. *Determine $Ker(f)$ and its dimension.*

3. *Determine $Im(f)$ and its dimension.*
4. *Is f injective, surjective?*

Exercise 6.4.3. *Let f be a linear mapping from $\mathbb{R}^2$ into $\mathbb{R}^3$, defined by (x,y) of $\mathbb{R}^2$:*

$$f(x,y) = (5x+3y, -2y+3x, -3x).$$

1. *Show that f is linear .*

2. *Determine $Ker(f)$ and its dimension.*

3. *Determine $Im(f)$ and its dimension.*
4. *Is f injective, surjective?*

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