Democratic and Popular Republic of Algeria
Ministry of Higher Education and Scientific Research
Ibn Khaldoun University of Tiaret

Faculty of Applied Sciences
Department of Science and Technology

Handout of Mathematics 03
Lessons and exercises

For Second-Year LMD Students in the Science and Technology Domain.

Presented by:

Dr. TABTI Hamza

Expertised by:

Dr. ELKHIRI Laid -Ibn Khaldoun University of Tiaret.

Dr. Tahar Mezeddek Mohamed -Mustapha STAMBOULI University of Mascara.

Academic Year: 2024/2025.

Contents

Introduction

This book is a collection of exercises and problems in mathematics for engineering students. It is intended for second-year Bachelor's students in the Science and Technology domain L2 and also for students from other fields or preparatory schools. This material contains essential theorems and exercises to illustrate them.
This handout emphasizes practical methods of computation (theorems, propositions, etc.) without providing proofs to focus directly on the main objectives. It also includes exercises with detailed solutions and additional unsolved problems to assess the reader's understanding. Readers are encouraged to solve the exercises independently before consulting the solutions.
The content of this handout is derived from the bibliography provided at the end.

Structure

1. The first chapter revisits results about single and multiple integrals.

2. The second chapter introduces improper integrals.

3. The third chapter focuses on solving differential equations.

4. The fourth chapter discusses infinite series, sequences, and series of functions, including different types of convergence, power series, their applications in solving differential equations, and Fourier series, a crucial tool for engineers across fields.

5. The fifth chapter covers Fourier transforms and their applications in solving differential equations.

6. The last chapter presents the Laplace transform and its applications.

I hope this document helps students mastering these areas of mathematics.

Chapter 1

Single and Multiple Integrals

1.1 Review of Riemann integrals and antiderivatives

The objective of these notes is to revisit the content of integration lessons. This quick review is crucial for refreshing memory before proceeding further.

1.1.1 Integrals of a Piecewise Continuous Function on a Segment

We begin by recalling a definition and the main properties of the Riemann integral for a real-valued function of one variable. We limit the discussion to functions that are continuous or piecewise continuous. Considering the broader class of Riemann integrable functions adds refinement but is not strictly necessary, especially when the Lebesgue integral is also available.
We start by recalling the definition and basic properties of piecewise continuous functions.

Definition 1.1.1. *Let $[a, b] \subset \mathbb{R}$; $a < b$ and f be a function defined on $[a, b]$. We say that f is piecewise continuous if there exist $n \in \mathbb{N}^*$ and points $x_0, ..., x_n \in [a, b]$ such that:*

$$a = x_0 < x_1 \cdots < x_n = b,$$

and for every $j \in [1, n]$ the restriction of f to the interval $]x_{j-1}, x_j[$ is continuous and admits finite limits at x_{j-1} and x_j.

The set of piecewise continuous functions is stable under addition, multiplication, and scalar multiplication. Furthermore, a piecewise continuous function on a segment is bounded.
It is also recalled that a function continuous on a segment is uniformly continuous (Heine's theorem).

Regardless of the chosen definition, the idea of the Riemann integral is to approximate a function by increasingly finer step functions (piecewise constant functions).

Definition 1.1.2. *Let $[a, b]$ be a segment of $\mathbb{R}$. A pointed subdivision σ of $[a, b]$ is defined as:*

- *an integer $n \in \mathbb{N}^*$,*
- *points $x_0, ..., x_n \in [a, b]$ such that $a = x_0 < x_1 \cdots < x_n = b$,*
- *and points $\xi_1 \in [x_0, x_1]$, $\xi_1 \in [x_1, x_2]$, ... ,$\xi_n \in [x_{n-1}, x_n]$.*

Furthermore, the step size of this subdivision σ is said to be less than or equal to $\delta > 0$ if

$$\forall j \in [1, n], \quad x_j - x_{j-1} \leq \delta.$$

Given a function f from $[a, b]$ to $\mathbb{R}$, we define

$$S_\sigma(f) = \sum_{j=1}^{n} f(\xi_j)\,(x_j - x_{j-1})\,.$$

This corresponds to approximating f by the function that takes the value $f(\xi_j)$ on $]x_{j-1}, x_j[$ for all $j \in [1, n]$.

Proposition 1.1.1. *Let f be a piecewise continuous function on $[a, b]$ mapping to $\mathbb{R}$. Then there exists a unique value $I(f)$ such that:*

For every $\varepsilon > 0$, there exists $\delta > 0$ such that for any partition σ of $[a, b]$ with mesh size less than or equal to δ, the following holds:

$$|S_\sigma(f) - I(f)| \leq \varepsilon.$$

where $S_\sigma(f)$ is the Riemann sum.
We denote this unique value as:

$$I(f) = \int_a^b f(x)\,dx.$$

Typically, a natural choice to obtain an increasingly fine subdivision of the segment $[a, b]$ s to use a uniform subdivision. Given $n \in \mathbb{N}^*$, we define, for all $j \in [1, n]$

$$x_j = a + \frac{j}{n}(b - a).$$

Natural choices for selecting ξ_j in $[x_{j-1}, x_j]$ are to take $\xi_j = x_{j-1}$ or $\xi_j = x_j$. Thus, for any piecewise continuous function f on $[a, b]$, we have:

$$\frac{b-a}{n} \sum_{j=0}^{n-1} f\left(a + \frac{j}{n}(b - a)\right) \underset{n\to+\infty}{\longrightarrow} \int_a^b f(x)\,dx.$$

1.1.2 Antiderivative of a continuous function

Let $I = [a, b] \subset \mathbb{R}$; $a < b$ and let $f : I \to \mathbb{R}$ be a continuous function.

Definition 1.1.3. *A function $F : I \to \mathbb{R}$ is an antiderivative of f if:*

1) *F is differentiable on I.*

2) $F' = f \ \forall x \in [a, b]$

Theorem 1.1.1. *Let f be continuous on $[a, b]$. For every antiderivative F of f, we have:*

$$\int_a^b f(x)\, dx = [F(x)]_a^b = F(b) - F(a).$$

In this notation, x is a dummy variable: it can be replaced by any other letter (avoiding one that is already used in the bounds), and we can thus also write

$$\int_a^b f(x)\, dx = \int_a^b f(t)\, dt = \int_a^b f(u)\, du = \int_a^b f(\tau)\, d\tau \ldots$$

Theorem 1.1.2. *Any real, continuous function f on I has at least one primitive.*

Notation 1.1.1. (Indefinite Integral)
Given a function f continuous on I and a primitive F of f, we denote:

$$F(x) = \int f(x)\, dx.$$

1.1.3 A list of Common Antiderivatives

Function f	Particular Antiderivative F	Interval
a	ax	$\mathbb{R}$
x^α, $\alpha \neq -1$	$\dfrac{x^{\alpha+1}}{\alpha+1}$	$\mathbb{R}^+$
$\dfrac{1}{x}$	$\ln\lvert x\rvert$	$]-\infty, 0[$ ou $]0, +\infty[$
e^x	e^x	$\mathbb{R}$
$\cos x$	$\sin x$	$\mathbb{R}$
$\sin x$	$-\cos x$	$\mathbb{R}$
$\cosh x$	$\sinh x$	$\mathbb{R}$
$\sinh x$	$\cosh x$	$\mathbb{R}$
$\dfrac{1}{\sqrt{1-x^2}}$	$\arcsin x$	$]-1, 1[$
$\dfrac{1}{1+x^2}$	$\arctan x$	$\mathbb{R}$

1.1.4 Some properties of integrals

Let f and g be continuous functions on $[a, b]$. We have:

1) Reversing the limits of integration:

$$\int_a^b f(x)\, dx = -\int_b^a f(x)\, dx.$$

2) Additivity with respect to intervals:

$$\int_a^b f(x)\, dx = \int_a^c f(x)\, dx + \int_c^b f(x)\, dx; \quad c \in [a, b].$$

3) Scaling by a constant:

$$\int_a^b \lambda \cdot f(x)\, dx = \lambda \int_a^b f(x)\, dx; \quad \lambda \in \mathbb{R}.$$

4) Linear combination:

$$\int_a^b [f(x) \pm g(x)]\, dx = \int_a^b f(x)\, dx \pm \int_a^b g(x)\, dx.$$

5) Integral over a point:

$$\int_a^a f(x)\, dx = 0.$$

1.1.5 General Integration Formulas

Let u be a differentiable function on $[a, b]$.

Function f	Particular Antiderivative F
$u'(x)u^{\alpha}(x)$, $\alpha \neq -1$	$\dfrac{u^{\alpha+1}(x)}{\alpha+1}$
$\dfrac{u'(x)}{u(x)}$	$\ln \lvert u(x) \rvert$
$u'(x)e^{u(x)}$	$e^{u(x)}$
$u'(x)\cos[u(x)]$	$\sin[u(x)]$
$u'(x)\sin[u(x)]$	$-\cos[u(x)]$
$\dfrac{u'(x)}{\sqrt{1-u^2(x)}}$	$\arcsin[u(x)]$
$\dfrac{u'(x)}{1+u^2(x)}$	$\arctan[u(x)]$

Example 1.1.1. *Evaluate the following indefinite integral:*

$$\int \left(5x^4 + e^{3x} + 9\sin 2x\right)\, dx$$

Solution. *We have:*

$$\begin{aligned} F(x) &= \int 5x^4\, dx + \int e^{3x}\, dx + 9\int \sin 2x\, dx \\ &= 5\int x^4\, dx + \frac{1}{3}\int 3e^{3x}\, dx + \frac{9}{2}\int 2\sin 2x\, dx \\ &= x^5 + \frac{1}{3}e^{3x} - \frac{9}{2}\cos 2x + c; \quad c \in \mathbb{R}. \end{aligned}$$

Example 1.1.2. *Calculate the following integral:*

$$I = \int_0^1 \left(3x^2 + 2xe^{x^2} + \frac{3x^2}{1+x^3}\right)\, dx$$

Solution. *We have:*

$$\begin{aligned} I &= \int_0^1 \left(3x^2 + 2xe^{x^2} + \frac{3x^2}{1+x^3}\right)\, dx \\ &= \left[x^3 + e^{x^2} + \ln(1+x^3)\right]_0^1 \\ &= e + \ln 2. \end{aligned}$$

1.1.6 General Methods for Computing Integrals

This part outlines the main techniques for calculating definite or indefinite integrals.

Integration by Parts

Let u and v be two continuous functions on $[a, b]$. We have:

$$\int_a^b uv'\, dx = [uv]_a^b - \int_a^b u'v\, dx.$$

Example 1.1.3. *Compute the following integrals:*
1) $\boldsymbol{A} = \int_1^2 \ln x\, dx$, ***2)*** $\boldsymbol{B} = \int_0^1 xe^x\, dx$, ***3)*** $\boldsymbol{C} = \int_0^\pi x \sin x\, dx$.

Solution. *Using the integration by parts formula:*
1) $\boldsymbol{A} = \int_1^2 \ln x\, dx$:

Let $u(x) = \ln x$, *so* $u'(x) = \frac{1}{x}$ *and* $v'(x) = 1$, *so* $v(x) = x$. *Then:*

$$\begin{aligned} A &= [x \ln x]_1^2 - \int_1^2 \frac{x}{x}\, dx \\ &= [x \ln x - x]_1^2 = 2\ln 2 - 1. \end{aligned}$$

2) $\boldsymbol{B} = \int_0^1 xe^x\, dx$:

Let $u(x) = x$, *so* $u'(x) = 1$ *and* $v'(x) = e^x$, *so* $v(x) = e^x$. *Then:*

$$\begin{aligned} B &= [xe^x]_0^1 - \int_0^1 e^x\, dx \\ &= [xe^x - e^x]_0^1 \\ &= 1. \end{aligned}$$

3) $\boldsymbol{C} = \int_0^\pi x \sin x\, dx$:

Let $u(x) = x$, so $u'(x) = 1$ and $v'(x) = \sin x$, so $v(x) = -\cos x$. Then:

$$\begin{aligned} C &= [-x\cos x]_0^\pi + \int_0^\pi \cos x\ dx \\ &= [\sin x - x\cos x]_0^\pi \\ &= \pi. \end{aligned}$$

Change of Variables

Let f be a continuous function on $[a,b]$, and u be a differentiable function. If we substitute $x = u(t)$, then:

$$\int_a^b f(x)\ dx = \int_{u^{-1}(a)}^{u^{-1}(b)} f\left[u(t)\right] u'(t)\ dt.$$

Example 1.1.4. *Compute the following integrals:*

1) $\boldsymbol{A} = \int_0^{\pi^2} \cos\left(\sqrt{x}\right)\ dx$, *2)* $\boldsymbol{B} = \int_0^{\ln 2} \sqrt{e^x - 1}\ dx$.

Solution. *Using the change of variables method:*

1) $\boldsymbol{A} = \int_0^{\pi^2} \cos\left(\sqrt{x}\right)\ dx$:

Let $\sqrt{x} = t \Rightarrow x = t^2$, so $dx = 2tdt$ with $t \in [0,\pi]$. Then, we obtain:

$$\int_0^{\pi^2} \cos\left(\sqrt{x}\right)\ dx = 2\int_0^\pi t\cos t\ dt.$$

Using integration by parts:

Let $u(t) = t$; so $u'(t) = 1$ and $v'(t) = \cos t$, so $v(t) = \sin t$. Then:

$$\begin{aligned} A &= 2\left[t\sin t\right]_0^\pi - 2\int_0^\pi \sin x\ dx \\ &= 2\left[t\sin t + \cos t\right]_0^\pi \\ &= -4. \end{aligned}$$

2) $\boldsymbol{B} = \int_0^{\ln 2} \sqrt{e^x - 1}\ dx$:

Let $\sqrt{e^x - 1} = t \Rightarrow x = \ln(t^2+1)$, so $dx = 2\dfrac{tdt}{t^2+1}$ with $t \in [0,1]$. Then, we obtain:

$$\begin{aligned} B &= \int_0^{\ln 2} \sqrt{e^x - 1}\ dx = 2\int_0^1 \frac{t^2}{t^2+1}\ dt \\ &= 2\int_0^1 \frac{t^2+1-1}{t^2+1}\ dt = \int_0^1 1\ dt - \int_0^1 \frac{1}{t^2+1}\ dt \\ &= 2\left[t - \arctan t\right]_0^1 \\ &= 2 - \frac{\pi}{2}. \end{aligned}$$

Transformations of Expressions

To obtain the primitives of a continuous function f, we can transform the expression of $f(x)$.

Example 1.1.5. *Calculate the following integrals:*

1) $\boldsymbol{A} = \displaystyle\int_1^2 \frac{1}{x^2+x}\, dx$ ***2)*** $\boldsymbol{B} = \displaystyle\int_0^{\frac{\pi}{2}} \cos^2 x\, dx.$

Solution. *We can transform the form of $f(x)$:*

1) $\boldsymbol{A} = \displaystyle\int_1^2 \frac{1}{x^2+x}\, dx$*:*

We decompose $f(x)$ into two simple fractions as follows:

$$\frac{1}{x^2+x} = \frac{1}{x(x+1)} = \frac{1}{x} - \frac{1}{x+1}.$$

$$\begin{aligned} A &= \int_1^2 \frac{1}{x^2+x}\, dx = \int_1^2 \left(\frac{1}{x} - \frac{1}{x+1}\right) dx \\ &= [\ln x - \ln(x+1)]_1^2 = \left[\ln\left(\frac{x}{x+1}\right)\right]_1^2 \\ &= \ln\frac{4}{3}. \end{aligned}$$

2) $\boldsymbol{B} = \displaystyle\int_0^{\frac{\pi}{2}} \cos^2 x\, dx$*:*

Here, we use trigonometric identities to rewrite $f(x)$. We have:

$$\cos^2 x = \frac{1+\cos 2x}{2}, \quad \forall x \in \mathbb{R}.$$

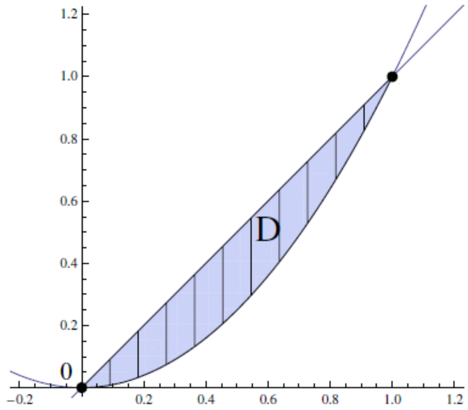
$$\begin{aligned} B &= \int_0^{\frac{\pi}{2}} \cos^2 x\, dx = \frac{1}{2}\int_0^{\frac{\pi}{2}} (1+\cos 2x)\, dx \\ &= \frac{1}{2}\left[x + \frac{1}{2}\sin 2x\right]_0^{\frac{\pi}{2}} \\ &= \frac{\pi}{4}. \end{aligned}$$

1.2 Double and triple integrals

1.2.1 Double Integrals

Let $f : D \subset \mathbb{R}^2 \to \mathbb{R}$ with $f(x, y)$ being a continuous function, and D being a domain bounded by simple curves in $\mathbb{R}^2$. Then, the double integral exists:

$$\iint_D f(x, y)\, dx\, dy.$$



Methods for Calculating Double Integrals

Proposition 1.2.1. *Let f, f_1 and f_2 three continuous functions;*
$f : D \subset \mathbb{R}^2 \to \mathbb{R}$
$(x, y) \mapsto f(x, y) = f_1(x) \cdot f_2(y)$. *Here* $D = [a, b] \times [c, d]$. *Then:*

$$\begin{aligned} \iint_D f(x, y)\, dx\, dy &= \iint\limits_{[a,b]\times[c,d]} f_1(x) \cdot f_2(y)\, dx\, dy \\ &= \left(\int_a^b f_1(x)\, dx\right)\left(\int_c^d f_2(y)\, dy\right). \end{aligned}$$

Example 1.2.1. *Compute the integral:*

$$\boldsymbol{I} = \iint_D 2x \sin y\, dx\, dy; \qquad D = [0, 1] \times [0, \pi].$$

Solution. *Since* $f(x, y) = 2x \cdot \sin y$, *the integrals can be separated:*

$$\begin{aligned} \boldsymbol{I} &= \iint_{[0,1]\times[0,\pi]} 2x \cdot \sin y\, dx\, dy \\ &= \left(\int_0^1 2x\, dx\right)\left(\int_0^\pi \sin y\, dy\right) \\ &= \left[x^2\right]_0^1 \times [-\cos y]_0^\pi \\ &= 2. \end{aligned}$$

Proposition 1.2.2. *If f is continuous on $D = [a, b] \times [c, d]$. So:*

$$\begin{aligned}\iint_D f(x,y)\,dx\,dy &= \iint_{[a,b]\times[c,d]} f(x,y)\,dx\,dy\\ &= \int_a^b \left(\int_c^d f(x,y)\,dy\right) dx\\ &= \int_c^d \left(\int_a^b f(x,y)\,dx\right) dy.\end{aligned}$$

Example 1.2.2. *Compute the integral:*

$$\boldsymbol{B} = \iint_D \frac{1}{(1+x+y)^2}\,dx\,dy; \qquad D = [0,1]\times[0,2].$$

Solution. *Split the integration into two steps:*

$$\begin{aligned}\boldsymbol{B} &= \int_0^2 \left[\int_0^1 \frac{1}{(1+x+y)^2}\,dx\right] dy = \int_0^2 \left[-\frac{1}{1+x+y}\right]_0^1 dy\\ &= \int_0^2 \left(-\frac{1}{2+y} + \frac{1}{1+y}\right) dy = [\ln(y+1) - \ln(y+2)]_0^2\\ &= \left[\ln\left(\frac{y+1}{y+2}\right)\right]_0^2 = \ln\frac{3}{2}.\end{aligned}$$

Proposition 1.2.3. *Let $f(x, y)$ be a continuous function over a region D described as:*

$$D = \left\{(x,y) \in \mathbb{R}^2 / a \le x \le b \quad \text{and} \quad \Phi(x) \le y \le \Psi(x)\right\}.$$

Here Φ and Ψ are continuous functions on $[a, b]$ according to Figure 1.1.

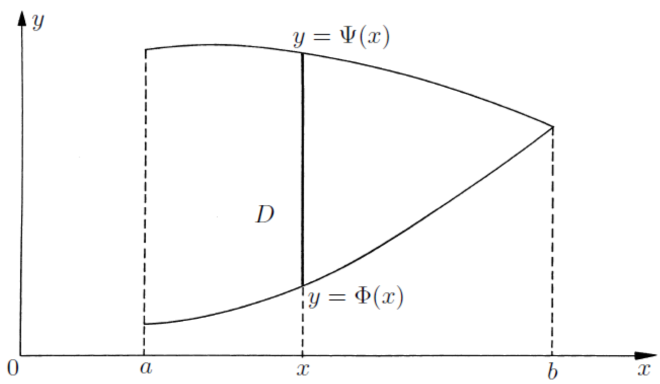


Figure 1.1: The region D

$$\Rightarrow \iint_D f(x,y)\,dx\,dy = \int_a^b \left(\int_{\Phi(x)}^{\Psi(x)} f(x,y)\,dy\right) dx.$$

Example 1.2.3. *Evaluate the following integral:*

$$\boldsymbol{A} = \iint_D xe^y \, dx \, dy; \quad D = \left\{(x, y) \in \mathbb{R}^2 / x^2 \leq y \leq x\right\}.$$

Solution. *We will find the bounds of integration, as follows:*
$(x, y) \in D \Longrightarrow x^2 \leq y \leq x$ *this implies that* $x^2 \leq x$, *thus* $x \in [0, 1]$.

$$D = \left\{(x, y) \in \mathbb{R}^2 / 0 \leq x \leq 1 \text{ and } x^2 \leq y \leq x\right\}$$

is represented on Figure 1.2.

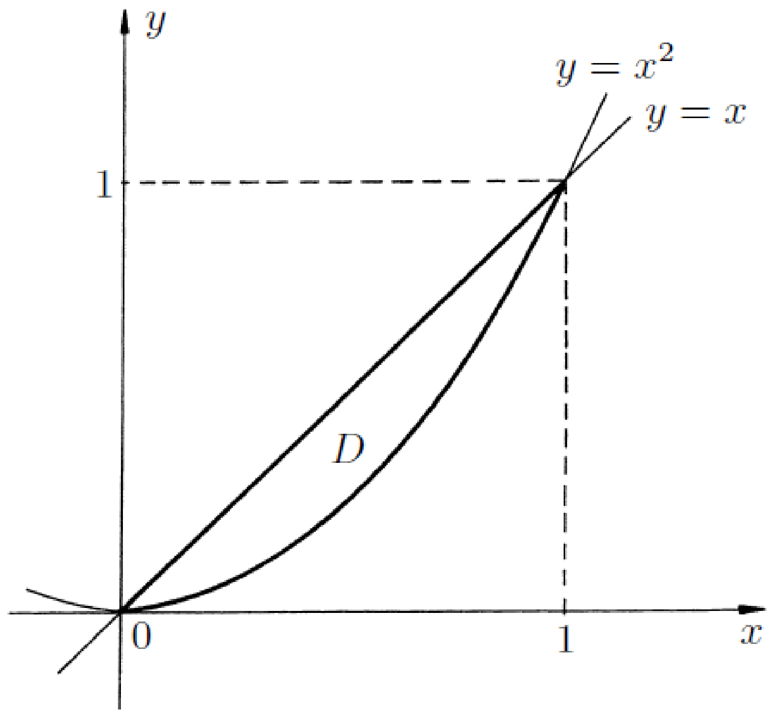


Figure 1.2: The region D

Now we can calculate the double integral A:

$$\begin{aligned}
\boldsymbol{A} &= \int_0^1 \left(\int_{x^2}^x xe^y \, dy\right) dx = \int_0^1 x \left[e^y\right]_{x^2}^x dx \\
&= \int_0^1 x \left(e^x - e^{x^2}\right) dx = \int_0^1 xe^x \, dx - \int_0^1 xe^{x^2} \, dx \\
&= \left[xe^x\right]_0^1 - \int_0^1 e^x \, dx - \frac{1}{2} \int_0^1 2xe^{x^2} \, dx \\
&= \left[xe^x - e^x - \frac{1}{2} e^{x^2}\right]_0^1 \\
&= \frac{3 - e}{2}.
\end{aligned}$$

Proposition 1.2.4. *Let $f(x,y)$ be a continuous function over a region D described as:*

$$D = \left\{(x,y) \in \mathbb{R}^2 / c \le y \le d \quad \text{and} \quad \Phi(y) \le x \le \Psi(y)\right\}.$$

Here Φ et Ψ are continuous functions on $[c,d]$ according to Figure 1.3.

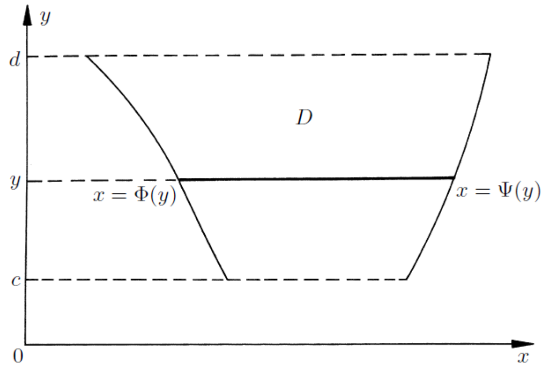


Figure 1.3: The region D

$$\Rightarrow \iint_D f(x,y)\,dx\,dy = \int_c^d \left(\int_{\Phi(y)}^{\Psi(y)} f(x,y)\,dx \right) dy.$$

Example 1.2.4. *Calculate the double integral:*

$$\boldsymbol{B} = \iint_D \frac{dx\,dy}{(x+y)^3}; \quad D = \left\{(x,y) \in \mathbb{R}^2 / x > 1,\ y > 1,\ x+y < 3\right\}.$$

Solution. *We will find the bounds of integration. We can write the domain of integration as follows:*
$D = \left\{(x,y) \in \mathbb{R}^2 / 1 \le y \le 2 \text{ et } 1 \le x \le 3 - y\right\}$ *is represented on Figure 1.4.*

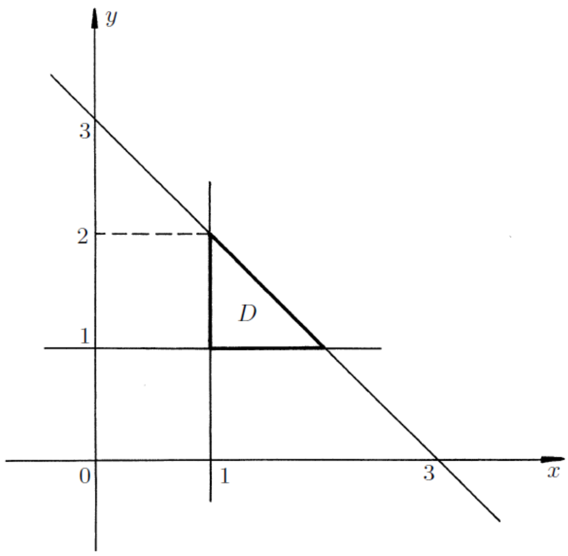


Figure 1.4: The region D

$$\begin{aligned}\Rightarrow \boldsymbol{B} &= \int_1^2 \left[\int_1^{3-y} \frac{1}{(x+y)^3}\, dx\right] dy = \int_1^2 \left[-\frac{1}{2(x+y)^2}\right]_1^{3-y} dy \\ &= -\frac{1}{2}\int_1^2 \left[\frac{1}{9} - \frac{1}{(y+1)^2}\right] dy = -\frac{1}{2}\left[\frac{y}{9} + \frac{1}{y+1}\right]_1^2 \\ &= \frac{1}{36}.\end{aligned}$$

Method of Variable Substitution

Let $f : D \subset \mathbb{R}^2 \to \mathbb{R}$
$(x, y) \mapsto f(x, y)$ be a continuous function on a closed and bounded domain. We define:

$$\begin{cases} x = x(u, v), \\ y = y(u, v). \end{cases}$$

1) $x(u, v)$ and $y(u, v)$ have continuous partial derivatives.

2) The Jacobian $\begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix} \neq 0.$

Theorem 1.2.1. *If $x(u, v)$ and $y(u, v)$ satisfy conditions 1) and 2), then:*

$$\iint_D f(x, y)\, dx\, dy = \iint_{D'} f\left(x(u, v), y(u, v)\right) \mid J \mid \; du\, dv.$$

Here $(x, y) \in D$ and $(u, v) \in D'$.

Example 1.2.5. *Calculate the following double integral using the change of variables (polar coordinates):*

$$\boldsymbol{I} = \iint_D \sqrt{x^2 + y^2}\, dx\, dy$$

and

$$D = \left\{(x, y) \in \mathbb{R}^2 / x^2 + y^2 \leq 1\right\}.$$

Solution. *To compute this double integral, we use polar coordinates as follows:*

$$\begin{cases} x = r\cos\theta, \\ y = r\sin\theta. \end{cases}$$

Here

$$J = \begin{vmatrix} \dfrac{\partial x}{\partial r} & \dfrac{\partial x}{\partial \theta} \\ \dfrac{\partial y}{\partial r} & \dfrac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \\ \sin\theta & r\cos\theta \end{vmatrix} = r\cos^2\theta + r\sin^2\theta = r.$$

Since, we have:

$$\iint_D f(x,y)\,dx\,dy = \iint_{D'} f\left(r\cos\theta, r\sin\theta\right) r\,dr\,d\theta.$$

In this example: $D' = [0,1]\times[0,2\pi]$ *according to Figure 1.5.*

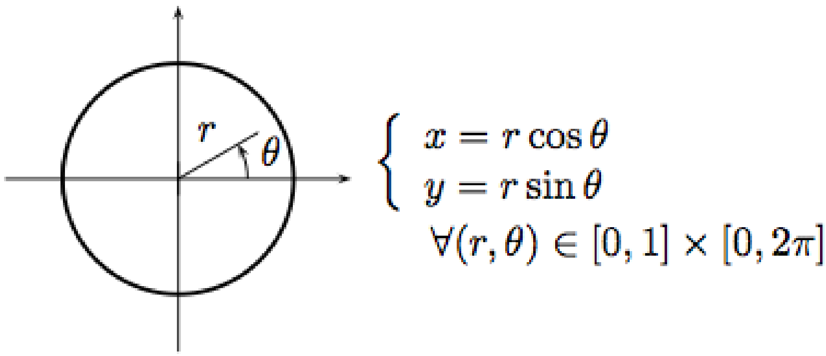


Figure 1.5: The region D'

Therefore:

$$\begin{aligned} \boldsymbol{I} &= \iint_{[0,1]\times[0,2\pi]} r^2\,dr\,d\theta \\ &= \left(\int_0^1 r^2\,dr\right)\left(\int_0^{2\pi} 1\,d\theta\right) \\ &= \left[\frac{r^3}{3}\right]_0^1 \times [\theta]_0^{2\pi} \\ &= \frac{2\pi}{3}. \end{aligned}$$

1.2.2 Les intégrales triples

Let $f : D \subset \mathbb{R}^3 \to \mathbb{R}$ with $f(x, y, z)$ being a continuous function, and D being a domain bounded by simple curves in $\mathbb{R}^3$. Then, the triple integral exists:

$$\iiint_D f(x, y, z)\, dx\, dy\, dz.$$

Methods for Calculating Triple Integrals

Proposition 1.2.5. *Let f, f_1, f_2 and f_3 are continuous functions;*
$f : D \to \mathbb{R}$
$(x, y) \mapsto f(x, y, z) = f_1(x) \cdot f_2(y) \cdot f_3(z)$ *with* $D = [a_1, b_1] \times [a_2, b_2] \times [a_3, b_3]$.

$$\begin{aligned}\iiint_D f(x, y)\, dx\, dy &= \iiint\limits_{[a_1,b_1]\times[a_2,b_2]\times[a_3,b_3]} f_1(x) \cdot f_2(y) \cdot f_3(z)\, dx\, dy\, dz \\ &= \left(\int_{a_1}^{b_1} f_1(x)\, dx\right)\left(\int_{a_2}^{b_2} f_2(y)\, dy\right)\left(\int_{a_3}^{b_3} f_3(z)\, dz\right).\end{aligned}$$

Example 1.2.6. *Compute this integral:*

$$\boldsymbol{I} = \iiint_D x^2 yz\, dx\, dy\, dz$$

et

$$D = [0, 1] \times [1, 2] \times [0, 2].$$

Solution. *Separate the variables and solve each term as follows:*

$$\begin{aligned}\boldsymbol{I} &= \iiint\limits_{[0,1]\times[1,2]\times[0,2]} x^2 yz\, dx\, dy\, dz \\ &= \left(\int_0^1 x^2\, dx\right)\left(\int_1^2 y\, dy\right)\left(\int_0^2 z\, dz\right) \\ &= \left[\frac{x^3}{3}\right]_0^1 \times \left[\frac{y^2}{2}\right]_1^2 \times \left[\frac{z^2}{2}\right]_0^1 = 1.\end{aligned}$$

Proposition 1.2.6. *Let f be a continuous function, let D be defined as:*

$$\Big\{(x, y, z) \in \mathbb{R}^3 / a \le x \le b,\ \Phi_1(x) \le y \le \Psi_1(x), \Phi_2(x, y) \le z \le \Psi_2(x, y)\Big\},$$

where Φ_1, Ψ_1, Φ_2 and Ψ_2 are continuous functions. Then:

$$\Rightarrow \iint_D f(x, y, z)\, dx\, dy\, dz = \int_a^b \left[\int\limits_{\Phi_1(x)}^{\Psi_2(x)} \left[\int\limits_{\Phi_2(x,y)}^{\Psi_2(x,y)} f(x, y, z)\, dz\right] dy\right] dx.$$

Example 1.2.7. *Compute the integral:*

$$\boldsymbol{I} = \iint_D \frac{1}{(1+x+y+z)^3}\, dx\, dy\, dz$$

with

$$D = \Big\{(x,y) \in \mathbb{R}^2 / x \geq 0,\ y \geq 0,\ z \geq 0 \text{ and } x+y+z \leq 1\Big\}.$$

Solution. *The region D can be described using the inequalities*

$$0 \leq x \leq 1, \quad 0 \leq y \leq 1-x, \quad 0 \leq z \leq 1-x-y.$$

Write the triple integral:

$$\boldsymbol{I} = \int_0^1 \left[\int\limits_0^{1-x} \left[\int\limits_0^{1-x-y} \frac{1}{(1+x+y+z)^3}\, dz \right] dy \right] dx$$

Compute the $z-$integral:

$$\begin{aligned} \int_0^{1-x-y} \frac{1}{(1+x+y+z)^3}\, dz &= \left[-\frac{1}{2(1+x+y+z)^2} \right]_0^{1-x-y} \\ &= -\frac{1}{8} + \frac{1}{2(1+x+y)^2}. \end{aligned}$$

Simplify:

$$\begin{aligned} \boldsymbol{I} &= \int_0^1 \left[\int\limits_0^{1-x} \left[\frac{1}{2(1+x+y)^2} - \frac{1}{8} \right] dy \right] dx \\ &= -\frac{1}{2} \int_0^1 \left[\frac{y}{4} + \frac{1}{1+x+y} \right]_0^{1-x} dx \\ &= -\frac{1}{2} \int_0^1 \left(-\frac{x}{4} - \frac{1}{1+x} + \frac{3}{4} \right) dx \\ &= \left[-\frac{x^2}{8} + \frac{3}{4}x - \ln(x+1) \right]_0^1 \\ &= \frac{\ln 2}{2} - \frac{5}{16}. \end{aligned}$$

1.3 Applications to area and volume computation

The area A of a domain D can be computed using double integrals:

1) With Cartesian coordinates:

$$A = \iint_D dx\,dy.$$

2) With polar coordinates:

$$A = \iint_D r\;dr\,d\theta.$$

Example 1.3.1. *Let D be the domain defined by:*

$$D = \left\{(x,y) \in \mathbb{R}^2 / -1 \leq x \leq 1 \quad \text{and} \quad x^2 \leq y \leq 4 - x^3\right\}.$$

Compute the area of D.

Solution. *The area can be computed as:*

$$\begin{aligned} area(D) &= \int_{-1}^{1}\int_{x^2}^{4-x^3} dy\;dx = \int_{-1}^{1}(4 - x^3 - x^2)\;dx \\ &= \left[4x - \frac{x^4}{4} - \frac{x^3}{3}\right]_{-1}^{1} \\ &= \frac{22}{3}. \end{aligned}$$

Example 1.3.2. *Let D be the disk centered at $(0,0)$ with radius R:*

$$D = \left\{(x,y) \in \mathbb{R}^2 / x^2 + y^2 \leq R^2\right\}.$$

Compute the area of D.

Solution. *Switch to polar coordinates:*

$$\begin{aligned} area(D) &= \int_0^{2\pi}\int_0^{R} r\;dr\,d\theta \\ &= \left(\int_0^R r\;dr\right)\left(\int_0^{2\pi} 1\;d\theta\right) \\ &= \left[\frac{r^2}{2}\right]_0^R \times [\theta]_0^{2\pi} \\ &= \pi R^2. \end{aligned}$$

Mass and Moment of Inertia

Let $\sigma(x,y)$ be the density of a material within a plane domain D:

- The **mass** is:

$$M = \iint_D \sigma(x,y)\; dx\, dy.$$

- The **moment of inertia** about the $x-$axis is:

$$I_{ox} = \iint_D y^2\sigma(x,y)\; dx\, dy.$$

- The **moment of inertia** about the $y-$axis is:

$$I_{oy} = \iint_D x^2\sigma(x,y)\; dx\, dy.$$

- The **moment of inertia** about the origin is:

$$I_0 = \iint_D (x^2+y^2)\sigma(x,y)\; dx\, dy.$$

Volume Calculations

The volume V of the domain $D \subset \mathbb{R}^3$ is given by:

$$V = \iiint_D \; dx\, dy\, dz.$$

Mass and Moment of Inertia

For $3D$ regions, the analogous formulas involve triple integrals.
Let $\sigma(x,y,z)$ be the density of a material within a plane domain D:

- The **mass** is:

$$M = \iint_D \sigma(x,y,z)\; dx\, dy\, dz.$$

- The **moment of inertia** about the $x-$axis is:

$$I_{ox} = \iint_D (y^2+z^2)\sigma(x,y,z)\; dx\, dy\, dz.$$

- The **moment of inertia** about the $y-$axis is:

$$I_{oy} = \iint_D (x^2+z^2)\sigma(x,y,z)\; dx\, dy\, dz.$$

- The **moment of inertia** about the origin is:

$$I_0 = \iint_D (x^2+y^2+z^2)\sigma(x,y,z)\; dx\, dy\, dz.$$

1.4 Supplementary exercises

Exercise 1.1. *Compute the following indefinite integrals:*

1) $\displaystyle\int \frac{x^2}{\sqrt{x^3+2}}\,dx$, *2)* $\displaystyle\int \frac{x+1}{x^2+2x+3}\,dx$, *3)* $\displaystyle\int (e^x\cos x - e^x \sin x)\,dx$,

4) $\displaystyle\int \sin\sqrt{x}\,dx$, *5)* $\displaystyle\int \frac{x^4}{1-x^2}\,dx$, *6)* $\displaystyle\int e^x(1/x+\ln x)\,dx$.

Exercise 1.2. *Evaluate the following definite integrals:*

1) $\displaystyle\int_{-1}^{3} e^{|x|}\,dx$, *2)* $\displaystyle\int_{2}^{e} \frac{\ln^2 x - 1}{x\ln^2 x}\,dx$, *3)* $\displaystyle\int_0^1 \frac{1}{1+\sqrt{x}}\,dx$,

4) $\displaystyle\int_0^{\pi/2} \sqrt{\sin x + 1}\,dx$, *5)* $\displaystyle\int_1^e x(\ln x)^2\,dx$, *6)* $\displaystyle\int_0^{\pi} \sin^2 x\cos^2 x\,dx$.

7) $\displaystyle\int_0^1 \sqrt{e^x+1}\,dx$, *8)* $\displaystyle\int_1^2 \frac{\ln x}{x^2}\,dx$, *9)* $\displaystyle\int_0^1 \sinh x\cosh^3 x\,dx$.

Exercise 1.3. *Compute the following double integrals:*

1) $\displaystyle I_1 = \iint_{D_1} \frac{1}{(x+y+1)^2}\,dx\,dy$ *where* $D_1 = [0,1]\times[0,1]$.

2) $\displaystyle I_2 = \iint_{D_2} \frac{x}{(1+x^2)(1+xy)}\,dx\,dy$ *where* $D_2 = [0,1]\times[0,1]$.

3) $\displaystyle I_3 = \iint_{D_3} \frac{1}{y\cos x+1}\,dx\,dy$ *where* $D_3 = \left[0,\frac{\pi}{2}\right]\times\left[0,\frac{1}{2}\right]$.

4) $\displaystyle I_4 = \iint_{D_4} \frac{x+y}{e^x e^y}\,dx\,dy$ *where* $D_4 = \left\{(x,y)\in\mathbb{R}^2/x,y\geq 0,\ \ x+y\leq 1\right\}$.

5) $\displaystyle I_5 = \iint_{D_5} \frac{1}{1+x^2+y^2}\,dx\,dy$ *where* $D_5 = \left\{(x,y)\in\mathbb{R}^2/\,x^2+y^2\leq 1\right\}$.

6) $\displaystyle I_6 = \iint_{D_6} \sqrt{x}\,dx\,dy$ *where* $D_6 = \left\{(x,y)\in\mathbb{R}^2/\ \ x^2\leq y\leq x\right\}$.

7) $\displaystyle I_7 = \iint_{D_7} x^2 y\,dx\,dy$ *with* $D_7 = \left\{(x,y)\in\mathbb{R}^2/y\geq 0,\ \ x^2+y^2-2x\leq 0\right\}$.

8) $\displaystyle I_8 = \iint_{D_8} dx\,dy$ *where* $D_8 = \left\{(x,y)\in\mathbb{R}^2/0\leq y\leq x,\ \ x^2+y^2\leq 4\right\}$.

Exercise 1.4. *Calculate the following triple integrals:*

1)

$$I_1 = \iiint_{D_1} \cos(x + y - z)\, dx\, dy\, dz$$

with

$$D_1 = \left[0, \frac{\pi}{2}\right] \times \left[0, \frac{\pi}{2}\right] \times \left[0, \frac{\pi}{2}\right].$$

2)

$$I_2 = \iiint_{D_1} \frac{1}{\sqrt{x + y + z}}\, dx\, dy\, dz$$

with

$$D_2 = [0, 1] \times [1, 2] \times [1, 2].$$

3)

$$I_3 = \iiint_{D_2} \frac{1}{(1 + x + y + z)^3}\, dx\, dy\, dz$$

with

$$D_3 = \left\{(x, y, z) \in \mathbb{R}^3 / x, y, z \geq 0, \quad x + y + z \leq 1\right\}.$$

Chapter 2

Improper Integrals

In this chapter, we extend the concept of integrals to functions defined on:

1) Unbounded intervals.

2) Intervals where the function has infinite values at one or more boundaries.

2.1 Integrals of Functions Defined on Unbounded Intervals

Definition 2.1.1. *Let f be a continuous function on $[a, b[\subset \mathbb{R}$. Then:*

$$\int_a^b f(x)\,dx = \lim_{\substack{t \to b \\ <}} \int_a^t f(x)\,dx.$$

This is called a improper integral. If F is an antiderivative of f, we can write:

$$\int_a^b f(x)\,dx = [F(x)]_a^b = \lim_{\substack{x \to b \\ <}} F(x) - F(a).$$

2.1.1 Convergence and Divergence

Let f be a continuous function on the interval $[a, b[\subset \mathbb{R}$.

1) The integral $\int_a^b f(x)\,dx$ converges if the limit exists and is finite.

2) The integral $\int_a^b f(x)\,dx$ diverges if the limit does not exist or is infinite.

2.2 Integrals of Functions with Infinite Values at Boundaries

Definition 2.2.1. *Let f be a continuous function on the interval $[a, +\infty[$. Hence, The integral is defined as:*

$$\int_a^{+\infty} f(x)\,dx = \lim_{t \to +\infty} \int_a^t f(x)\,dx.$$

This is called also a improper integral. We can write:

$$\int_a^{+\infty} f(x)\,dx = [F(x)]_a^{+\infty} = \lim_{x\to+\infty} F(x) - F(a).$$

Similarly, integrals on other intervals, such as $]a,b]$, $]a,b[$, $]-\infty,a]$ or $]-\infty,+\infty[$, can be defined in an analogous manner.

2.2.1 Calcul pratique des intégrales généralisées

The aim of this chapter is to define the nature of the improper integral. One of the methods is to determine whether it exists or not, as follows.

Using an Antiderivative

Let f be a continuous function on the interval $]a,b]$, and let F be an antiderivative of f. Then:

$$\int_a^b f(x)\,dx = [F(x)]_a^b = F(b) - \lim_{\substack{x\to a\\ >}} F(x).$$

Example 2.2.1. *Compute the following improper integrals*

1) $\boldsymbol{A} = \int_0^5 \frac{1}{\sqrt{x}}\,dx$, *2)* $\boldsymbol{B} = \int_0^1 \frac{1}{x}\,dx$, *3)* $\boldsymbol{C} = \int_0^{+\infty} \frac{1}{1+x^2}\,dx$.

Solution. *We will use the previous chapter to find the antiderivatives:*

1) $\boldsymbol{A} = \int_0^5 \frac{1}{\sqrt{x}}\,dx$*:*

Using the formula for powers of x; $\int x^\alpha\,dx = \frac{x^{\alpha+1}}{\alpha+1}$; $\alpha \neq -1$.

$$\begin{aligned} A &= \int_0^5 \frac{1}{\sqrt{x}}\,dx = \int_0^5 x^{-\frac{1}{2}}\,dx = \left[2\sqrt{x}\right]_0^5 \\ &= 2\sqrt{5} - 2\lim_{\substack{x\to 0\\ >}} \sqrt{x} = 2\sqrt{5}. \end{aligned}$$

Therefore, the integral $\int_0^5 \frac{1}{\sqrt{x}}\,dx$ *converges.*

2) $\boldsymbol{B} = \int_0^1 \frac{1}{x}\,dx$*:*

$$B = \int_0^1 \frac{1}{x}\,dx = \left[\ln x\right]_0^1 = \ln 1 - \lim_{\substack{x\to 0\\ >}} \ln x = +\infty.$$

Thus, the integral $\int_0^1 \frac{1}{x}\,dx$ *diverges.*

3) $\boldsymbol{C} = \displaystyle\int_0^{+\infty} \frac{1}{1+x^2}\,dx$*:*

$$C = \int_0^{+\infty} \frac{1}{1+x^2}\,dx = \Big[\arctan x\Big]_0^{+\infty} = \lim_{x\to+\infty} \arctan x - \arctan 0 = \frac{\pi}{2}.$$

Hence, the integral $\displaystyle\int_0^{+\infty} \frac{1}{1+x^2}\,dx$ *converges.*

Intégration par parties

Let u and v are continuous functions, then:

$$\int_a^b uv'\,dx = [uv]_a^b - \int_a^b u'v\,dx.$$

Example 2.2.2. *Compute the following improper integrals:*

1) $\boldsymbol{A} = \displaystyle\int_0^1 \ln x\,dx$*,* *2)* $\boldsymbol{B} = \displaystyle\int_0^{+\infty} xe^{-x}\,dx$*.*

Solution. *We will use the method of integration by parts:*

1) $\boldsymbol{A} = \displaystyle\int_0^1 \ln x\,dx$*:*

$u(x) = \ln x$ *d'où* $u'(x) = \dfrac{1}{x}$ *et* $v'(x) = 1$ *d'où* $v(x) = x$*. Alors:*

$$\begin{aligned} A &= [x\ln x]_0^1 - \int_0^1 \frac{x}{x}\,dx \\ &= [x\ln x - x]_0^1 = -1 - \lim_{\substack{x\to 0 \\ >}} (x\ln x - x) \\ &= -1. \end{aligned}$$

Therefore, the integral $\displaystyle\int_0^1 \ln x\,dx$ *converges.*

2) $\boldsymbol{B} = \displaystyle\int_0^{+\infty} xe^{-x}\,dx$*:*

$u(x) = x$ *d'où* $u'(x) = 1$ *et* $v'(x) = e^{-x}$ *d'où* $v(x) = -e^{-x}$*. Alors:*

$$\begin{aligned} B &= \left[-xe^{-x}\right]_0^{+\infty} + \int_0^{+\infty} e^{-x}\,dx \\ &= \left[-xe^{-x} - e^{-x}\right]_0^{+\infty} = -\lim_{x\to+\infty} (xe^{-x} + e^{-x}) + 1 \\ &= 1. \end{aligned}$$

Thus, the integral $\displaystyle\int_0^{+\infty} xe^{-x}\,dx$ *converges.*

Change of variables

Let f be a continuous function on $]a, b]$, and u be a differentiable function. If we substitute $x = u(t)$; then we obtain:

$$\int_a^b f(x)\,dx = \int_{u^{-1}(a)}^{u^{-1}(b)} f\left[u(t)\right] u'(t)\,dt.$$

Example 2.2.3. *Evaluate the following improper integrals:*

$$\boldsymbol{A} = \int_3^{+\infty} \frac{1}{x\sqrt{x+1}}\,dx.$$

Solution. *We will utilize the method of change of variables:*

Let $\sqrt{x+1} = t \Rightarrow x = t^2 - 1$, *so* $dx = 2tdt$ *with* $t \in [2, +\infty[$. *Then:*

$$\int_3^{+\infty} \frac{1}{x\sqrt{x+1}}\,dx = 2\int_2^{+\infty} \frac{1}{t^2-1}\,dt.$$

We can decompose $f(t)$ *into two simple terms as follows:*
$\dfrac{2}{t^2-1} = \dfrac{2}{(t+1)(t-1)} = \dfrac{1}{t-1} - \dfrac{1}{t+1}$. *Thus:*

$$\begin{aligned}
A &= 2\int_2^{+\infty} \frac{1}{t^2-1}\,dt = \int_2^{+\infty} \left(\frac{1}{t-1} - \frac{1}{t+1}\right) dt \\
&= \left[\ln(t-1) - \ln(t+1)\right]_2^{+\infty} = \left[\ln\left(\frac{t-1}{t+1}\right)\right]_2^{+\infty} \\
&= \lim_{t\to+\infty} \ln\left(\frac{t-1}{t+1}\right) - \ln\frac{1}{3} = \ln 3.
\end{aligned}$$

Hence, the integral $\displaystyle\int_3^{+\infty} \frac{1}{x\sqrt{x+1}}\,dx$ *converges.*

2.2.2 Integration of Positive Functions

If f is negative on I, then $-f$ is positive on I, and the convergence of the integral $\displaystyle\int_a^b f(x)\,dx$ reduces to that of the integral $\displaystyle\int_a^b -f(x)\,dx$.
The following study will be limited exclusively to positive functions.

2.2.3 Riemann's integral

Let $f : [1, +\infty[\to \mathbb{R}_*^+$

$$x \mapsto f(x) = \frac{1}{x^\alpha} \text{ be a continuous function; } \alpha \in \mathbb{R}.$$

First case. If $\alpha = 1$:

$$\begin{aligned} I_1 &= \int_1^{+\infty} \frac{1}{x}\, dx = [\ln x]_1^{+\infty} \\ &= \lim_{x\to+\infty} \ln x - \ln 1 \\ &= +\infty. \end{aligned}$$

Thus, the integral $\displaystyle\int_1^{+\infty} \frac{1}{x}\, dx$ diverges.

Second case. If $\alpha \neq 1$:

$$\begin{aligned} I_\alpha &= \int_1^{+\infty} \frac{1}{x^\alpha}\, dx = \int_1^{+\infty} x^{-\alpha}\, dx = \left[\frac{x^{1-\alpha}}{1-\alpha}\right]_1^{+\infty} \\ &= \lim_{x\to+\infty} \frac{x^{1-\alpha}}{1-\alpha} - \frac{1}{1-\alpha} \\ &= \begin{cases} \dfrac{1}{\alpha-1} & \text{si} \quad \alpha > 1, \\ +\infty & \text{si} \quad \alpha < 1. \end{cases} \end{aligned}$$

Therefore, the integral $\displaystyle\int_1^{+\infty} \frac{1}{x^\alpha}\, dx \begin{cases} \text{converges} & \text{if} \quad \alpha > 1, \\ \text{diverges} & \text{if} \quad \alpha \leq 1. \end{cases}$

In the general case, there are two types of Riemann integrals; $a > 0$:

First type: $\displaystyle\int_a^{+\infty} \frac{1}{x^\alpha}\, dx$ converges if $\alpha > 1$.

Second type: $\displaystyle\int_0^{a} \frac{1}{x^\alpha}\, dx$ converges if $\alpha < 1$.

Test of Comparaison

Theorem 2.2.1. *Let f and g be two continuous functions on the interval $[a, +\infty[$, such that:*

$$0 \leq f(x) \leq g(x), \quad \forall x \in [a, +\infty[.$$

Then:

1) *If* $\displaystyle\int_a^{+\infty} g(x)\, dx$ *converges* $\Rightarrow \displaystyle\int_a^{+\infty} f(x)\, dx$ *converges.*

2) *If* $\displaystyle\int_a^{+\infty} f(x)\, dx$ *diverges* $\Rightarrow \displaystyle\int_a^{+\infty} g(x)\, dx$ *diverges.*

Example 2.2.4. *determine whether the following integrals converge or diverge by using the test of Comparaison:*

1) $\boldsymbol{A} = \displaystyle\int_1^{+\infty} e^{-x^2}\,dx$, *2)* $\boldsymbol{B} = \displaystyle\int_2^{+\infty} \frac{2+\sin x}{x^3}\,dx$, *3)* $\boldsymbol{C} = \displaystyle\int_0^{1} \frac{e^x}{x}\,dx$.

Solution. *1)* $\boldsymbol{A} = \displaystyle\int_1^{+\infty} e^{-x^2}\,dx$:

We have: $x \geq 1 \Rightarrow x^2 \geq x \Rightarrow -x^2 \leq -x$. *Then:* $e^{-x^2} \leq e^{-x} \quad \forall x \geq 1$.

Since lhe integral $\displaystyle\int_1^{+\infty} e^{-x}\,dx$ *converges* $\left(\displaystyle\int_1^{+\infty} e^{-x}\,dx = \left[-e^{-x}\right]_1^{+\infty} = e^{-1}\right)$

thus, our integral $\displaystyle\int_1^{+\infty} e^{-x^2}\,dx$ *converges also by the test of Comparaison.*

2) $\boldsymbol{B} = \displaystyle\int_2^{+\infty} \frac{2+\sin x}{x^3}\,dx$:

We know that: $-1 \leq \sin x \leq 1$. *So:* $\dfrac{2+\sin x}{x^3} \leq \dfrac{3}{x^3} \quad \forall x \geq 2$.

Since $\displaystyle\int_2^{+\infty} \frac{1}{x^3}\,dx$ *is convergent (Riemann's integral* 1^{st} *type* $\alpha = 3 > 1$*),*

then the integral $\displaystyle\int_2^{+\infty} \frac{2+\sin x}{x^3}\,dx$ *converges by the test of Comparaison.*

3) $\boldsymbol{C} = \displaystyle\int_0^{1} \frac{e^x}{x}\,dx$:

We have: $1 < e^x \leq e$. *Then:* $\dfrac{1}{x} \leq \dfrac{e^x}{x} \quad \forall x > 0$.

Since the integral $\displaystyle\int_0^{1} \frac{1}{x}\,dx$ *is divergent (Riemann's integral* 2^{nd} *type* $\alpha = 1$*),*

therefore the integral $\displaystyle\int_0^{1} \frac{e^x}{x}\,dx$ *diverges by the test of Comparaison.*

Limit comparison test

Theorem 2.2.2. *Let f and g be two continuous functions on the interval* $]a, b]$. *We say that:*

$$f \underset{a}{\sim} g \Leftrightarrow \lim_{\substack{x \to a \\ >}} \frac{f(x)}{g(x)} = 1.$$

Thus, $\displaystyle\int_a^b f(x)\,dx$ *and* $\displaystyle\int_a^b g(x)\,dx$ *are either converge or diverge:*

1) *If* $\displaystyle\int_a^b g(x)\,dx$ *is convergent* $\Rightarrow \displaystyle\int_a^b f(x)\,dx$ *converges .*

2) *If* $\displaystyle\int_a^b g(x)\,dx$ *is divergent* $\Rightarrow \displaystyle\int_a^b f(x)\,dx$ *diverges.*

Example 2.2.5. *determine whether the following integrals converge or diverge by using the limit comparison test:*

1) $\boldsymbol{A} = \displaystyle\int_1^{+\infty} \frac{x+3}{x^3+x+1}\,dx,$ *2)* $\boldsymbol{B} = \displaystyle\int_0^{2} \frac{\sin x}{x\sqrt{x}}\,dx.$

Solution. *1)* $\boldsymbol{A} = \displaystyle\int_1^{+\infty} \frac{x+3}{x^3+x+1}\,dx$*:*

$$f(x) = \frac{x+3}{x^3+x+1} \underset{+\infty}{\sim} g(x) = \frac{x}{x^3} = \frac{1}{x^2} \quad \text{with} \lim_{x\longrightarrow+\infty} \frac{f(x)}{g(x)} = 1.$$

$\displaystyle\int_1^{+\infty} \frac{1}{x^2}\,dx$ *is convergent (It is a Riemann's integral* 1^{st} *type* $\alpha = 2 > 1$*),* *we deduce that* $\displaystyle\int_1^{+\infty} \frac{x+3}{x^3+x+1}\,dx$ *converges by the limit comparison test.*

2) $\boldsymbol{B} = \displaystyle\int_0^{2} \frac{\sin x}{x\sqrt{x}}\,dx$*:*

- *The function* $x \longmapsto \dfrac{\sin x}{x\sqrt{x}}$ *is continuous and positive on* $]0,2]$*.*

We have: $\sin x \underset{0}{\sim} x$ *when* $x \longrightarrow 0$*, hence:*

$$f(x) = \frac{\sin x}{x\sqrt{x}} \underset{0}{\sim} g(x) = \frac{x}{x\sqrt{x}} = \frac{1}{\sqrt{x}} \quad \text{with} \lim_{x\longrightarrow 0} \frac{f(x)}{g(x)} = 1.$$

Since $\displaystyle\int_0^{2} \frac{1}{\sqrt{x}}\,dx$ *converges (Riemann's integral* 2^{nd} *type* $\alpha = 1/2 < 1$*),* *then the integral* $\displaystyle\int_0^{2} \frac{\sin x}{x\sqrt{x}}\,dx$ *converges also by the limit comparison test.*

2.2.4 Integration of Functions with Arbitrary Signs

Definition 2.2.2. *Let* f *be a continuous function on* $]a,b]$*.*

If $\displaystyle\int_a^b |f(x)|\,dx$ *converges* $\Rightarrow \displaystyle\int_a^b f(x)\,dx$ *is said to be absolutely convergent.*

$$\Rightarrow \int_a^b f(x)\,dx \text{ is convergent.}$$

Example 2.2.6. *Investigate the convergence of this integral:*

$$\int_1^{+\infty} \frac{\cos x}{x\sqrt{x}}\,dx.$$

Solution. *We will study the integral with the absolute value:*

We know that: $-1 \le \cos x \le 1$*, so:* $|\cos x| \le 1 \quad \forall x \ge 1$*. Then:*

$$\left|\frac{\cos x}{x\sqrt{x}}\right| = \frac{|\cos x|}{x\sqrt{x}} \le \frac{1}{x\sqrt{x}} = \frac{1}{x^{3/2}}, \quad \forall x \ge 1.$$

Since $\int_1^{+\infty} \frac{1}{x^{3/2}}\, dx$ *is convergent (Riemann's integral* 1^{st} *type* $\alpha = 3/2 > 1$*),* *thus the integral* $\int_1^{+\infty} \left|\frac{\cos x}{x\sqrt{x}}\right| dx$ *converges by the test of Comparaison.*

$$\Rightarrow \int_1^{+\infty} \frac{\cos x}{x\sqrt{x}}\, dx \text{ is absolutely convergent.}$$

$$\Rightarrow \int_1^{+\infty} \frac{\cos x}{x\sqrt{x}}\, dx \text{ converges.}$$

Abel's test

Theorem 2.2.3. *Let f and g be two continuous functions on the interval $[a, b[$. Then*

1) *If f be monotonic with* $\lim\limits_{\substack{x \to b \\ <}} f(x) = 0.$

and

2) *There exists $M > 0$ such that for all $t \in [a, b[$:* $\left|\int_a^t g(x)\, dx\right| \leq M.$

Therefore $\int_a^b f(x)g(x)\, dx$ *converges.*

Example 2.2.7. *Study the convergence of this integral by using the Abel's test:*

$$\int_1^{+\infty} \frac{\sin x}{x}\, dx.$$

Solution. *Indeed: The function* $f(x) = \frac{1}{x}$ *is continuous and decreasing on* $[1, +\infty[$ *and* $\lim\limits_{x \to +\infty} f(x) = 0.$

On the other hand, for all $x \in [1, +\infty[$:

$$\left|\int_1^t \sin x\, dx\right| = \left|\cos t - \cos 1\right| \leq 2.$$

Thus, according to Abel's test, the improper integral

$$\int_1^{+\infty} \frac{\sin x}{x}\, dx$$

is convergent.

2.3 Supplementary exercises

Exercise 2.1. *Compute the following improper integrals:*

1) $\displaystyle\int_1^{+\infty} \frac{1}{x\sqrt{\ln x}}\,dx$, ***2)*** $\displaystyle\int_0^{+\infty} x^3 e^{-x^2}\,dx$, ***3)*** $\displaystyle\int_0^{441} \frac{\pi\sin(\pi\sqrt{x})}{\sqrt{x}}\,dx$,

4) $\displaystyle\int_1^{+\infty} \frac{1}{x(x^2+1)}\,dx$, ***5)*** $\displaystyle\int_0^{+\infty} \frac{1}{\sqrt{e^x+1}}\,dx$, ***6)*** $\displaystyle\int_1^{2} \frac{1}{x\sqrt{x^2-1}}\,dx$.

Exercise 2.2. *Determine if the following integrals is convergent or divergent and if it's convergent find its value:*

1) $\displaystyle\int_0^{+\infty} \frac{x^2}{x^{17/5}+1}\,dx$, ***2)*** $\displaystyle\int_1^{+\infty} \frac{e^{\cos x}}{x}\,dx$, ***3)*** $\displaystyle\int_1^{e} \frac{1}{\ln x}\,dx$,

4) $\displaystyle\int_1^{+\infty} \frac{1}{2^{\sqrt{x}}}\,dx$, ***5)*** $\displaystyle\int_2^{+\infty} \frac{\sin x}{x^2\sqrt{x}}\,dx$, ***6)*** $\displaystyle\int_0^{1} \frac{\cos x}{\sqrt{x}}\,dx$.

Exercise 2.3. *Investigate if the following integrals is convergent or divergent and if it's convergent find its value:*

1) $\displaystyle\int_1^{+\infty} \frac{x}{x^4+1}\,dx$, ***2)*** $\displaystyle\int_2^{+\infty} \frac{\ln x}{x\sqrt{x}}\,dx$, ***3)*** $\displaystyle\int_0^{1} \frac{\ln(1+x)}{x^2}\,dx$,

4) $\displaystyle\int_1^{+\infty} \frac{1}{\sqrt{x}e^{\sqrt{x}}}\,dx$, ***5)*** $\displaystyle\int_1^{+\infty} \frac{\arctan x}{x^3}\,dx$, ***6)*** $\displaystyle\int_{-\infty}^{0} \frac{e^{1/x}}{x^2}\,dx$.

Chapter 3

Differential Equations

Differential equations are of great utility in various engineering fields. In chemical engineering, they are essential for modeling reaction kinetics and process dynamics, especially in scenarios such as mixing problems involving multiple tanks and substances, which are critical for reactor design and process optimization. In civil engineering, differential equation models are indispensable for assessing the safety and longevity of structures under various loading conditions, such as in the analysis of the seismic resistance of multi-story buildings

3.1 Review of Ordinary Differential Equations

Definition 3.1.1. *A **differential equation** is an equation that establishes a relationship between the independent variable x, the unknown function $y = y(x)$, and some of its derivatives. It is expressed as:*

$$F(x, y, y', \cdots, y^{(n)}) = g(x).$$

Remark 3.1.1. *If the unknown function in the differential equation depends on a single variable, the equation is called an ordinary differential equation (ODE).*

Definition 3.1.2. *The **order** of a differential equation is defined as the order of the highest derivative appearing in the equation.*

Definition 3.1.3. *The integral or **solution** of a differential equation is any function that satisfies the equation.*

3.1.1 First-Order Linear Differential Equations

Definition 3.1.4. *A **first-order linear differential equation** can be written in the form:*

$$a(x)y' + b(x)y = c(x),$$

where a, b and c are continuous functions on a given interval with $a \neq 0$.

Homogeneous Equation (Without Second Member)

Consider the following differential equation:

$$a(x)y' + b(x)y = 0. \tag{3.1}$$

The solutions of (3.1) are $y(x) = 0$ and nonzero solutions satisfying $\dfrac{y'(x)}{y(x)} = -\dfrac{b(x)}{a(x)}$ with $a(x) \neq 0$.
Since a and b are continuous functions, it follows that $\ln|y(x)| = R(x) + \lambda_1$ where R is an antiderivative of $-\dfrac{b(x)}{a(x)}$.
Thus, the solutions to (3.1) are given by:

$$y(x) = \lambda e^{R(x)}.$$

If a condition $x = x_0$ is fixed, the solution of (3.1) is unique:

$$y(x_0) = \lambda e^{R(x_0)}$$

In particular, if y vanishes at a point, then y is identically zero.

Non-Homogeneous Equation (With Second Member)

Consider the following differential equation:

$$a(x)y' + b(x)y = c(x). \tag{3.2}$$

The resolution method is divided into two steps:

1) Solve the homogeneous equation. The general solution is:

$$y_g(x) = \lambda \exp\left(\int \frac{-b(x)}{a(x)}\, dx \right), a \neq 0 \quad \text{et} \quad \lambda \in \mathbb{R}.$$

2) Find a particular solution by considering λ as a function of x (hence the name **method of variation of parameters**):

$$y_p(x) = \lambda(x) \exp\left(\int \frac{-b(x)}{a(x)}\, dx \right)$$

with

$$\lambda(x) = \int \frac{c(x)}{a(x)} \exp\left(\int \frac{b(x)}{a(x)}\, dx \right) dx.$$

Thus, the solution to (3.2) is given by:

$$y(x) = y_g(x) + y_p(x).$$

Example 3.1.1. *Solve the following differential equation:*

$$y' + (\tan x)y = \cos x; \quad \forall x \in \left[0, \frac{\pi}{2}\right[$$

Solution. *Consider the equation:* $y' + (\tan x)y = \cos x$*;* $x \in [0, \pi/2[$.

- *The functions* $a(x) = 1$*,* $b(x) = \tan x$ *and* $c(x) = \cos x$ *are continuous on* $[0, \pi/2[$*. Since* $a(x) \neq 0$*, then we have*

$$f(x) = \frac{b(x)}{a(x)} = \frac{\tan x}{1} = \tan x = \frac{\sin x}{\cos x}.$$

The general solution of the homogeneous equation is:

$$y_g(x) = \lambda \exp\left(-\int f(x)\, dx\right) = \lambda \exp\left(\int \frac{-\sin x}{\cos x}\, dx\right) = \lambda e^{\ln(\cos x)} = \lambda \cos x.$$

Thus, $y_g(x) = \lambda \cos x$*;* $\lambda \in \mathbb{R}$*.*
Finding a particular solution, we write:

$$y_p(x) = \lambda(x) \cos x.$$

$$\begin{aligned}
\lambda(x) &= \int \frac{c(x)}{a(x)} \exp\left(\int f(x)\, dx\right)\, dx = \int \cos x \cdot \exp\left(\int \frac{\sin x}{\cos x}\, dx\right)\, dx \\
&= \int \cos x \cdot e^{-\ln(\cos x)}\, dx = \int \cos x \cdot \exp\left[\ln\left(\frac{1}{\cos x}\right)\right]\, dx \\
&= \int \frac{\cos x}{\cos x}\, dx = x.
\end{aligned}$$

$\Longrightarrow y_p(x) = x \cos x$*. Finally:*
$\Longrightarrow$ *the solutions are:* $y(x) = y_g(x) + y_p(x) = (x + \lambda) \cos x; \quad \lambda \in \mathbb{R}$*.*

3.1.2 Second-Order Linear Differential Equations with Constant Coefficients

Homogeneous Differential Equations with Constant Coefficients

Definition 3.1.5. *A* ***second-order linear homogeneous differential equation with constant coefficients*** *is any equation of the form:*

$$ay'' + by' + cy = 0 \tag{3.3}$$

where a,b and c are real constants, and $a \neq 0$*.*

We search for a solution of the form:

$$y(x) = e^{rx}.$$

The development of the differential equation with this function y leads to the following characteristic equation:

$$ar^2 + br + c = 0.$$

Depending on the sign of the discriminant ($\Delta = b^2 - 4ac$),then the solutions of (3.3) are:

- $\Delta > 0$: Distinct Real Roots r_1 and r_2

$$y(x) = c_1 e^{r_1 x} + c_2 e^{r_2 x}.$$

- $\Delta < 0$: Complex Roots $r_1 = \alpha + i\beta$ and $r_2 = \alpha - i\beta$

$$y(x) = c_1 e^{r_1 x} + c_2 e^{r_2 x} = e^{\alpha x}\left[A\cos(\beta x) + B\sin(\beta x)\right].$$

- $\Delta = 0$: Repeated Root r

$$y(x) = e^{rx}(c_1 x + c_2).$$

Here, c_1 and c_2 are arbitrary constants determined by initial or boundary conditions.

Differential Equations with Constant Coefficients and a Non-Homogeneous Term

Here we aim to determine the solution of differential equations of the form:

$$ay'' + by' + cy = f(x). \tag{3.4}$$

where a,b and c are real constants and $a \neq 0$.

The resolution method is divided into two steps:

1) We first solve the homogeneous equation: we obtain the general solution, denoted by y_g.

2) We then look for a particular solution of the non-homogeneous equation, denoted by y_p.

The solution of the equation (3.4) is then:

$$y(x) = y_g(x) + y_p(x).$$

Below, we present the form of the particular solutions for several specific cases of the function $f(x)$. In the entire table P_n, Q_n and Z_n denote polynomials, all of degree n. Additionally k, α and β are real numbers, and (E_c) is the characteristic equation associated with (3.4).

Form of the non-homogeneous term $f(x)$	Form of the particular solution y_p
$P_n(x)$ with $c \neq 0$	$Q_n(x)$
$P_n(x)$ with $c = 0$ et $b \neq 0$	$xQ_n(x)$
$P_n(x)$ with $c = b = 0$	$x^2Q_n(x)$
$P_n(x)e^{kx}$ with k not a root of (E_c)	$Q_n(x)e^{kx}$
$P_n(x)e^{kx}$ with k a simple root of (E_c)	$xQ_n(x)e^{kx}$
$P_n(x)e^{kx}$ with k a double root of (E_c)	$x^2Q_n(x)e^{kx}$
$P_n(x)e^{\alpha x}\cos(\beta x)$, $\alpha + i\beta$ not a root of (E_c)	$e^{\alpha x}\left[Q_n(x)\cos(\beta x) + Z_n(x)\sin(\beta x)\right]$
$P_n(x)e^{\alpha x}\sin(\beta x)$, $\alpha + i\beta$ not a root of (E_c)	$e^{\alpha x}\left[Q_n(x)\cos(\beta x) + Z_n(x)\sin(\beta x)\right]$
$P_n(x)e^{\alpha x}\cos(\beta x)$, $\alpha + i\beta$ a double root	$xe^{\alpha x}\left[Q_n(x)\cos(\beta x) + Z_n(x)\sin(\beta x)\right]$
$P_n(x)e^{\alpha x}\sin(\beta x)$, $\alpha + i\beta$ a double root	$xe^{\alpha x}\left[Q_n(x)\cos(\beta x) + Z_n(x)\sin(\beta x)\right]$

Example 3.1.2. *Solve the following differential equation:*

$$y'' - 5y' + 6y = e^{2x}.$$

Solution. *First, calculate the general solution of the homogeneous equation:* $y'' - 5y' + 6y = 0$.
The characteristic equation is $r^2 - 5r + 6 = (r-2)(r-3) = 0$.
$\Longrightarrow r_1 = 2$ *and* $r_2 = 3$. *The general solution of the previous homogeneous equation is:*

$$y_g(x) = c_1e^{2x} + c_2e^{3x};\ c_1, c_1 \in \mathbb{R}.$$

The right-hand side is of the form Ae^{kx}*, so we seek a particular solution of the form* axe^{kx} *because* $r_1 = k = 2$ *and* $r_2 \neq k$. *Therefore:* $y_p(x) = axe^{2x}$.

$$\begin{aligned} y_p \ \textit{solution} \quad &\Longleftrightarrow\quad a(4x+4)e^{2x} - 5a(1+2x)e^{2x} + 6axe^{2x} = e^{2x} \\ &\Longleftrightarrow\quad a = -1. \end{aligned}$$

$\Longrightarrow y_p(x) = -xe^{2x}$. *We deduce that the solution are:*
$y(x) = y_g(x) + y_p(x) = c_1e^{2x} + c_2e^{3x} - xe^{2x};\quad c_1, c_1 \in \mathbb{R}.$

3.2 Partial differential equations

In the first section, we dealt with solving differential equations where the unknown function depends on a single variable, which we referred to as an ordinary differential equation (O.D.E). In this section, we will extend our study to cases where the unknown function depends on multiple variables. In such cases, we are tasked with solving partial differential equations (P.D.E). We will focus on the study and solution of first- and second-order equations.

3.2.1 Generalities

Let be u a function depending on n independent variables $x_1, x_2, \cdots, x_n$.

Definition 3.2.1. *A P.D.E. (partial differential equation) is any relation involving an unknown function u, the variables $x_1, x_2, \cdots, x_n$ and its partial derivatives. The general form is given by:*

$$F\left(x_1, \cdots, x_n, u, \frac{\partial u}{\partial x_1}, \cdots, \frac{\partial^n u}{\partial x_1^n}, \cdots, \frac{\partial^n u}{\partial x_1^{m_1} \cdots \partial x_k^{m_k}}\right) = g(x_1, \cdots, x_n, u)$$

where $1 \leq k \leq n$ and $m_1 + m_2 + \cdots + m_k = n$.

Definition 3.2.2. *The order of a partial differential equation (P.D.E.) is the order of the highest partial derivative it contains.*

Definition 3.2.3. *A P.D.E. is said to be linear if it is linear with respect to u and all its partial derivatives.*

Definition 3.2.4. *A P.D.E. is said to be homogeneous if $g(x_1, \cdots, x_n, u) = 0$.*

3.2.2 First-Order P.D.E.

Definition 3.2.5. *A first-order partial differential equation (P.D.E.) is any equation of the form:*

$$\sum_{i=1}^{n} a_i(x_1, \cdots, x_n, u)\frac{\partial u}{\partial x_i} = g(x_1, \cdots, x_n, u). \tag{3.5}$$

$\{a_i\}_{i=1,\ldots,n}$: referred to as the coefficients, and g may explicitly depend on the function u.

In this study, we will focus on first-order P.D.E.s with 2 or 3 independent variables.

Method of Characteristics

Consider the following problem:

$$F\left(x, u(x), \nabla u(x)\right) = 0$$

where $F \in \mathcal{C}^1\left(\Omega \times \mathbb{R} \times \mathbb{R}^n\right)$, $x \in \Omega \subset \mathbb{R}$ (*i.e.* $x = (x_1, x_2, x_3)$) and $\nabla u(x)$ is the gradient of u, defined as:

$$\nabla u = \left(\frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}, \frac{\partial u}{\partial x_3}\right).$$

First, we consider the case where the P.D.E. depends on two independent variables. Let us examine the following P.D.E.:

$$a(x, y, u)\frac{\partial u}{\partial x} + b(x, y, u)\frac{\partial u}{\partial y} \ = \ c(x, y, u). \tag{3.6}$$

At every point in the space (x, y, u), there exists a direction whose direction cosines are proportional to a, b and c. This field of directions defines a family of curves, such that the tangent to each curve aligns with the direction of the field at the point of contact. These curves are determined by solving the following system of ordinary differential equations (O.D.E.s):

$$\frac{dx}{a(x, y, u)} = \frac{dy}{b(x, y, u)} \ = \ \frac{du}{c(x, y, u)}. \tag{3.7}$$

Let ds denote the common value of these ratios; equation (3.7) then becomes:

$$\begin{cases} \dfrac{dx}{ds} = a(x, y, u) \\ \dfrac{dy}{ds} = b(x, y, u) \\ \dfrac{du}{ds} = c(x, y, u). \end{cases}$$

Thus, the integral surface formed by the characteristics of equation (3.7) is the desired solution, and we write:

$$\Phi(c_1, c_2) = 0 \quad ou \quad c_1 = \Phi(c_2). \tag{3.8}$$

Hence the name of the method.

Example 3.2.1. *Solve the following P.D.E.:*

$$xy^2\frac{\partial u}{\partial x} + x^2 y\frac{\partial u}{\partial y} \ = \ (x^2 + y^2)u. \tag{3.9}$$

Solution. *We associate with equation (3.8) the following system:*

$$\frac{dx}{xy^2} = \frac{dy}{x^2y} = \frac{du}{(x^2+y^2)u}.$$

On one hand, we have:

$$\frac{dx}{xy^2} = \frac{dy}{x^2y} \Rightarrow ydy = xdx \Rightarrow y^2 - x^2 = c_2.$$

On the other hand

$$\begin{aligned}
x^2\frac{dy}{x^2y} + y^2\frac{dx}{xy^2} &= x^2\frac{du}{(x^2+y^2)u} + y^2\frac{du}{(x^2+y^2)u} \\
\frac{xdy}{xy} + \frac{ydx}{xy} &= \frac{du}{u} \\
\frac{d(xy)}{xy} &= \frac{du}{u} \Rightarrow \frac{u}{xy} = c_1.
\end{aligned}$$

Thus, the solution of (3.9) is:

$$\frac{u}{xy} = \Phi(y^2 - x^2) \ \Rightarrow \ u = xy\Phi(y^2 - x^2). \tag{3.10}$$

3.2.3 Second-Order P.D.E.

Definition 3.2.6. *A second-order partial differential equation (P.D.E.) is any equation of the form:*

$$F\left(x_1, \cdots, x_n, \frac{\partial u}{\partial x_1}, \cdots, \frac{\partial u}{\partial x_n}, \frac{\partial^2 u}{\partial x_1^2}, \cdots, \frac{\partial^2 u}{\partial x_n^2}, \frac{\partial^2 u}{\partial x_1 \partial x_2}, \cdots, \frac{\partial^2 u}{\partial x_{n-1}\partial x_n}\right) = 0,$$

or

$$\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}(x_1, \cdots, x_n, u)\frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i(x_1, \cdots, x_n, u)\frac{\partial u}{\partial x_i} = h(x_1, \cdots, x_n, u).$$

$\{a_{ij}\}_{i,j=1,\ldots,n}$ and $\{b_i\}_{i=1,\ldots,n}$: are called coefficients, and h may explicitly depend on the function u.

We will limit our study to second-order linear and quasilinear equations with two independent variables, that is, equations of the form:

$$\begin{aligned}
a(x,y)\frac{\partial^2 u}{\partial x^2} &+ b(x,y)\frac{\partial^2 u}{\partial x \partial y} + c(x,y)\frac{\partial^2 u}{\partial y^2} + p(x,y)\frac{\partial u}{\partial x} \\
&+ q(x,y)\frac{\partial u}{\partial y} + g(x,y)u = f(x,y),
\end{aligned} \tag{3.11}$$

or

$$a\left(x,y,u,\frac{\partial u}{\partial x},\frac{\partial u}{\partial y}\right)\frac{\partial^2 u}{\partial x^2} + b\left(x,y,u,\frac{\partial u}{\partial x},\frac{\partial u}{\partial y}\right)\frac{\partial^2 u}{\partial x \partial y} + c\left(x,y,u,\frac{\partial u}{\partial x},\frac{\partial u}{\partial y}\right)\frac{\partial^2 u}{\partial y^2} = R\left(x,y,u,\frac{\partial u}{\partial x},\frac{\partial u}{\partial y}\right). \tag{3.12}$$

Classification of Second-Order P.D.E.s

The type of the partial differential equations (3.11) and (3.12) depends on their discriminant $\Delta = b^2 - 4ac$.

Definition 3.2.7. *The equation is of the hyperbolic type if and only if* $\Delta > 0$.

Example 3.2.2. *Consider the following partial differential equation:*

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + 4\frac{\partial^2 u}{\partial x \partial y} = 0$$

defined on $\Omega = \mathbb{R}^2$.
$\Delta = b^2 - 4ac = 12 > 0$ *far all* x, y *of* Ω. *So, this is an equation of the hyperbolic type.*

Definition 3.2.8. *The equation is of the parabolic type if and only if* $\Delta = 0$.

Example 3.2.3. *Consider this partial differential equation:*

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + 2\frac{\partial^2 u}{\partial x \partial y} = 0$$

defined on $\Omega = \mathbb{R}^2$.
$\Delta = b^2 - 4ac = 0$ *far all* x, y *of* Ω. *So, this is an equation of the parabolic type.*

Definition 3.2.9. *The equation is of the elliptic type if and only if the discriminant* $\Delta < 0$.

Example 3.2.4. *Let's Consider the following partial differential equation:*

$$x\frac{\partial^2 u}{\partial x^2} + y\frac{\partial^2 u}{\partial y^2} + \frac{\partial u}{\partial x} = 0$$

defined on $\Omega = \mathbb{R}_+^* \times \mathbb{R}_+^*$.
$\Delta = b^2 - 4ac = -4xy < 0$ *far all* x, y *of* Ω. *So, this is an equation of the elliptic type.*

Main Equations of Physics

Laplace's Equation It is the equation of the form:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0. \tag{3.13}$$

Equation (3.13) is of the elliptic type.

Heat Equation (Diffusion Equation) It is the equation of the form:

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{a}\frac{\partial u}{\partial t}. \tag{3.14}$$

Equation (3.14) is of the parabolic type. The term $\frac{\partial u}{\partial t}$ is called the diffusion term.

Tricomi's Equation It is the equation of the form:

$$\frac{\partial^2 u}{\partial y^2} = y\frac{\partial^2 u}{\partial x^2}. \tag{3.15}$$

3.2.4 Standard Form of Second-Order P.D.E.s (Method of Characteristics)

In this subsection, we explore the standard form of second-order partial differential equations (P.D.E.s) and the method of characteristics used to solve them

Theorem 3.2.1. *The characteristic curves (3.11) are the solutions of the equation:*

$$a(x,y)\left(\frac{dy}{dx}\right)^2 + b(x,y)\frac{dy}{dx} + c(x,y) = 0 \tag{3.16}$$

with $a(x,y) \neq 0$.
If $a(x,y) = 0$ and $c(x,y) \neq 0$. Then the characteristic curves (3.11) are the solutions of the following equation:

$$c(x,y)\left(\frac{dy}{dx}\right)^2 + b(x,y)\frac{dy}{dx} + a(x,y) = 0 \tag{3.17}$$

In the case where $a(x,y) = 0$ and $c(x,y) = 0$, the characteristic curves (3.11) are straight lines with the equations:

$$x = k_1 \quad et \quad y = k_2$$

Theorem 3.2.2. *If (3.11) is of the hyperbolic type, we take as a change of variables the characteristic curves ($\varphi_1(x,y) = k_1$ and $\varphi_1(x,y) = k_2$) for:*

$$\begin{cases} x_1 = \varphi_1(x,y) \\ \\ x_2 = \varphi_2(x,y), \end{cases}$$

The equation (3.11) becomes:

$$\frac{\partial^2 u}{\partial x_1 \partial x_2} = G\left(x_1, x_2, u, \frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}\right). \tag{3.18}$$

Moreover, if we take

$$\begin{cases} y_1 = \varphi_1(x,y) + \varphi_2(x,y) \\ \\ y_2 = \varphi_1(x,y) - \varphi_2(x,y), \end{cases}$$

then (3.11) will be written as:

$$\frac{\partial^2 u}{\partial y_1^2} - \frac{\partial^2 u}{\partial y_2^2} = H\left(y_1, y_2, u, \frac{\partial u}{\partial y_1}, \frac{\partial u}{\partial y_2}\right). \tag{3.19}$$

Theorem 3.2.3. *If (3.11) is of the parabolic type, we take as a change of variables the characteristic curves $x_1 = \varphi_1(x,y)$ and x_2 any independent function with φ_1, the equation (3.11) becomes:*

$$\frac{\partial^2 u}{\partial x_2^2} = G\left(x_1, x_2, u, \frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}\right). \tag{3.20}$$

Theorem 3.2.4. *If (3.11) is of the elliptic type, we take as a change of variables the following characteristic curves:*
($\eta_1(x,y) + i\psi_1(x,y) = \lambda_1 \in \mathbb{C}, \eta_2(x,y) + i\psi_2(x,y) = \lambda_2 \in \mathbb{C}$) for

$$\begin{cases} x_1 = \eta_1(x,y) + i\psi_1(x,y) \\ \\ x_2 = \eta_2(x,y) + i\psi_2(x,y), \end{cases}$$

The term involving the mixed derivative in equation (3.11) will vanish.

Example 3.2.5. *Consider the following Cauchy problem:*

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} = a^2 \dfrac{\partial^2 u}{\partial x^2} \\ u(x,0) = \varphi_1(x) \\ \frac{\partial u}{\partial t}(x,0) = \varphi_2(x). \end{cases} \tag{3.21}$$

where a is a constant. Find the solution to (3.21) using the method of characteristics.

Solution. $\dfrac{\partial^2 u}{\partial t^2} = a^2 \dfrac{\partial^2 u}{\partial x^2}$ *It is the equation of the vibrating string, the discriminant is $\Delta = 4a^2 > 0$, thus, it is an equation of the hyperbolic type. The characteristic curves (3.21) are the solutions of the equation:*

$$\left(\frac{dx}{dt}\right)^2 - a^2 = 0$$

This leads to:

$$(dx - adt)(dx + adt) = 0.$$

Thus $x - at = c_1$ and $x + at = c_2$. We take as a change of variables $z(x,t) = x - at$ and $y(x,t) = x + at$, we find

$$\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2} \implies \frac{\partial^2 u}{\partial z \partial y} = 0. \tag{3.22}$$

The expression (3.22) implies:

$$u(z,y) = \Psi_1(z) + \Psi_2(y),$$

From which it follows:

$$u(x,t) = \Psi_1(x - at) + \Psi_2(x + at),$$

According to the initial conditions, we have:

$$u(x,0) = \Psi_1(x) + \Psi_2(x) = \varphi_1(x) \tag{3.23}$$

and

$$\frac{\partial u}{\partial t}(x,0) = -a\Psi_1'(x) + a\Psi_2'(x) = \varphi_2(x). \tag{3.24}$$

Integrating (3.24), *we obtain:*

$$-\Psi_1(x) + \Psi_2(x) = \frac{1}{a}\int_0^x \varphi_2(\tau)\, d\tau. \tag{3.25}$$

Adding (3.23) *and* (3.25), *we find:*

$$\Psi_2(x) = \frac{1}{2}\varphi_1(x) + \frac{1}{2a}\int_0^x \varphi_2(\tau)\, d\tau.$$

The subtraction between (3.23) *and* (3.25) *gives:*

$$\Psi_1(x) = \frac{1}{2}\varphi_1(x) - \frac{1}{2a}\int_0^x \varphi_2(\tau)\, d\tau.$$

Thus, the solution of (3.21) *is:*

$$\begin{aligned} u(x,t) &= \frac{\varphi_1(x-at)+\varphi_1(x+at)}{2} + \frac{1}{2a}\left[\int_0^{x+at} \varphi_2(\tau)\, d\tau - \int_0^{x-at} \varphi_2(\tau)\, d\tau\right] \\ &= \frac{\varphi_1(x-at)+\varphi_1(x+at)}{2} + \frac{1}{2a}\int_{x-at}^{x+at} \varphi_2(\tau)\, d\tau. \end{aligned} \tag{3.26}$$

3.3 Special Functions

In this section, we will present the definitions and properties of some special functions that play an important role in various areas of mathematics, such as asymptotic series, number theory, and fractional calculus theory.

3.3.1 The Gamma Function

One of the fundamental functions in fractional calculus is the Gamma function $\Gamma(x)$. This function generalizes the factorial notion $n!$ and allows n to take non-integer values.

Definition 3.3.1. *(The Gamma Function). The function* $\Gamma : (0, +\infty) \longrightarrow \mathbb{R}$, *defined by*

$$\Gamma(x) = \int_0^{\infty} t^{x-1} e^{-t}\, dt, \tag{3.27}$$

is called Euler's Gamma function (or Euler's integral of the second kind). Gamma function defined by (3.27) *has the following:*

(i) $\Gamma(x+1) = x\Gamma(x)$ *for* $x > 0$.

(ii) $\Gamma(n+1) = n!$ *for* $n \in \mathbb{N}$.

The Gamma function also satisfies the following reflection formula:

$$\Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin(\pi x)}; \quad 0 < x < 1. \tag{3.28}$$

Taking $x = 1/2$ we obtain from (3.28) a useful particular value of the Gamma function:

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}.$$

3.3.2 The Beta function

In many cases, it is more convenient to use the Beta function instead of a certain combination of the values of the Gamma function.

Definition 3.3.2. *(Beta function). For every $x > 0$, $y > 0$, the Beta function is defined by*

$$B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}\, dt. \tag{3.29}$$

This function is related to the Gamma function by the following identity:

$$B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}; \quad x > 0,\ y > 0.$$

It should also be mentioned that the Beta function is symmetric, meaning that:

$$B(x,y) = B(y,x); \quad \forall x > 0,\ \forall y > 0.$$

Example 3.3.1. *Let the following improper integral defined by:*

$$\Gamma(n+1) = \int_0^{+\infty} x^n e^{-x}\, dx; \qquad \forall n \in \mathbb{N}.$$

1) *Prove that this integral is convergent.*

2) *Deduce a relation between $\Gamma(n+2)$ and $\Gamma(n+1)$.*

3) *Calculate $\Gamma(1)$ and $\Gamma(2)$.*

4) *Deduce the value of $\Gamma(n+1)$ for all $n \in \mathbb{N}$.*

Solution. *1) the convergence of $\displaystyle\int_0^{+\infty} x^n e^{-x}\, dx$; $n \in \mathbb{N}$.*

- *The function $x \longmapsto f(x) = x^n e^{-x}$ is continuous and positive on $[0, +\infty[$ $\forall n \in \mathbb{N}$. Thus, we have a problem at $+\infty$. We can write the integral in the form:*

$$\int_0^{+\infty} x^n e^{-x}\, dx = \int_0^{x_0} x^n e^{-x}\, dx + \int_{x_0}^{+\infty} x^n e^{-x}\, dx.$$

Instead of studying the entire integral, it is sufficient to study the two integrals.

1-a) $\displaystyle\int_0^{x_0} x^n e^{-x}\,dx.$

Since f *is continuous on* $[0, x_0]$, *hence* $\displaystyle\int_0^{x_0} x^n e^{-x}\,dx$ *exists (it is a simple integral)* $\forall n \in \mathbb{N}$.

1-b) $\displaystyle\int_{x_0}^{+\infty} x^n e^{-x}\,dx.$

We know that $\displaystyle\lim_{x \longrightarrow +\infty} x^{n+2} e^{-x} = 0$, $\forall n \in \mathbb{N}$. *Thus, from* x_0, *we have*

$$x^{n+2} e^{-x} < 1 \Longrightarrow f(x) < \frac{1}{x^2}.$$

such that $x \geq x_0$ *with* $n \in \mathbb{N}$.

$\displaystyle\int_{x_0}^{+\infty} \frac{1}{x^2}\,dx$ *converges (Riemann's integral* 1^{st} *type* $\alpha = 2 > 1$*), it follows that* $\displaystyle\int_{x_0}^{+\infty} x^n e^{-x}\,dx$ *converges by the test of comparaison. Therefore:*

$\displaystyle\int_0^{+\infty} x^n e^{-x}\,dx$ *converges* $\forall n \in \mathbb{N}$ *(The sum of the two converge integrals).*

2) *The relation between* $\Gamma(n+2)$ *and* $\Gamma(n+1)$*:*

We have: $\Gamma(n+2) = \displaystyle\int_0^{+\infty} x^{n+1} e^{-x}\,dx$. *We set:* $u(x) = x^{n+1}$, *it follows that* $u'(x) = (n+1)x^n$ *and* $v'(x) = e^{-x}$, *then* $v(x) = -e^{-x}$. *The functions* u *and* v *being of class* $\mathcal{C}^1$, *We can perform integration by parts:*

$$\begin{aligned}
\Gamma(n+2) &= \int_0^{+\infty} x^{n+1} e^{-x}\,dx = \Big[-x^{n+1} e^{-x}\Big]_0^{+\infty} + (n+1)\int_0^{+\infty} x^n e^{-x}\,dx \\
&= -\lim_{x \longrightarrow +\infty} x^{n+1} e^{-x} + (n+1)\Gamma(n+1) \\
&= (n+1)\Gamma(n+1) \Rightarrow \Gamma(n+2) = (n+1)\Gamma(n+1).
\end{aligned}$$

3) *Calculation of* $\Gamma(1)$ *and* $\Gamma(2)$*:*

$\Gamma(1) = \displaystyle\int_0^{+\infty} x^0 e^{-x}\,dx = \int_0^{+\infty} e^{-x}\,dx = \Big[-e^{-x}\Big]_0^{+\infty} = -\lim_{x \longrightarrow +\infty} e^{-x} + 1 = 1.$

With the previous relation or through integration by parts, we obtain:

$$\Gamma(2) = \int_0^{+\infty} x e^{-x}\,dx = 1 = \Gamma(0).$$

4) *Deduce the value of* $\Gamma(n+1)$ *for all* $n \in \mathbb{N}$*:*

We know that: $\Gamma(n+1) = n\Gamma(n) = n(n-1)\Gamma(n-1) = n(n-1) \times \cdots \times 2 \times 1$
$\Longrightarrow \Gamma(n+1) = n!$.

Example 3.3.2. *Let the Beta function be defined by*

$$B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}\,dt.$$

1) *Compute* $\Gamma\left(\frac{1}{2}\right)$ *using the Beta function.*

2) *Deduce the value of* $\int_{-\infty}^{+\infty} e^{-x^2}\,dx$.

Solution. *1) Calculation of* $\Gamma\left(\frac{1}{2}\right)$.

We know that:

$$\left[\Gamma\left(\frac{1}{2}\right)\right]^2 = \frac{\Gamma(1/2)\Gamma(1/2)}{\Gamma(1)} = B\left(\frac{1}{2},\frac{1}{2}\right).$$

By definition

$$B\left(\frac{1}{2},\frac{1}{2}\right) = \int_0^1 t^{-1/2}(1-t)^{-1/2}\,dt. \tag{3.30}$$

Using the substitution $t = x^2$ *in* (3.30)*, we will find:*

$$B\left(\frac{1}{2},\frac{1}{2}\right) = 2\int_0^1 \frac{dx}{\sqrt{1-x^2}} = 2\Big[\arcsin x\Big]_0^1 = 2\left(\arcsin 1 - \arcsin 0\right) = \pi.$$

Thus

$$\Gamma\left(\frac{1}{2}\right) = \int_0^{+\infty} t^{-1/2}e^{-t}\,dt = \sqrt{\pi}.$$

2) The value of $\int_{-\infty}^{+\infty} e^{-x^2}\,dx$.

We observe that

$$\int_0^{+\infty} t^{-1/2}e^{-t}\,dt = 2\int_0^{+\infty} e^{-x^2}\,dx = \int_{-\infty}^{+\infty} e^{-x^2}\,dx.$$

Therefore

$$\int_{-\infty}^{+\infty} e^{-x^2}\,dx = \sqrt{\pi}.$$

3.4 Supplementary Exercises

Exercise 3.1. *Solve the following first-order differential equations:*
1) $(x+1)y' + xy = (x+1)^2$; $x > 0$.
2) $(x^2+1)y' + xy = 1$.
3) $(e^x+1)y' - y = \dfrac{e^x}{x^2+1}$.
4) $y'' + y' - 2y = 2x + 1$.

Exercise 3.2. *Solve the following second-order differential equations:*
1) $y'' + y' - 2y = 2x + 1$.
2) $y'' - 5y' + 6y = e^{2x}$.
3) $y^{''} + 2y^{'} + y = x^2 + x + 1$.

Exercise 3.3. *Let g_0 and g_1 be two functions of a real variable, of class $\mathcal{C}^2$ on $\mathbb{R}$, Define the function u on $\mathbb{R}_+^* \times \mathbb{R}$ by:*

$$u(x,y) \;=\; g_0\left(\frac{y}{x}\right) + x g_1\left(\frac{y}{x}\right).$$

- *Justify that u is of class $\mathcal{C}^2$, then prove that*

$$x^2\frac{\partial^2 u}{\partial x^2}(x,y) + 2xy\frac{\partial^2 u}{\partial x \partial y}(x,y) + y^2\frac{\partial^2 u}{\partial y^2}(x,y) \;=\; 0.$$

Exercise 3.4. *Let $c \neq 0$. Find the $\mathcal{C}^2-$class solutions of the following partial differential equation:*

$$c^2\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2},$$

using a change of variables of the form $\alpha = x + at$ and $\beta = x + bt$.

Exercise 3.5. *Consider the following improper integrals:*

$$I = \int_0^{+\infty} e^{-x^2}\,dx \qquad \text{and} \qquad J = \int_0^{+\infty} e^{-x^4}\,dx.$$

1) *Justify the convergence of I and J.*
2) *Express I and J, using the Gamma function Γ.*

Exercise 3.6. *Verify that for all $x > 0$ and $y > 0$, we have:*

1) $B(x+1,y) = \dfrac{x}{x+y}B(x,y)$.

2) $B(1,y) = \dfrac{1}{y}$.

Chapter 4

Series

4.1 Infinite series

The goal of this section is to understand the meaning of such an infinite sum and to develop methods to calculate it. Since there are infinitely many terms to add in an infinite series, we cannot just keep adding to see what,comes out. Instead we look at whatwe get by summing the first n terms of the sequence and stopping. The sum of the first n termsis an ordinary finite sum and can be calculated by normal addition. It is called the nth partial sum. As n gets larger, we expect the partial sums to get closer and closer to a limiting value in the same sense that the terms of a sequence approach a limit.

Definition 4.1.1. *Let $(u_n)_n$ a sequence of numbers. It is denoted by:*

$$\begin{aligned} S_n &= u_0 + u_1 + \cdots\cdots + u_n \\ &= \sum_{k=0}^{n} u_k. \end{aligned}$$

S_n: is called the partial sum of order n.

Definition 4.1.2. *Given a sequence of numbers $(u_n)_n$. It is called an* ***infinite series*** *with $n^{th}-$term u_n:*

$$\sum_{n=0}^{+\infty} u_n = u_0 + u_1 + \cdots\cdots + u_n + u_{n+1} + \cdots$$

Notation 4.1.1. *Here are some commonly used notations:*
$\sum_{n=0}^{+\infty} u_n$, $\sum u_n$ or $\sum_{n\geq 0} u_n$: it is the infinite series.
$(S_n)_n$: is the sequence of partial sums of the series.
u_n: is the $n^{th}-$term of the infinite series $\sum u_n$.
S: the sum of the infinite series $\sum_{n=0}^{+\infty} u_n$ if it is converges.

4.1.1 Convergent Series

Let $\sum u_n$ is an infinite series, we say that the series converges and that its sum is S. In this case, we also write

$$\begin{aligned}\sum_{n=0}^{+\infty} u_n &= u_0 + u_1 + \cdots\cdots + u_n + u_{n+1} + \cdots \\ &= \lim_{n\longrightarrow+\infty} S_n = S.\end{aligned}$$

1) If $(S_n)_n$ converges $\Rightarrow \sum_{n=0}^{+\infty} u_n$ converges.

2) If $(S_n)_n$ diverges $\Rightarrow \sum_{n=0}^{+\infty} u_n$ diverges.

Example 4.1.1. *Find the sum of the series:*
1) $\sum_{n=1}^{+\infty} \frac{1}{n(n+1)}$, ***2)*** $\sum_{n=1}^{+\infty} \ln\left(1+\frac{1}{n}\right)$, ***3)*** $\sum_{n=0}^{+\infty} (-1)^n$.

Solution. *We look for a pattern in the sequence of partial sums that might lead to a formula for S_n. The key observation is the partial fraction decomposition:*

1) $\sum_{n=1}^{+\infty} \frac{1}{n(n+1)}$*:*

$$u_n = \frac{1}{n(n+1)} = \frac{a}{n} + \frac{b}{n+1} = \frac{(a+b)n+a}{n(n+1)}.$$

$$\Longrightarrow \begin{cases} a+b=0 \\ a=1 \end{cases} \Longrightarrow a=1 \textit{ and } b=-1.$$

Thus, we can easily see that:

$$u_n = \frac{1}{n} - \frac{1}{n+1}; \quad n \geq 1.$$

1-a) *The partial sum S_n:*

$$\begin{aligned} S_n &= u_1 + u_2 + \cdot\cdot\cdot\cdot\cdot \quad \cdot\cdot\cdot\cdot + u_{n-1} + u_n \\ &= \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \cdot\cdot\cdot\cdot \quad \cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot \\ &\quad \cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot + \left(\frac{1}{n-1} - \frac{1}{n}\right) + \left(\frac{1}{n} - \frac{1}{n+1}\right) \\ &= 1 - \frac{1}{n+1}. \end{aligned}$$

1-b) *the sum S of the series:*

$$S = \lim_{n \longrightarrow +\infty} S_n = \lim_{n \longrightarrow +\infty} \left(1 - \frac{1}{n+1}\right) = 1 - 0 = 1.$$

Hence, the series converges, and its sum is 1:

$$\sum_{n=1}^{+\infty} \frac{1}{n(n+1)} = 1.$$

2) $\displaystyle\sum_{n=1}^{+\infty} \ln\left(1 + \frac{1}{n}\right)$*:*

$u_n = \ln\left(1 + \frac{1}{n}\right)$; $n \geq 1$. *We can write the term u_n in the following form:*

$$u_n = \ln\left(1 + \frac{1}{n}\right) = \ln\left(\frac{n+1}{n}\right) = \ln(n+1) - \ln n \quad \forall n \geq 1.$$

2-a) *The partial sum S_n:*

$$\begin{aligned} S_n &= u_1 + u_2 + \cdot\cdot\cdot\cdot\cdot \quad \cdot\cdot\cdot\cdot + u_{n-1} + u_n \\ &= (\ln 2 - \ln 1) + (\ln 3 - \ln 2) + \cdot\cdot\cdot\cdot \quad \cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot \\ &\quad \cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot + [\ln n - \ln(n-1)] + [\ln(n+1) - \ln n] \\ &= \ln(n+1) - \ln 1 = \ln(n+1). \end{aligned}$$

2-b) *the sum S of the series:*

$$S = \lim_{n \longrightarrow +\infty} S_n = \lim_{n \longrightarrow +\infty} \ln(n+1) = +\infty.$$

Since the sequence of partial sums of the series does not converge, we say that the series diverges.

3) $\sum_{n=0}^{+\infty}(-1)^n$*:*
$S_0 = 1,\ S_1 = 1-1 = 0,\ S_2 = 1-1+1 = 1,\ \cdots$
The partial sums alternate between 1 *and* 0*, so the sequence* (S_n) *does not converges to a limit, therefore the series is divergent and has no sum.*

Geometric Series

We will study the geometric series, which is defined as follows:

$$\sum_{n=0}^{+\infty} q^n, \quad q \in \mathbb{R}^+.$$

First case. If $q = 1$:

$$\begin{aligned} S_n &= u_0 + u_1 + \cdots\cdots + u_{n-1} + u_n \\ &= 1 + 1 + \cdots\cdots + 1 + 1 = n + 1. \end{aligned}$$

Since $\lim\limits_{n\longrightarrow+\infty} S_n = \lim\limits_{n\longrightarrow+\infty} n+1 = +\infty$.
Then: $\sum 1^n$ diverges.

Second case. If $q \neq 1$:
2-a) The partial sum S_n:
We know that:

$$\begin{aligned} S_n &= u_0 + u_1 + \cdots\cdots + u_n \\ &= 1 + q + \cdots\cdots + q^n. \end{aligned}$$

Hence:

$$\begin{aligned} S_{n+1} &= 1 + q + \cdots\cdots q^n + q^{n+1} \\ &= S_n + q^{n+1}. \end{aligned} \tag{4.1}$$

On one hand, on the other hand we have:

$$\begin{aligned} S_{n+1} &= u_0 + u_1 + \cdots\cdots + u_{n+1} \\ &= 1 + q + \cdots\cdots + q^{n+1} \\ &= 1 + q\,(1 + \cdots\cdots + q^n) \\ &= 1 + q \cdot S_n. \end{aligned} \tag{4.2}$$

Thus, according to (4.1) and (4.2), we conclude that:
$S_n(1-q) = 1 - q^{n+1} \Rightarrow S_n = \dfrac{1-q^{n+1}}{1-q} \quad \forall q \neq 1.$

2-b) the sum S of the series:
We calculate the limit of S_n and obtain the following:

$$S = \lim_{n\longrightarrow+\infty} S_n = \begin{cases} \dfrac{1}{1-q} & \text{if} \quad 0 \le q < 1, \\ +\infty & \text{if} \quad q > 1. \end{cases}$$

$$\Rightarrow \sum_{n=0}^{+\infty} q^n \begin{cases} \text{converges} & \text{if} \quad 0 \le q < 1, \\ \text{diverges} & \text{if} \quad q \ge 1. \end{cases}$$

Proposition 4.1.1. *Let $\sum u_n$ and $\sum v_n$ two infinite series and $\lambda \in \mathbb{R}_*$. Then, we have:*

1) *$\sum u_n$ and $\sum(\lambda \cdot u_n)$ either both converge or both diverge.*

2) *If $\sum u_n$ converges and $\sum v_n$ converges; so $\sum(u_n+v_n)$ converges also.*

3) *If $\sum u_n$ converges and $\sum v_n$ diverges; then $\sum(u_n+v_n)$ diverges.*

4) *If $\sum u_n$ diverges and $\sum v_n$ diverges; then, we cannot conclude anything about the series $\sum(u_n+v_n)$.*

Divergence Test

Remark 4.1.1. *Let $\sum u_n$ an infinite series.*
If $\lim\limits_{n\longrightarrow+\infty} u_n \neq 0 \Rightarrow \sum u_n$ diverges.

Example 4.1.2. *Applying the Divergence Test:*

1) $\displaystyle\sum_{n=0}^{+\infty} \frac{3n+1}{2n+5}$, *2)* $\displaystyle\sum_{n=1}^{+\infty} \frac{e^n}{n}$.

Solution. *Here, we calculate the limit of the $n^{th}-$term:*

1) $\displaystyle\sum_{n=0}^{+\infty} \frac{3n+1}{2n+5}$:

We have $\displaystyle\lim_{n\longrightarrow+\infty} u_n = \lim_{n\longrightarrow+\infty} \frac{3n+1}{2n+5} = \frac{3}{2} \neq 0 \Longrightarrow \sum_{n=0}^{+\infty} u_n$ *diverges.*

2) $\displaystyle\sum_{n=1}^{+\infty} \frac{e^n}{n}$:

We have $\displaystyle\lim_{n\longrightarrow+\infty} u_n = \lim_{n\longrightarrow+\infty} \frac{e^n}{n} = +\infty \neq 0 \Longrightarrow \sum_{n=1}^{+\infty} u_n$ *diverges.*

Remark 4.1.2. $u_n \longrightarrow 0$ *but the Series Diverges.*

The series $\sum_{n=1}^{+\infty} \frac{1}{n}$ *is called the* ***harmonic series****, It is a divergent series. For the proof, see:* **Example** *4.1.3.*

4.1.2 Series with Positive Terms

In the study of the convergence of a series, it is not always possible to calculate its sum. However, we can determine its behavior using other techniques. For this, we need additional tools (the tests).

The Integral Test

Theorem 4.1.1. *We consider the function defined by:*
$f : [N, +\infty[\to \mathbb{R}^+$ *continuous, positive and decreasing function. We set:*

$$u_n = f(n), \quad \forall n \geq N \quad (N \, a \text{ positive integer}).$$

Then the series $\sum_{n=1}^{+\infty} u_n$ *and the integral* $\int_1^{+\infty} f(x)\,dx$ *both converge or both diverge.*

1) *If* $\int_N^{+\infty} f(x)\,dx$ *converges* $\Rightarrow \sum_{n=N}^{+\infty} u_n$ *converges.*

2) *If* $\int_N^{+\infty} f(x)\,dx$ *diverges* $\Rightarrow \sum_{n=N}^{+\infty} u_n$ *diverges.*

Example 4.1.3. *Show that the harmonic series:*
$\sum_{n=1}^{+\infty} \frac{1}{n}$ *is divergent.*

Solution. *We may apply the Integral Test:*
We have $f(x) = \frac{1}{x}$*;* $x \in [1, +\infty[$*. This function is continuous, positive and decreasing. Since* $\int_1^{+\infty} \frac{1}{x}\,dx$ *diverges (Riemann's integral* 1^{st} *type* $\alpha = 1$*), then the series* $\sum_{n=1}^{+\infty} \frac{1}{n}$ *diverges also.*

The Riemann series

A Riemann series is any series of the form:

$$\sum_{n=1}^{+\infty} \frac{1}{n^\alpha}; \quad \alpha \in \mathbb{R}.$$

We notice that:

$$\lim_{n \longrightarrow +\infty} u_n = \begin{cases} 0 \quad \text{si} \quad \alpha > 0, \\ 1 \quad \text{si} \quad \alpha = 0 \\ +\infty \quad \text{si} \quad \alpha < 0. \end{cases}$$

Thus $\sum_{n=1}^{+\infty} \frac{1}{n^\alpha}$ diverges if $\alpha \leq 0$. For $\alpha > 0$, we can apply the Integral Test.

First case. If $\alpha = 1$:

It is the harmonic series which is divergent.

Second case. If $\alpha \neq 1$:

We set $f(x) = \frac{1}{x^\alpha}$; $x \in [1, +\infty[$. This function is continuous, positive and decreasing and we have also:

$$\int_1^{+\infty} \frac{1}{x^\alpha}\, dx \begin{cases} \text{converges} \quad \text{if} \quad \alpha > 1, \\ \text{diverges} \quad \text{if} \quad \alpha < 1. \end{cases} \Leftrightarrow \sum_{n=1}^{+\infty} \frac{1}{n^\alpha} \begin{cases} \text{converges} \quad \text{if} \quad \alpha > 1, \\ \text{diverges} \quad \text{if} \quad \alpha < 1. \end{cases}$$

$$\Rightarrow \sum_{n=1}^{+\infty} \frac{1}{n^\alpha} \begin{cases} \text{converges} \quad \text{if} \quad \alpha > 1, \\ \text{diverges} \quad \text{if} \quad \alpha \leq 1. \end{cases}$$

The Comparison Test

Theorem 4.1.2. *Let $\sum_{n=0}^{+\infty} u_n$ and $\sum_{n=0}^{+\infty} v_n$ two series with positive terms. We assume that:*

$$0 \leq u_n \leq v_n \quad \forall n \geq N \quad (N \textit{ a positive integer}).$$

1) *If $\sum_{n=0}^{+\infty} v_n$ converges $\Rightarrow \sum_{n=0}^{+\infty} u_n$ converges.*

2) *If $\sum_{n=0}^{+\infty} u_n$ diverges $\Rightarrow \sum_{n=0}^{+\infty} v_n$ diverges.*

Example 4.1.4. *Which of the following series converge, and which diverge? Give reasons for your answers.*
:

1) $\displaystyle\sum_{n=2}^{+\infty} \frac{1}{n^n}$, *2)* $\displaystyle\sum_{n=2}^{+\infty} \frac{1}{\sqrt{n(n-1)}}$, *3)* $\displaystyle\sum_{n=2}^{+\infty} \frac{\ln n}{n^3}$.

Solution. *We can apply the Comparison Test:*

1) $\displaystyle\sum_{n=2}^{+\infty} \frac{1}{n^n}$*:*
We konw that: $n \geq 2 \Rightarrow n^n \geq 2^n$. *Then:*

$$\frac{1}{n^n} \leq \frac{1}{2^n} = \left(\frac{1}{2}\right)^n \quad \forall n \geq 2.$$

$\displaystyle\sum_{n=2}^{+\infty} \left(\frac{1}{2}\right)^n$ *converges (it is a geometric series with ratio* $0 < q = 1/2 < 1$*), hence the series* $\displaystyle\sum_{n=2}^{+\infty} \frac{1}{n^n}$ *converges according to the Comparison Test.*

2) $\displaystyle\sum_{n=2}^{+\infty} \frac{1}{\sqrt{n(n-1)}}$*:*
We have: $n^2 \geq n(n-1) \Rightarrow n \geq \sqrt{n(n-1)}$. *So:*

$$\frac{1}{n} \leq \frac{1}{\sqrt{n(n-1)}} \quad \forall n \geq 2.$$

Since $\displaystyle\sum_{n=2}^{+\infty} \frac{1}{n}$ *is divergent (it is the harmonic series), therefore the series* $\displaystyle\sum_{n=2}^{+\infty} \frac{1}{\sqrt{n(n-1)}}$ *diverges by the Comparison Test.*

3) $\displaystyle\sum_{n=2}^{+\infty} \frac{\ln n}{n^3}$*:*
We know that: $\ln n \leq n$. *Then:* $\displaystyle\frac{\ln n}{n^3} \leq \frac{n}{n^3} = \frac{1}{n^2} \quad \forall n \geq 2.$
Since $\displaystyle\sum_{n=2}^{+\infty} \frac{1}{n^2}$ *converges (it is a Riemann series with* $\alpha = 2 > 1$*), thus this series* $\displaystyle\sum_{n=2}^{+\infty} \frac{\ln n}{n^3}$ *converges according to the Comparison Test.*

The Limit Comparison Test

Theorem 4.1.3. *Let* $\sum_{n=0}^{+\infty} u_n$ *and* $\sum_{n=0}^{+\infty} v_n$ *two series with strictly positive terms. We say that:*

$$u_n \sim v_n \Leftrightarrow \lim_{n\to+\infty} \frac{u_n}{v_n} = 1.$$

Then the series $\sum_{n=0}^{+\infty} u_n$ *and* $\sum_{n=0}^{+\infty} v_n$ *both converge or both diverge:*

1) *If* $\sum_{n=0}^{+\infty} v_n$ *converges* $\Rightarrow \sum_{n=0}^{+\infty} u_n$ *converges.*

2) *If* $\sum_{n=0}^{+\infty} v_n$ *diverges* $\Rightarrow \sum_{n=0}^{+\infty} u_n$ *diverges.*

Example 4.1.5. *Study the convergence of the following series:*

1) $\sum_{n=0}^{+\infty} \frac{2^n+n+3}{3^n+n^2+5}$, ***2)*** $\sum_{n=1}^{+\infty} \ln\left(1+\frac{1}{n^3}\right)$.

Solution. *We may apply the Limit Comparison Test:*

1) $\sum_{n=0}^{+\infty} \frac{2^n+n+3}{3^n+n^2+5}$:

$$u_n = \frac{2^n+n+3}{3^n+n^2+5} \underset{+\infty}{\sim} v_n = \frac{2^n}{3^n} = \left(\frac{2}{3}\right)^n \quad \text{with} \lim_{n\to+\infty} \frac{u_n}{v_n} = 1.$$

$\sum_{n=0}^{+\infty} \left(\frac{2}{3}\right)^n$ *converges (a geometric series with ratio* $0 < q = 2/3 < 1$*), we deduce that* $\sum_{n=0}^{+\infty} \frac{2^n+n+3}{3^n+n^2+5}$ *converges by the Limit Comparison Test.*

2) $\sum_{n=1}^{+\infty} \ln\left(1+\frac{1}{n^3}\right)$:

We know that $\ln(1+x) \underset{0}{\sim} x$ *when* $x \longrightarrow 0$*, then:*

$u_n = \ln\left(1+\frac{1}{n^3}\right) \underset{+\infty}{\sim} v_n = \frac{1}{n^3}$ *because* $\lim_{n\longrightarrow+\infty} \frac{u_n}{v_n} = 1.$

Since $\sum_{n=1}^{+\infty} \frac{1}{n^3}$ *is convergent (a Riemann series with* $\alpha = 3 > 1$*), we conclude that* $\sum_{n=1}^{+\infty} \ln\left(1+\frac{1}{n^3}\right)$ *converges according to the Limit Comparison Test.*

The Root Test

Theorem 4.1.4. *Let* $\sum_{n=0}^{+\infty} u_n$ *be a series with positive terms, we suppose that:*

$$\lim_{n\longrightarrow+\infty} \sqrt[n]{u_n} = l; \quad (\textit{finite or infinite}).$$

Then:

1) *If* $l < 1 \Rightarrow \sum_{n=0}^{+\infty} u_n$ *converges.*

2) *If* $l > 1 \Rightarrow \sum_{n=0}^{+\infty} u_n$ *diverges.*

3) *If* $l = 1 \Rightarrow$ *the test is inconclusive .*

Example 4.1.6. *Investigate the convergence of the following series:*

1) $\sum_{n=0}^{+\infty} \left(\frac{3n+2}{5n+7}\right)^n$, *2)* $\sum_{n=1}^{+\infty} \left(1+\frac{2}{n}\right)^{n^2}$, *3)* $\sum_{n=1}^{+\infty} \left(1+\frac{1}{n}\right)^n$.

Solution. *We shall apply the Root Test:*

1) $\sum_{n=0}^{+\infty} \left(\frac{3n+2}{5n+7}\right)^n$*:*

$$\sqrt[n]{u_n} = \left(\frac{3n+2}{5n+7}\right)^{\frac{n}{n}} = \frac{3n+2}{5n+7} \Longrightarrow \lim_{n\longrightarrow+\infty} \sqrt[n]{u_n} = \lim_{n\longrightarrow+\infty} \frac{3n+2}{5n+7} = \frac{3}{5} < 1.$$

Thus: $\sum_{n=0}^{+\infty} \left(\frac{3n+2}{5n+7}\right)^n$ *converges by the Root Test.*

2) $\sum_{n=1}^{+\infty} \left(1+\frac{2}{n}\right)^{n^2}$*:*

$$\sqrt[n]{u_n} = \left(1+\frac{2}{n}\right)^n \Longrightarrow \lim_{n\longrightarrow+\infty} \sqrt[n]{u_n} = \lim_{n\longrightarrow+\infty} \left(1+\frac{2}{n}\right)^n = e^2 > 1.$$

Hence: $\sum_{n=1}^{+\infty} \left(1+\frac{2}{n}\right)^{n^2}$ *diverges according to the Root Test.*

3) $\sum_{n=1}^{+\infty} \left(1+\frac{1}{n}\right)^n$ *:* ***The Divergence Test***

$$\lim_{n\longrightarrow+\infty} u_n = \lim_{n\longrightarrow+\infty} \left(1+\frac{1}{n}\right)^n = e^1 \neq 0 \Longrightarrow \sum_{n=1}^{+\infty} \left(1+\frac{1}{n}\right)^n \textit{ diverges.}$$

The Ratio Test

Theorem 4.1.5. *Let* $\sum_{n=0}^{+\infty} u_n$ *be a series with strictly positive terms, we assume that:*

$$\lim_{n \longrightarrow +\infty} \frac{u_{n+1}}{u_n} = l; \quad (\text{finite or infinite}).$$

1) *If* $l < 1 \Rightarrow \sum_{n=0}^{+\infty} u_n$ *converges.*

2) *If* $l > 1 \Rightarrow \sum_{n=0}^{+\infty} u_n$ *diverges.*

3) *If* $l = 1 \Rightarrow$ *the test is inconclusive .*

Example 4.1.7. *Study the convergence of the following series:*

1) $\sum_{n=0}^{+\infty} \frac{2^n}{n!}$, *2)* $\sum_{n=1}^{+\infty} \frac{n!}{n^n}$, *3)* $\sum_{n=0}^{+\infty} \frac{1 \times 6 \times \cdots \times (5n+1)}{4^n \cdot n!}$.

Solution. *We can apply the Ratio Test:*

1) $\sum_{n=0}^{+\infty} \frac{2^n}{n!}$:

We know that : $u_{n+1} = \frac{2^{n+1}}{(n+1)!} = \frac{2^n \cdot 2}{n!(n+1)}$. *So:*

$$\lim_{n \longrightarrow +\infty} \frac{u_{n+1}}{u_n} = \lim_{n \longrightarrow +\infty} \frac{2^n \cdot 2}{n! \cdot (n+1)} \times \frac{n!}{2^n} = \lim_{n \longrightarrow +\infty} \frac{2}{n+1} = 0 < 1.$$

Therefore: $\sum_{n=0}^{+\infty} \frac{2^n}{n!}$ *converges by the Ratio Test.*

2) $\sum_{n=1}^{+\infty} \frac{n!}{n^n}$:

We have: $u_{n+1} = \frac{(n+1)!}{(n+1)^{n+1}} = \frac{n!(n+1)}{(n+1)^n(n+1)} = \frac{n!}{(n+1)^n}$. *Then:*

$$\begin{aligned} \lim_{n \longrightarrow +\infty} \frac{u_{n+1}}{u_n} &= \lim_{n \longrightarrow +\infty} \frac{n!}{(n+1)^n} \times \frac{n^n}{n!} = \lim_{n \longrightarrow +\infty} \left(\frac{n}{n+1}\right)^n \\ &= \lim_{n \longrightarrow +\infty} \left(1 + \frac{1}{n}\right)^{-n} = e^{-1} < 1. \end{aligned}$$

Thus: $\sum_{n=1}^{+\infty} \frac{n!}{n^n}$ *converges according to the Ratio Test.*

3) $\displaystyle\sum_{n=0}^{+\infty} \frac{1\times 6\times\cdots\times(5n+1)}{4^n\cdot n!}$*:*

$u_{n+1} = \dfrac{1\times 6\times\cdots\times(5n+6)}{4^{n+1}\cdot(n+1)!} = \dfrac{1\times 6\times\cdots\times(5n+1)\times(5n+6)}{4\cdot 4^n\cdot n!\cdot(n+1)}$. *So:*

$$\begin{aligned}\lim_{n\longrightarrow+\infty}\frac{u_{n+1}}{u_n} &= \lim_{n\longrightarrow+\infty}\frac{1\times\cdot\cdot\times(5n+1)\times(5n+6)}{4\cdot 4^n\cdot n!\cdot(n+1)}\times\frac{4^n\cdot n!}{1\times\cdot\cdot\times(5n+1)}\\ &= \lim_{n\longrightarrow+\infty}\frac{5n+6}{4n+4}\\ &= \frac{5}{4}>1.\end{aligned}$$

Hence: $\displaystyle\sum_{n=0}^{+\infty} \frac{1\times 6\times\cdots\times(5n+1)}{4^n\cdot n!}$ *diverges by the Ratio Test.*

4.1.3 Series with some negative terms

Definition 4.1.3. *The series* $\displaystyle\sum_{n=0}^{+\infty} u_n$ *converges absolutely if the corresponding series of absolute values* $\displaystyle\sum_{n=0}^{+\infty} |u_n|$ *converges.*

Theorem 4.1.6. *any infinite series absolutely converges is converges.*

Example 4.1.8. *Investigate the absolutely convergence of the following series:*

1) $\displaystyle\sum_{n=1}^{+\infty} \frac{(-1)^n}{n^2+1}$, *2)* $\displaystyle\sum_{n=0}^{+\infty} \frac{\cos n}{3^n}$.

Solution. *we will study the convergence of the series* $\sum |u_n|$*:*

1) $\displaystyle\sum_{n=1}^{+\infty} \frac{(-1)^n}{n^2+1}$*:*

The term u_n *is not positive, immediately, we have:*

$|u_n| = \left|\dfrac{(-1)^n}{n^2+1}\right| = \dfrac{1}{n^2+1}$. *Here, we can use the Limit Comparison Test:*

$|u_n| = \dfrac{1}{n^2+1} \sim v_n = \dfrac{1}{n^2}$ *because* $\displaystyle\lim_{n\longrightarrow+\infty}\frac{|u_n|}{v_n} = 1$.

Since $\displaystyle\sum_{n=1}^{+\infty}\frac{1}{n^2}$ *is convergent (it is a Riemann series with* $\alpha = 2 > 1$*).*

we deduce that $\displaystyle\sum_{n=1}^{+\infty}\left|\frac{(-1)^n}{n^2+1}\right|$ *converges by the Limit Comparison Test.*

$$\Rightarrow \sum_{n=1}^{+\infty}\frac{(-1)^n}{n^2+1} \text{ is absolutely convergent.}$$

$$\Rightarrow \sum_{n=1}^{+\infty}\frac{(-1)^n}{n^2+1} \text{ converges.}$$

2) $\displaystyle\sum_{n=0}^{+\infty}\frac{\cos n}{3^n}$*:*

The term u_n *is not always positive, then we have:*

$|u_n| = \left|\dfrac{\cos n}{3^n}\right| = \dfrac{|\cos n|}{3^n}$. *Here, we can apply the Comparison Test:*

$$\frac{|\cos n|}{3^n} \leq \frac{1}{3^n} = \left(\frac{1}{3}\right)^n \quad \forall n \in \mathbb{N}.$$

$\displaystyle\sum_{n=0}^{+\infty}\left(\frac{1}{3}\right)^n$ *is convergent (a geometric series with ratio* $0 < q = 1/3 < 1$*).*

We conclude that $\displaystyle\sum_{n=1}^{+\infty}\left|\frac{\cos n}{3^n}\right|$ *converges according to the Comparison Test.*

$$\Rightarrow \sum_{n=1}^{+\infty}\frac{\cos n}{3^n} \text{ eis absolutely convergent} \Rightarrow \sum_{n=1}^{+\infty}\frac{\cos n}{3^n} \text{ converges.}$$

4.1.4 The Alternating Series

Definition 4.1.4. *A series* $\displaystyle\sum_{n=0}^{+\infty}u_n$ *is said to be* ***alternating series*** *if:*

$$u_n = (-1)^n b_n; \quad b_n \geq 0 \quad \forall n.$$

Theorem 4.1.7. (Leibniz's Theorem)

$$\sum_{n=0}^{+\infty}u_n \text{ is alternating series. If: } \begin{cases} \lim\limits_{n\longrightarrow+\infty} b_n = 0 \\ \text{and} \\ (b_n)_n \searrow \end{cases} \Longrightarrow \sum_{n=0}^{+\infty}u_n \text{ converges.}$$

Example 4.1.9. *Study the convergence of the following series:*

1) $\displaystyle\sum_{n=1}^{+\infty}\frac{(-1)^n}{n}$, *2)* $\displaystyle\sum_{n=2}^{+\infty}\frac{(-1)^n}{\ln n}$.

Solution. *1)* $\displaystyle\sum_{n=1}^{+\infty}\frac{(-1)^n}{n}$. *Here:* $b_n = \dfrac{1}{n} > 0 \quad \forall n \geq 1$.

$$\begin{cases} \lim\limits_{n\longrightarrow+\infty} b_n = 0 \\ \qquad\text{and} \\ (b_n)_n \searrow \end{cases} \Longrightarrow \sum_{n=1}^{+\infty}\frac{(-1)^n}{n} \text{ converges by Leibniz's Theorem.}$$

2) $\displaystyle\sum_{n=2}^{+\infty}\frac{(-1)^n}{\ln n}$. *Here:* $b_n = \dfrac{1}{\ln n} > 0 \quad \forall n \geq 2$.

$$\begin{cases} \lim\limits_{n\longrightarrow+\infty} b_n = 0 \\ \qquad\text{and} \\ (b_n)_n \searrow \end{cases} \Longrightarrow \sum_{n=2}^{+\infty}\frac{(-1)^n}{\ln n} \text{ converges according to the Theorem of}$$

Leibniz.

Definition 4.1.5. *A series* $\displaystyle\sum_{n=0}^{+\infty} u_n$ *is said to be* ***converges conditionally*** *if it is converges and* $\displaystyle\sum_{n=0}^{+\infty} |u_n|$ *diverges.*

Example 4.1.10. *the following infinite series:*

1) $\displaystyle\sum_{n=1}^{+\infty}\frac{(-1)^n}{n}$, *2)* $\displaystyle\sum_{n=2}^{+\infty}\frac{(-1)^n}{\ln n}$.

are converge conditionally.

4.2 Sequences and Series of Functions

4.2.1 Sequences of Functions

Definition 4.2.1. *Let* $E \subset \mathbb{R}$ *and* $n \in \mathbb{N}$.

$f_n : E \to \mathbb{R}$

$\qquad x \mapsto f_n(x)$ *a function.*

$(f_n)_n$*: is called a sequence of functions defined on* E.

The types of convergence

Definition 4.2.2. **(Pointwise Convergence)**: *Let $(f_n)_n$ a sequence of functions defined on E.*
We say that $(f_n)_n$ converges pointwise on E if $\forall x \in E$ the sequence $(f_n(x))_n$ converges to a finite limit, which we denote by $f(x)$.

$$\lim_{n \longrightarrow +\infty} f_n(x) = f(x), \quad \forall x \in E \subseteq \mathbb{R}.$$

We write:

$$f_n \to f; \quad \forall x \in E.$$

Example 4.2.1. *Study the pointwise convergence of the following sequences of functions:*
1) $f_n(x) = \dfrac{1}{n+x}$*; $n \geq 1$ with $x \in \mathbb{R}$* ***2)*** $f_n(x) = x^n$*; $n \in \mathbb{N}$ with $x \in \mathbb{R}^+$*
3) $f_n(x) = \dfrac{nx}{nx+1}$*; $n \in \mathbb{N}$ $x \in \mathbb{R}^+$* ***4)*** $f_n(x) = ne^{-nx}$*; $n \in \mathbb{N}$ with $x \in \mathbb{R}$.*

Solution. *We will calculate the limit of $f_n(x)$:*
1) $f_n(x) = \dfrac{1}{n+x}$*:*

$$\lim_{n \longrightarrow +\infty} f_n(x) = 0, \quad \forall x \in \mathbb{R}.$$

Thus, $f_n \longrightarrow f$; $f(x) = 0$ $\forall x \in \mathbb{R}$.

2) $f_n(x) = x^n$*:*

$$\lim_{n \longrightarrow +\infty} f_n(x) = \begin{cases} 0 & \text{if} \quad 0 \leq x < 1 \\ 1 & \text{if} \quad x = 1 \\ +\infty & \text{if} \quad x > 1. \end{cases}$$

Hence, $(f_n)_n$ diverges on $\mathbb{R}^+$.
3) $f_n(x) = \dfrac{nx}{nx+1}$*:*

$$\lim_{n \longrightarrow +\infty} f_n(x) = \begin{cases} 0 & \text{if} \quad x = 0 \\ 1 & \text{if} \quad x > 0. \end{cases}$$

$$\text{Therefore, } f_n \longrightarrow f;\ f(x) = \begin{cases} 0 & \text{if} \quad x = 0 \\ 1 & \text{if} \quad x > 0 \end{cases}$$

4) $f_n(x) = ne^{-nx}$:

$$\lim_{n\longrightarrow+\infty} f_n(x) = \begin{cases} +\infty & if \quad x<0 \\ +\infty & if \quad x=0 \\ 0 & if \quad x>0. \end{cases}$$

Thus, the sequence $(f_n)_n$ diverges on $\mathbb{R}^+$.

Definition 4.2.3. (**The domain of Convergence**)*: Let $(f_n)_n$ a sequence of functions defined on E. We call the domain of convergence D the set of "x" where the sequence of functions converges pointwise.*

Example 4.2.2. *We will conclude the domain of convergence for the sequences of functions mentioned above:*

1) $f_n(x) = \dfrac{1}{n+x}$; $\quad n \geq 1$ *with* $x \in \mathbb{R} \Rightarrow D = \mathbb{R}$.

2) $f_n(x) = x^n$; $\;n \in \mathbb{N}$ *with* $x \in \mathbb{R}^+ \Rightarrow D = [0, 1]$.

3) $f_n(x) = ne^{-nx}$; $\;n \in \mathbb{N}$ *with* $x \in \mathbb{R} \Rightarrow D =]0, +\infty[$.
4) $f_n(x) = nx + 1$; $\;n \in \mathbb{N}$ *with* $x \in \mathbb{R} \Rightarrow D = \{0\}$.

Definition 4.2.4. (**The uniform convergence**)*: Let $(f_n)_n$ a sequence of functions; $f_n \underset{C \cdot P}{\rightarrow} f$; $\forall x \in E$.*

$$(f_n)_n \text{ converges uniformly to} f \Leftrightarrow \lim_{n\longrightarrow+\infty} \sup_{x\in E} |f_n(x) - f(x)| = 0.$$

We denote:

$$f_n \rightrightarrows f; \quad \forall x \in E.$$

Example 4.2.3. *Investigate the pointwise convergence and the uniform convergence of the following sequences of functions:*
1) $f_n(x) = \dfrac{1}{n+x}$; $n \geq 1$ *with* $x \in [0, 2]$,
2) $f_n(x) = x^n$; $n \in \mathbb{N}$ *with* $x \in \left[0, \dfrac{1}{2}\right]$,
3) $f_n(x) = xe^{-nx}$; $n \geq 1$ *with* $x \in \mathbb{R}^+$.

Solution. *We will first study the pointwise convergence and then the uniform convergence:*

1) $f_n(x) = \dfrac{1}{n+x}$*:*

1-a) *The pointwise convergence:*

$$\lim_{n \longrightarrow +\infty} f_n(x) = 0, \quad \forall x \in \mathbb{R}.$$

Then: $f_n \longrightarrow f$*;* $f(x) = 0 \ \forall x \in [0, 2]$*.*

1-b) *The uniform convergence:*

$|f_n(x) - f(x)| = \left| \dfrac{1}{n+x} - 0 \right| = \dfrac{1}{n+x}.$

We have:

$$g(x) = \frac{1}{n+x}; \qquad x \in [0, 2].$$

We know that g *is continuous and differentiable on* $[0, 2]$*:*

$$g'(x) = -\frac{1}{(n+x)^2} \Rightarrow g'(x) < 0 \Rightarrow g \searrow .$$

x	0 $\qquad\qquad$ 2
$g'(x)$	$-$
g	$\searrow$

The study of the monotonicity of the function g *on* $[0, 2]$ *give us:*
$\sup\limits_{x \in [0,2]} g(x) = \sup\limits_{x \in [0,2]} |f_n(x) - f(x)| = \dfrac{1}{n}$*. We deduce that:*

$$\begin{aligned}
\lim_{n \longrightarrow +\infty} \sup_{x \in [0,2]} |f_n(x) - f(x)| &= \lim_{n \longrightarrow +\infty} \frac{1}{n} \\
&= 0 \\
&\Rightarrow f_n \rightrightarrows f \quad \forall x \in [0, 2].
\end{aligned}$$

2) $f_n(x) = x^n$*:*

2-a) *The pointwise convergence:*

$$\lim_{n \longrightarrow +\infty} f_n(x) = \begin{cases} 0 & \textit{if} \quad x = 0 \\ 0 & \textit{if} \quad 0 \le x < \dfrac{1}{2}. \end{cases}$$

Thus: $f_n \longrightarrow f$*;* $f(x) = 0 \ \forall x \in \left[0, \dfrac{1}{2}\right]$*.*

2-b) *The uniform convergence:*

$|f_n(x) - f(x)| = |x^n - 0| = x^n.$

We set:

$$g(x) = x^n; \qquad x \in \left[0, \frac{1}{2}\right].$$

We know that g is continuous and differentiable on $\left[0, \frac{1}{2}\right]$:

$$g'(x) = nx^{n-1} \Rightarrow g'(x) \geq 0 \Rightarrow g \nearrow.$$

x	0 $\qquad\qquad$ $1/2$
$g'(x)$	$+$
g	$\nearrow$

The investigation of the monotonicity of the function g on $\left[0, \frac{1}{2}\right]$ give us:

$\sup\limits_{x\in[0,1/2]} g(x) = \sup\limits_{x\in[0,1/2]} |f_n(x) - f(x)| = \frac{1}{2^n}$. *We conclude that:*

$$\begin{aligned}
\lim_{n\longrightarrow+\infty} \sup_{x\in[0,1/2]} |f_n(x) - f(x)| &= \lim_{n\longrightarrow+\infty} \frac{1}{2^n} \\
&= 0 \\
&\Rightarrow f_n \rightrightarrows f \quad \forall x \in [0, 1/2].
\end{aligned}$$

3) $f_n(x) = xe^{-nx}$:

3-a) *The pointwise convergence:*

$$\lim_{n\longrightarrow+\infty} f_n(x) = \begin{cases} 0 & \text{if} \quad x = 0 \\ \\ 0 & \text{if} \quad x > 0. \end{cases}$$

Hence: $f_n \longrightarrow f$; $f(x) = 0 \ \forall x \in \mathbb{R}^+$.

3-b) *The uniform convergence:*

$|f_n(x) - f(x)| = \left|xe^{-nx} - 0\right| = xe^{-nx}.$

We set:

$$g(x) = xe^{-nx}; \qquad x \in \mathbb{R}^+.$$

We know that g is continuous and differentiable on $\mathbb{R}^+$:

$$g'(x) = e^{-nx} - nxe^{-nx} = (1 - nx)e^{-nx}.$$

$$g'(x) = 0 \Longleftrightarrow 1 - nx = 0 \Longleftrightarrow x = \frac{1}{n};\ n \geq 1.$$

x	0		$\frac{1}{n}$		$+\infty$
$g'(x)$		$+$	0	$-$	
g	0	$\nearrow$	$\frac{1}{ne}$	$\searrow$	0

The study of the monotonicity of the function g on $\mathbb{R}^+$ give us:

$\sup\limits_{x\in\mathbb{R}^+} g(x) = \sup\limits_{x\in\mathbb{R}^+} |f_n(x) - f(x)| = \dfrac{1}{ne}$. *We deduce that:*

$$\begin{aligned}
\lim_{n\longrightarrow+\infty} \sup_{x\in\mathbb{R}^+} |f_n(x) - f(x)| &= \lim_{n\longrightarrow+\infty} \frac{1}{ne} \\
&= 0 \\
&\Rightarrow f_n \rightrightarrows f \quad \forall x \in \mathbb{R}^+.
\end{aligned}$$

Theorem 4.2.1. *Let $(f_n)_n$ be a sequence of functions defined on E.*

$$\textit{If} \quad f_n \rightrightarrows f \Rightarrow f_n \longrightarrow f \quad \forall x \in E.$$

The converse is false.

Proof. We suppose that $(f_n)_n$ converges uniformly to f on E. Then, we have:

$$\begin{aligned}
f_n \rightrightarrows f \quad &\Rightarrow \lim_{n\longrightarrow+\infty} \sup_{x\in E} |f_n(x) - f(x)| = 0 \\
&\Rightarrow \lim_{n\longrightarrow+\infty} |f_n(x) - f(x)| = 0, \quad \forall x \in E \\
&\Rightarrow \lim_{n\longrightarrow+\infty} f_n(x) = f(x), \quad \forall x \in E \\
&\Rightarrow f_n \longrightarrow f; \quad \forall x \in E.
\end{aligned}$$

□

Theorem 4.2.2. (**Continuity and uniform convergence**)*: Let $(f_n)_n$ be a sequence of continuous functions defined on E.*

$$\textit{If} \quad f_n \rightrightarrows f \Rightarrow f \ \textit{ is continuous on } E.$$

Remark 4.2.1. *If $(f_n)_n$ are continuous and f is not continuous. Therefore: $(f_n)_n$ does not converges uniformly to f.*

Example 4.2.4. *Study the pointwise convergence and the uniform convergence of this sequence of functions:*

$$f_n(x) = \frac{1}{1+nx}; \quad n \geq 1 \quad \textit{and} \quad x \in \mathbb{R}^+.$$

Solution. *We will first study the pointwise convergence:*

1) $\lim\limits_{n\longrightarrow+\infty} f_n(x) = \begin{cases} 1 & si \quad x=0 \\ \\ 0 & si \quad x>0. \end{cases}$

$$Thus,\ f_n \longrightarrow f;\ f(x) = \begin{cases} 1 & si \quad x=0 \\ \\ 0 & si \quad x>0. \end{cases}$$

2) *Since the* $(f_n)_n$ *are continuous* $\forall n \in \mathbb{N}^*$ *and* f *is not continuous at* 0. *Hence* $(f_n)_n$ *does not converges uniformly to* f *on* $\mathbb{R}^+$.

Theorem 4.2.3. (**Integration and uniform convergence**)*: We set* $E=[a,b]$ *and* $(f_n)_n$ *are continuous defined on* E. *If* $f_n \rightrightarrows f;\ \forall x \in E$. *Then:*

$$\lim_{n\longrightarrow+\infty} \int_a^b f_n(x)\,dx = \int_a^b \lim_{n\longrightarrow+\infty} f_n(x)\,dx = \int_a^b f(x)\,dx.$$

Example 4.2.5. *Calculate the following expression using two methods:*

$$\lim_{n\longrightarrow+\infty} \int_0^1 \frac{x}{1+n^2x^2}\,dx.$$

Solution. *We can calculate the previous expression using two methods: directly and by uniform convergence.*
Method 01*(Direct):*

$$\begin{aligned} \int_0^1 \frac{x}{1+n^2x^2}\,dx &= \frac{1}{2n^2}\int_0^1 \frac{2n^2x}{1+n^2x^2}\,dx \\ &= \frac{1}{2n^2}\Big[\ln(1+n^2x^2)\Big]_0^1 = \frac{\ln(1+n^2)}{2n^2}. \end{aligned}$$

$$\Longrightarrow \lim_{n\longrightarrow+\infty} \int_0^1 \frac{x}{1+n^2x^2}\,dx = \lim_{n\longrightarrow+\infty} \frac{\ln(1+n^2)}{2n^2} = 0.$$

Method 02 *(The uniform convergence):*
Let $f_n(x) = \dfrac{x}{1+n^2x^2};\ n \geq 1$ *and* $x \in \mathbb{R}^+$.
a) *The pointwise convergence:*

$$\lim_{n\longrightarrow+\infty} f_n(x) = \begin{cases} 0 & if \quad x=0 \\ \\ 0 & if \quad x>0. \end{cases} \Longrightarrow f_n \longrightarrow f;\ f(x)=0\ \forall x \in \mathbb{R}^+.$$

b) *The uniform convergence:*

$$|f_n(x) - f(x)| = \left|\frac{x}{1+n^2x^2} - 0\right| = \frac{x}{1+n^2x^2}.$$

We set:

$$g(x) = \frac{x}{1+n^2x^2}; \quad x \in \mathbb{R}^+.$$

We know that g is continuous and differentiable on $\mathbb{R}^+$:

$$g'(x) = \frac{1+n^2x^2 - 2n^2x^2}{(1+n^2x^2)^2} = \frac{1-n^2x^2}{(1+n^2x^2)^2}.$$

$$g'(x) = 0 \Longleftrightarrow 1 - n^2x^2 = 0 \Longleftrightarrow x = \pm\frac{1}{n};\ n \geq 1.$$

x	$-\infty$		$-\frac{1}{n}$		*0*		$+\frac{1}{n}$		$+\infty$
$g'(x)$		$-$	0		$+$		0	$-$	
g		↘		↗	*0*	↗		↘	

The investigation of the monotonicity of the function g on $\mathbb{R}^+$ gives the following result:

$\sup\limits_{x\in\mathbb{R}^+} g(x) = \sup\limits_{x\in\mathbb{R}^+} |f_n(x) - f(x)| = \frac{1}{2n}$. *We conclude that:*

$$\begin{aligned}
\lim_{n\longrightarrow+\infty} \sup_{x\in\mathbb{R}^+} |f_n(x) - f(x)| &= \lim_{n\longrightarrow+\infty} \frac{1}{2n} \\
&= 0 \\
&\Rightarrow f_n \rightrightarrows f \quad \forall x \in \mathbb{R}^+.
\end{aligned}$$

Then, we can interchange the limit and the integral as follows:

$$\lim_{n\longrightarrow+\infty} \int_0^1 f_n(x)\,dx = \int_0^1 \lim_{n\longrightarrow+\infty} f_n(x)\,dx = \int_0^1 0\,dx = 0.$$

4.2.2 Series of Functions

Definition 4.2.5. *Let $(f_n)_n$ a sequence of functions. It is denoted by:*

$$\begin{aligned} S_n(x) &= f_0(x) + f_1(x) + \cdots \cdots + f_n(x) \\ &= \sum_{k=0}^{n} f_k(x). \end{aligned}$$

S_n: is called the partial sum of order n.

Definition 4.2.6. *A **series of functions** $\sum_{n=0}^{+\infty} f_n$ with $x \in E$ is a series of the form:*

$$\sum_{n=0}^{+\infty} f_n = f_0 + f_1 + \cdots \cdots + f_n + f_{n+1} + \cdots$$

It may converges for certain values of x and diverges for others. Let $D \subseteq E$ be the set of x in E for which the series converges. Then:

$$\begin{aligned} \sum_{n=0}^{+\infty} f_n &= f_0 + f_1 + \cdots \cdots + f_n + f_{n+1} + \cdots \\ &= \lim_{n \longrightarrow +\infty} S_n = S \quad \forall x \in D. \end{aligned}$$

D: The domain of convergence.

Notation 4.2.1. *Here are some commonly used notations:*
$\sum_{n=0}^{+\infty} f_n$, $\sum f_n$ *or* $\sum_{n \geq 0} f_n$*: it is a series of functions.*
$(S_n)_n$: is the sequence of functions of partial sums of the series $\sum f_n$.
f_n: is the $n^{th}-$term of the series of functions $\sum f_n$.
S: the function sum of the series of functions $\sum_{n=0}^{+\infty} f_n$ if it is pointwise converges.

The types of convergence

Definition 4.2.7. (**Pointwise Convergence**)*:*

$$\begin{aligned} \sum_{n=0}^{+\infty} f_n \text{ converges pointwise } &\Leftrightarrow (S_n)_n \text{ converges pointwise to } S; \quad \forall x \in D \\ &\Leftrightarrow S_n \longrightarrow S; \quad \forall x \in D. \end{aligned}$$

Example 4.2.6. *Study the pointwise convergence of the following series of functions:*

1) $\sum_{n=0}^{+\infty} x^n;\ x \in \mathbb{R}$, *2)* $\sum_{n=1}^{+\infty} \frac{1}{n^x};\ x \in \mathbb{R}$,

3) $\sum_{n=1}^{+\infty} \left(1+\frac{x}{n}\right)^{n^2};\ x \in \mathbb{R}$, *4)* $\sum_{n=1}^{+\infty} \frac{x^{2n}}{n!};\ x \in \mathbb{R}$.

Solution. *We fix* x *and study the series* $\sum f_n(x)$:

1) $\sum_{n=0}^{+\infty} x^n$*:*

It is a geometric series with ratio $q = x$*:*

$$\sum_{n=0}^{+\infty} x^n \begin{cases} \textit{converges} & \textit{if} \quad |x| < 1, \\ \textit{diverges} & \textit{if} \quad |x| \geq 1. \end{cases} \Rightarrow D =]-1, 1[.$$

2) $\sum_{n=1}^{+\infty} \frac{1}{n^x}$*:*

It is a Riemann series with; $\alpha = x$*:*

$$\sum_{n=1}^{+\infty} \frac{1}{n^x} \begin{cases} \textit{converges} & \textit{if} \quad x > 1, \\ \textit{diverges} & \textit{if} \quad x \leq 1. \end{cases} \Rightarrow D =]1, +\infty[.$$

3) $\sum_{n=1}^{+\infty} \left(1+\frac{x}{n}\right)^{n^2}$*:*

$$\sqrt[n]{u_n} = \left(1+\frac{x}{n}\right)^n \Longrightarrow \lim_{n \longrightarrow +\infty} \sqrt[n]{u_n} = \lim_{n \longrightarrow +\infty} \left(1+\frac{x}{n}\right)^n = e^x.$$

First case. *If* $l < 1$*:*

Si $l < 1 \Longleftrightarrow e^x < 1 \Longleftrightarrow x < 0 \Longleftrightarrow \sum_{n=1}^{+\infty} f_n(x)$ *converges by the Root Test.*

Second case. *If* $l > 1$*:*

If $l > 1 \Longleftrightarrow x > 0 \Longleftrightarrow \sum_{n=1}^{+\infty} f_n(x)$ *diverges by the Root Test.*

Third case. *If* $l = 1$*:*

$l = 1 \Longrightarrow e^x = 1 \Longrightarrow x = 0 \Longrightarrow f_n(0) = 1^{n^2} = 1.$

Then it diverges because $\lim_{n \longrightarrow +\infty} 1 = 1 \neq 0.$

$$\Longrightarrow \sum_{n=1}^{+\infty} \left(1+\frac{x}{n}\right)^{n^2} \begin{cases} \textit{converges} & \textit{if} \quad x < 0, \\ \textit{diverges} & \textit{if} \quad x \geq 0. \end{cases} \Rightarrow D =]-\infty, 0[.$$

4) $\displaystyle\sum_{n=0}^{+\infty} \frac{x^{2n}}{n!}$*:*
First case. *If $x = 0$:*
$\displaystyle\sum_{n=0}^{+\infty} f_n(0) = \sum_{n=0}^{+\infty} 0 = 0$. *Therefore the series converges for $x = 0$.*
Second case. *If $x \neq 0$:*
We can apply the Ratio Test. We have: $u_{n+1} = \dfrac{x^{2n+2}}{(n+1)!} = \dfrac{x^{2n} \cdot x^2}{n!(n+1)}$.

$$\lim_{n \longrightarrow +\infty} \frac{u_{n+1}}{u_n} = \lim_{n \longrightarrow +\infty} \frac{x^{2n} \cdot x^2}{n! \cdot (n+1)} \times \frac{n!}{x^{2n}} = \lim_{n \longrightarrow +\infty} \frac{x^2}{n+1} = 0 < 1.$$

Thus, $\displaystyle\sum_{n=0}^{+\infty} \frac{x^{2n}}{n!}$ *converges; $\forall x \in \mathbb{R} \Rightarrow D = \mathbb{R}$.*

Definition 4.2.8. (**The uniform convergence**)*:*

$$\begin{aligned} \sum_{n=0}^{+\infty} f_n \text{ converges uniformly } &\Leftrightarrow (S_n)_n \text{ converges uniformly to } S \\ &\Leftrightarrow S_n \rightrightarrows S; \quad \forall x \in D. \end{aligned}$$

Remark 4.2.2. *Let* $\displaystyle\sum_{n=0}^{+\infty} f_n$ *be a series of functions defined on D.*
If $\displaystyle\sum_{n=0}^{+\infty} f_n$ *converges uniformly* $\Rightarrow \displaystyle\sum_{n=0}^{+\infty} f_n$ *converges pointwise on D.*
The converse is false.

Example 4.2.7. *Investigate the uniform convergence of the following series of functions:*
1) $\displaystyle\sum_{n=1}^{+\infty} \frac{1}{(n+x)(n+x+1)}$; $x \in \mathbb{R}^+$, ***2)*** $\displaystyle\sum_{n=1}^{+\infty} \frac{(-1)^n}{n+x^2}$; $x \in \mathbb{R}^+$.

Solution. *The goal is to prove the uniform convergence of S_n to S:*

1) $\displaystyle\sum_{n=1}^{+\infty} \frac{1}{(n+x)(n+x+1)}$*:*

- *Soit* $f_n(x) = \dfrac{1}{(n+x)(n+x+1)}$; $n \geq 1$. *We can write the term f_n in the following form:*

$$f_n(x) = \frac{1}{(n+x)(n+x+1)} = \frac{a}{n+x} + \frac{b}{n+x+1} = \frac{(a+b)n + (a+b)x + a}{(n+x)(n+x+1)}.$$

$$\Longrightarrow \begin{cases} a+b=0 \\ \\ a=1 \end{cases} \Longrightarrow a=1 \ et \ b=-1.$$

Thus, we can easily see that: $f_n(x)=\dfrac{1}{n+x}-\dfrac{1}{n+x+1}; \quad n\geq 1.$

1-a) *The partial sum* S_n*:*

$$\begin{aligned} S_n(x) &= f_1(x)+f_2(x)+\cdots\cdots+f_{n-1}(x)+f_n(x) \\ &= \left(\frac{1}{1+x}-\frac{1}{2+x}\right)+\left(\frac{1}{2+x}-\frac{1}{3+x}\right)+\cdots\cdots \\ &\quad \cdots\cdots+\left(\frac{1}{n+x-1}-\frac{1}{n+x}\right)+\left(\frac{1}{n+x}-\frac{1}{n+x+1}\right) \\ &= \frac{1}{1+x}-\frac{1}{n+x+1}. \end{aligned}$$

1-b) *the function sum* S *of the series:*

$$S(x) = \lim_{n\longrightarrow+\infty} S_n(x)=\lim_{n\longrightarrow+\infty}\left(\frac{1}{1+x}-\frac{1}{n+x+1}\right)=\frac{1}{1+x}.$$

Hence, $S_n\longrightarrow S$*;* $S(x)=\dfrac{1}{1+x}\ \forall x\in\mathbb{R}^+.$

1-c) *The uniform convergence:*

$|S_n(x)-S(x)|=\left|\dfrac{1}{1+x}-\dfrac{1}{n+x+1}-\dfrac{1}{1+x}\right|=\dfrac{1}{n+x+1}.$

We set:

$$g(x)=\frac{1}{n+x+1}; \qquad x\in\mathbb{R}^+.$$

We know that g *is continuous and differentiable on* $\mathbb{R}^+$*:*

$$g'(x)=-\frac{1}{(n+x+1)^2}\Rightarrow g'(x)<0\Rightarrow g\searrow.$$

x	$0 \qquad +\infty$
$g'(x)$	$-$
g	$\searrow$

The study of the monotonicity of the function g on $\mathbb{R}^+$ give us:

$\sup\limits_{x\in\mathbb{R}^+} g(x) = \sup\limits_{x\in\mathbb{R}^+} |S_n(x) - S(x)| = \dfrac{1}{n+1}$. *We conclude that:*

$$\begin{aligned}\lim_{n\longrightarrow+\infty} \sup_{x\in\mathbb{R}^+} |S_n(x) - S(x)| &= \lim_{n\longrightarrow+\infty} \frac{1}{n+1} = 0\\ &\Rightarrow S_n \rightrightarrows S \quad \forall x \in \mathbb{R}^+.\end{aligned}$$

Then, $\displaystyle\sum_{n=1}^{+\infty} \frac{1}{(n+x)(n+x+1)}$ *converges uniformly on* $\mathbb{R}^+$.

2) $\displaystyle\sum_{n=1}^{+\infty} \frac{(-1)^n}{n+x^2}$*:*

In the general case, to prove the uniform convergence of a series of functions that satisfies the conditions of the Leibniz's Theorem, it is necessary to use a more significant result, which is:

$$|S_n - S| \leq b_{n+1}; \quad \forall x \in D \quad \textit{and} \quad \forall n \in \mathbb{N}.$$

2-a) *The pointwise convergence:*

We have : $f_n(x) = \dfrac{(-1)^n}{n+x^2} \Rightarrow b_n(x) = \dfrac{1}{n+x^2} > 0 \quad \forall n \geq 1.$

Since $\begin{cases} \lim\limits_{n\longrightarrow+\infty} b_n(x) = 0 \ \checkmark \\ \\ (b_n)_n \searrow \checkmark \end{cases} \Longrightarrow \displaystyle\sum_{n=1}^{+\infty} \frac{(-1)^n}{n+x^2}$ *converges pointwise on* $\mathbb{R}^+$.

2-b) *The uniform convergence:*

Since $\displaystyle\sum_{n=1}^{+\infty} \frac{(-1)^n}{n+x^2}$ *satisfies the conditions of Leibniz's Theorem, therefore:*

$$|S_n(x) - S(x)| \leq b_{n+1}(x); \quad \forall x \in \mathbb{R}^+ \quad \textit{and} \quad \forall n \geq 1. \quad \textit{So}:$$

$$\begin{aligned}|S_n(x) - S(x)| \leq \frac{1}{n+x^2+1} \quad &\Rightarrow \sup_{x\in\mathbb{R}^+} |S_n(x) - S(x)| \leq \sup_{x\in\mathbb{R}^+} \frac{1}{n+x^2+1}\\ &\Rightarrow \sup_{x\in\mathbb{R}^+} |S_n(x) - S(x)| \leq \frac{1}{n+1}\\ &\Rightarrow \lim_{n\longrightarrow+\infty} \sup_{x\in\mathbb{R}^+} |S_n(x) - S(x)| \leq \lim_{n\longrightarrow+\infty} \frac{1}{n+1} = 0\\ &\Rightarrow \lim_{n\longrightarrow+\infty} \sup_{x\in\mathbb{R}^+} |S_n(x) - S(x)| = 0\\ &\Rightarrow S_n \rightrightarrows S \quad \forall x \in \mathbb{R}^+.\end{aligned}$$

Thus, $\displaystyle\sum_{n=1}^{+\infty} \frac{(-1)^n}{n+x^2}$ *converges uniformly on* $\mathbb{R}^+$.

Definition 4.2.9. (**Normal convergence**)*:*
Let $\sum_{n=0}^{+\infty} f_n$ *be a series of functions defined on* D*.*

$$\textit{If}\quad \sum_{n=0}^{+\infty} \sup_{x\in D} |f_n(x)| \quad \textit{converges} \quad \Leftrightarrow \quad \sum_{n=0}^{+\infty} f_n \quad \textit{converges normally}, \ \forall x \in D.$$

Practical method: Let $\sum_{n=0}^{+\infty} f_n$ be a series of functions defined on D. If there exists an infinite series with positive term a_n convergent and verifies:

$$|f_n(x)| \leq a_n; \quad \forall x \in D \quad \text{and} \quad \forall n \in \mathbb{N}.$$

Then, the series $\sum_{n=0}^{+\infty} f_n$ converges normally on D.
a_n: does not depend of x.

Remark 4.2.3. *Let* $\sum_{n=0}^{+\infty} f_n$ *be a series of functions defined on* D*.*
If $\sum_{n=0}^{+\infty} f_n$ *converges normally* $\Rightarrow \sum_{n=0}^{+\infty} f_n$ *converges uniformly on* D*.*
The converse is false.

Example 4.2.8. *Study the normal convergence of the following series of functions:*
1) $\sum_{n=1}^{+\infty} \frac{x}{1+n^4x^2}$; $x \in \mathbb{R}^+$, ***2)*** $\sum_{n=0}^{+\infty} \frac{\sin(nx)}{2^n}$; $x \in \mathbb{R}$.

Solution. ***1)*** $\sum_{n=1}^{+\infty} \frac{x}{1+n^4x^2}$*:*
Let $f_n(x) = \frac{x}{1+n^4x^2}$; $n \geq 1$ *and* $x \in \mathbb{R}^+$.
1-a) *the limit of the* $n^{th}-$*term:*

$$\lim_{n\longrightarrow+\infty} f_n(x) = \begin{cases} 0 & \textit{if} \quad x = 0 \\ 0 & \textit{if} \quad x > 0. \end{cases}$$

1-b) *Calculating* $\sup_{x\in\mathbb{R}^+} |f_n(x)|$*:*

$$|f_n(x)| = \left| \frac{x}{1+n^4x^2} - 0 \right| = \frac{x}{1+n^4x^2}.$$

We set:

$$g(x) = \frac{x}{1+n^4x^2}; \qquad x \in \mathbb{R}^+.$$

We know that g *is continuous and differentiable on* $\mathbb{R}^+$*:*

$$g'(x) = \frac{1+n^4x^2-2n^4x^2}{(1+n^4x^2)^2} = \frac{1-n^4x^2}{(1+n^4x^2)^2}.$$

$$g'(x) = 0 \iff 1-n^4x^2 = 0 \iff x = \pm\frac{1}{n^2};\ n \geq 1.$$

x	$-\infty$		$-\frac{1}{n^2}$		0		$+\frac{1}{n^2}$		$+\infty$
$g'(x)$		$-$	0		$+$		0	$-$	
g		$\searrow$		$\nearrow$	0	$\nearrow$		$\searrow$	

The investigation of the monotonicity of the function g *on* $\mathbb{R}^+$ *give us:*

$$\sup_{x\in\mathbb{R}^+} g(x) = \sup_{x\in\mathbb{R}^+} |f_n(x)| = \frac{1}{2n^2}.$$

Since $\displaystyle\sum_{n=1}^{+\infty} \frac{1}{n^2}$ *is convergent (it is a Riemann series:* $\alpha = 2 > 1$*), we deduce that* $\displaystyle\sum_{n=1}^{+\infty} \frac{x}{1+n^4x^2}$ *converges normally on* $\mathbb{R}^+$*.*

2) $\displaystyle\sum_{n=0}^{+\infty} \frac{\sin(nx)}{2^n}$*:*

Let $f_n(x) = \dfrac{\sin(nx)}{2^n}$*;* $n \in \mathbb{N}$ *and* $x \in \mathbb{R}$*.*

We know that: $|\sin(nx)| \leq 1$*. Then:*

$$\left|\frac{\sin(nx)}{2^n}\right| \leq \frac{1}{2^n}; \quad \forall x \in \mathbb{R} \quad \textit{and} \quad \forall n \in \mathbb{N}.$$

$\displaystyle\sum_{n=0}^{+\infty} \left(\frac{1}{2}\right)^n$ *converges (it is a geometric series with ratio* $0 < q = 1/2 < 1$*)*

thus, the series $\displaystyle\sum_{n=0}^{+\infty} \sup_{x\in\mathbb{R}} \left|\frac{\sin(nx)}{2^n}\right|$ *converges by the Comparison Test.*

We conclude that $\displaystyle\sum_{n=0}^{+\infty} \frac{\sin(nx)}{2^n}$ *converges normally on* $\mathbb{R}$*.*

Definition 4.2.10. (**Absolute Convergence**)*:*
The series of functions $\sum_{n=0}^{+\infty} f_n$ *converges absolutely if the corresponding series of absolute values* $\sum_{n=0}^{+\infty} |f_n|$ *converges pointwise.*

Example 4.2.9. *Investigate the absolute convergence of the following series of functions:*

1) $\sum_{n=1}^{+\infty} \frac{(-1)^n}{n\sqrt{n}+x}$*;* $x \in \mathbb{R}^+$*,* ***2)*** $\sum_{n=0}^{+\infty} \frac{\cos(nx)}{3^n}$*;* $x \in \mathbb{R}$*.*

Solution. *We study the corresponding series of absolute values* $\sum |f_n|$*:*

1) $\sum_{n=1}^{+\infty} \frac{(-1)^n}{n\sqrt{n}+x}$*:*

$|f_n(x)| = \left|\frac{(-1)^n}{n\sqrt{n}+x}\right| = \frac{1}{n\sqrt{n}+x}$*. We will get:*

$|f_n(x)| = \frac{1}{n\sqrt{n}+x} \sim v_n = \frac{1}{n\sqrt{n}}$ *because* $\lim_{n\longrightarrow+\infty} \frac{|f_n(x)|}{v_n} = 1$*.*

$\sum_{n=1}^{+\infty} \frac{1}{n^{3/2}}$ *is convergent (it is a Riemann series:* $\alpha = 3/2 > 1$*), we deduce that* $\sum_{n=1}^{+\infty} \left|\frac{(-1)^n}{n\sqrt{n}+x}\right|$ *converges pointwise by the Limit Comparison Test.*

$$\Rightarrow \sum_{n=1}^{+\infty} \frac{(-1)^n}{n\sqrt{n}+x} \text{ converges absolutely on } \mathbb{R}^+.$$

$$\Rightarrow \sum_{n=1}^{+\infty} \frac{(-1)^n}{n\sqrt{n}+x} \text{ converges pointwise on } \mathbb{R}^+.$$

2) $\sum_{n=0}^{+\infty} \frac{\cos(nx)}{3^n}$*:*

$$|f_n(x)| = \left|\frac{\cos(nx)}{3^n}\right| = \frac{|\cos(nx)|}{3^n}$$

We will obtain:

$$\frac{|\cos(nx)|}{3^n} \leq \frac{1}{3^n} = \left(\frac{1}{3}\right)^n \quad \forall n \in \mathbb{N}.$$

$\sum_{n=0}^{+\infty} \left(\frac{1}{3}\right)^n$ *converges (it is a geometric series with ratio* $0 < q = 1/3 < 1$*).*

Then, $\displaystyle\sum_{n=0}^{+\infty}\left|\frac{\cos(nx)}{3^n}\right|$ *converges pointwise by the Comparison Test.*

$\Rightarrow \displaystyle\sum_{n=0}^{+\infty}\frac{\cos(nx)}{3^n}$ *converges absolutely on* $\mathbb{R}$*.*

$\Rightarrow \displaystyle\sum_{n=0}^{+\infty}\frac{\cos(nx)}{3^n}$ *converges pointwise on* $\mathbb{R}$*.*

Remark 4.2.4. *Let* $\displaystyle\sum_{n=0}^{+\infty} f_n$ *be a series of functions defined on* D*.*

If $\displaystyle\sum_{n=0}^{+\infty} f_n$ *converges normally* $\Rightarrow$ $\displaystyle\sum_{n=0}^{+\infty} f_n$ *converges absolutely on* D*.*
The converse is false.

Properties of the sum of a series of functions related to the uniform convergence

Theorem 4.2.4. (**The Continuity**)*:*
Let $\displaystyle\sum_{n=0}^{+\infty} f_n$ *be a series of continuous functions defined on* E*, If this series converges uniformly, then its sum* S *is continuous on* E*:*

$$\sum_{n=0}^{+\infty} f_n = \lim_{n\longrightarrow+\infty} S_n = S, \quad \forall x \in E.$$

Theorem 4.2.5. (**The integration**)*:*
Similarly, one often wants to exchange integrals and limit processes. For the Riemann integral, this can be done if uniform convergence is assumed.

Let $\displaystyle\sum_{n=0}^{+\infty} f_n$ *be a series of continuous functions defined on* $E=[a,b]$*. If this series* $\displaystyle\sum_{n=0}^{+\infty} f_n$ *converges uniformly on* E*, we have:*

$$\int_a^b \sum_{n=0}^{+\infty} f_n(x)\,dx = \sum_{n=0}^{+\infty}\int_a^b f_n(x)\,dx = \int_a^b S(x)\,dx, \quad \forall x \in E.$$

Remark 4.2.5. *To obtain the previous properties, it is sufficient to prove:*

1) *The normal convergence.*

2) *The uniform convergence of the series* $\displaystyle\sum_{n=0}^{+\infty} f_n$*.*

Example 4.2.10. *We consider the following series of functions:*

$$\sum_{n=1}^{+\infty} \frac{\sin nx}{n^2}; \qquad \forall x \in \mathbb{R}.$$

1) *Study the normal convergence of this series on* $\mathbb{R}$*.*
2) *Show that the sum of the series, denoted by* S *is a continuous function.*
3) *Show that:*

$$\int_0^{\pi} S(x)\, dx = \sum_{n=1}^{+\infty} \frac{2}{(2n-1)^3}.$$

Solution. *We first prove the normal convergence:*
1) $\displaystyle\sum_{n=1}^{+\infty} \frac{\sin(nx)}{n^2}$*:*
Let $f_n(x) = \dfrac{\sin(nx)}{n^2}$*;* $n \in \mathbb{N}^*$ *with* $x \in \mathbb{R}$*.*
We know that: $|\sin(nx)| \leq 1$*. Then:*

$$\left|\frac{\sin(nx)}{n^2}\right| \leq \frac{1}{n^2}; \quad \forall x \in \mathbb{R} \quad \textit{and} \quad \forall n \in \mathbb{N}^*.$$

Since $\displaystyle\sum_{n=1}^{+\infty} \frac{1}{n^2}$ *is convergent (it is a Riemann series:* $\alpha = 2 > 1$*), we deduce that* $\displaystyle\sum_{n=1}^{+\infty} \frac{\sin(nx)}{n^2}$ *converges normally on* $\mathbb{R}$*.*
2) *The continuity of the function sum* S*:*
The $(f_n)_n$ *are continuous for all* $n \in \mathbb{N}^*$*, and they converge uniformly on* $\mathbb{R}$*, Therefore, the function sum* S *is continuous on* $\mathbb{R}$*;*

$$\sum_{n=1}^{+\infty} \frac{\sin nx}{n^2} = S(x) \qquad \forall x \in \mathbb{R}.$$

3) *The integral of* S*:*
The convergence being uniform on $[0, \pi]$ *and the* $(f_n)_n$ *are continuous, then:*
$\displaystyle\int_0^{\pi} S(x)\, dx = \int_0^{\pi} \sum_{n=1}^{+\infty} \frac{\sin nx}{n^2}\, dx = \sum_{n=1}^{+\infty} \int_0^{\pi} \frac{\sin nx}{n^2}\, dx$*. we calculate:*

$$\begin{aligned}
\int_0^{\pi} \frac{\sin nx}{n^2}\, dx &= \frac{1}{n^2} \int_0^{\pi} \sin(nx)\, dx = \frac{1}{n^2}\left[-\frac{\cos nx}{n}\right]_0^{\pi} \\
&= \frac{1}{n^3}\left[-\cos(n\pi) + 1\right] \\
&= \frac{1}{n^3}\left[1 - (-1)^n\right].
\end{aligned}$$

Depending on whether n is even or odd $1-(-1)^n$ *is either zero or equals* 2*, We will split the sum into two parts, indeed:*

$$\begin{aligned}\sum_{n=1}^{+\infty}\int_0^{\pi}\frac{\sin nx}{n^2}\,dx &= \sum_{n=1}^{+\infty}\frac{(1-(-1)^n)}{n^3}=\sum_{p=1}^{+\infty}\frac{\left(1-(-1)^{2p}\right)}{(2p)^3}+\sum_{p=0}^{+\infty}\frac{\left(1-(-1)^{2p+1}\right)}{(2p+1)^3}\\ &= \sum_{p=0}^{+\infty}\frac{2}{(2p+1)^3}.\end{aligned}$$

Then, we set: $n=p+1$*, hence:*

$$\int_0^{\pi}S(x)\,dx=\sum_{p=0}^{+\infty}\frac{2}{(2p+1)^3}=\sum_{n=1}^{+\infty}\frac{2}{(2n-1)^3}.$$

4.3 Power series and Fourier series

4.3.1 Power series

Definition 4.3.1. *A real power series is a series of the form*

$$\sum_{n=0}^{+\infty}a_nx^n.$$

a_n*: called the coefficient of the series and* x *is a real variable.*

Definition 4.3.2. *let* $\sum_{n=0}^{+\infty}a_nx^n$ *A real power series;* $x\in\mathbb{R}$.

$D=\left\{x\in\mathbb{R}/\sum_{n=0}^{+\infty}a_nx^n \ \ \text{converges}\right\}$ *is called the domain of convergence of the previous power series.*

The Radius of Convergence R

Theorem 4.3.1. *Let* $\sum_{n=0}^{+\infty}a_nx^n$ *be the power series. There exists a unique positive real number, finite or infinite* R *that satisfies the following properties:*

$$\begin{cases} Si\ \ |x|<R, & \text{the series converges}\\ \\ Si\ \ |x|>R, & \text{the series diverges}\\ \\ Si\ \ |x|=R, & \text{nothing can be concluded.}\end{cases}$$

For the third case, we need to study the convergence of the series $\sum_{n=0}^{+\infty} a_n x^n$ *for* $x = +R$ *and* $x = -R$.

Definition 4.3.3. **(Determination of the radius of convergence)***:*
Let $\sum_{n=0}^{+\infty} a_n x^n$ *be the power series.*

$$\text{If} \begin{cases} \lim_{n \longrightarrow +\infty} \left| \frac{a_{n+1}}{a_n} \right| = l \\ \qquad \text{or} \\ \lim_{n \longrightarrow +\infty} \sqrt[n]{|a_n|} = l \end{cases} \Longrightarrow R = \frac{1}{l}.$$

Theorem 4.3.2. *Let* R *be the radius of convergence of* $\sum_{n=0}^{+\infty} a_n x^n$*, then:*

1) *If* $R = 0 \Rightarrow D = \{0\}$,

2) *If* $R = +\infty \Rightarrow D = \mathbb{R}$.

Example 4.3.1. *determine the radius of convergence* R *for the following power series:*

1) $\sum_{n=0}^{+\infty} \frac{x^n}{2^n}$, ***2)*** $\sum_{n=0}^{+\infty} \frac{x^n}{(n+1)!}$, ***3)*** $\sum_{n=0}^{+\infty} n! \cdot x^n$,
4) $\sum_{n=1}^{+\infty} \frac{(-1)^n}{n} \cdot x^n$, ***5)*** $\sum_{n=2}^{+\infty} \frac{\ln n}{n^3} \cdot x^n$, ***6)*** $\sum_{n=2}^{+\infty} \frac{\ln n}{\sqrt{n}} \cdot x^n$.

Solution. *The goal is to calculate the radius of convergence* R*:*
1) $\sum_{n=0}^{+\infty} \frac{x^n}{2^n}$*:*

$$\frac{1}{R} = \lim_{n \longrightarrow +\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \longrightarrow +\infty} \frac{2^n}{2^{n+1}} = \frac{1}{2} \Longrightarrow R = 2.$$

2) $\sum_{n=0}^{+\infty} \frac{x^n}{(n+1)!}$*:*

$$\lim_{n \longrightarrow +\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \longrightarrow +\infty} \frac{(n+1)!}{(n+2)(n+1)!} = \lim_{n \longrightarrow +\infty} \frac{1}{n+2} = 0 \Longrightarrow R = +\infty.$$

3) $\displaystyle\sum_{n=0}^{+\infty} n! \cdot x^n$*:*

$$\lim_{n\longrightarrow+\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\longrightarrow+\infty}\frac{n!(n+1)}{n!} = \lim_{n\longrightarrow+\infty}(n+1) = +\infty \Longrightarrow R = 0.$$

4) $\displaystyle\sum_{n=1}^{+\infty} \frac{(-1)^n}{n} \cdot x^n$*:*

$$\frac{1}{R} = \lim_{n\longrightarrow+\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\longrightarrow+\infty}\frac{n}{n+1} = 1 \Longrightarrow R = 1.$$

5) $\displaystyle\sum_{n=2}^{+\infty} \frac{\ln n}{n^3} \cdot x^n$*:*

$$\lim_{n\longrightarrow+\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\longrightarrow+\infty}\frac{\ln(n+1)}{\ln n} \times \left(\frac{n}{n+1}\right)^3 = 1 \Longrightarrow R = 1.$$

6) $\displaystyle\sum_{n=2}^{+\infty} \frac{\ln n}{\sqrt{n}} \cdot x^n$*:*

$$\lim_{n\longrightarrow+\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\longrightarrow+\infty}\frac{\ln(n+1)}{\ln n} \times \sqrt{\frac{n}{n+1}} = 1 \Longrightarrow R = 1.$$

Properties of power series

Proposition 4.3.1. *Let* $\displaystyle\sum_{n=0}^{+\infty} a_n x^n$ *be a power series with a radius of convergence* $R > 0$ *and let:*
$f :]-R, +R[\to \mathbb{R}$

$$x \mapsto f(x) = \sum_{n=0}^{+\infty} a_n x^n. \text{ Therefore:}$$

1) *f is continuous on* $]-R, +R[$.

2) *The series* $\displaystyle\sum_{n=1}^{+\infty} n a_n x^{n-1}$ *and* $\displaystyle\sum_{n=0}^{+\infty} a_n \frac{x^{n+1}}{n+1}$ *(obtained by differentiating and integrating the series* $\displaystyle\sum_{n=0}^{+\infty} a_n x^n$*) have the same radius R as the series* $\displaystyle\sum_{n=0}^{+\infty} a_n x^n$.

3) $f'(x) = \displaystyle\sum_{n=1}^{+\infty} n a_n x^{n-1}$ *and* $\displaystyle\int_0^x f(t)\,dt = \sum_{n=1}^{+\infty} a_n \frac{x^{n+1}}{n+1}$*;* $\forall x \in]-R, +R[$.

Example 4.3.2. *Calculate the sum of the following power series on their domains of convergence:*

1) $\sum_{n=0}^{+\infty} x^n$, ***2)*** $\sum_{n=1}^{+\infty} nx^{n-1}$, ***3)*** $\sum_{n=0}^{+\infty} \frac{x^{n+1}}{n+1}$,

Solution. *We will calculate the function sum S on the domain of convergence:*

1) $\sum_{n=0}^{+\infty} x^n$*:*

It is a geometric series with ratio $q = x$ that converges $\forall x \in]-1, +1[$.

$$\sum_{n=0}^{+\infty} x^n = \frac{1}{1-x}; \quad \forall |x| < 1.$$

2) $\sum_{n=1}^{+\infty} nx^{n-1}$*:*

We know that: $\sum_{n=0}^{+\infty} x^n = \frac{1}{1-x}$ $\forall x \in]-1, +1[$. *Thus:*

$$\frac{d}{dx}\left(\sum_{n=0}^{+\infty} x^n\right) = \frac{d}{dx}\left(\frac{1}{1-x}\right) \Rightarrow \sum_{n=0}^{+\infty} \frac{d}{dx}(x^n) = \sum_{n=1}^{+\infty} nx^{n-1} = \frac{1}{(1-x)^2}.$$

3) $\sum_{n=0}^{+\infty} \frac{x^{n+1}}{n+1}$*:*

$$\int_0^x \sum_{n=0}^{+\infty} t^n \, dt = \int_0^x \frac{1}{1-t} \, dt \Rightarrow \sum_{n=0}^{+\infty} \int_0^x t^n \, dt = \sum_{n=0}^{+\infty} \frac{x^{n+1}}{n+1} = -\ln(1-x).$$

Proposition 4.3.2. *Let* $\sum_{n=0}^{+\infty} a_n x^n$ *be a power series with a radius of convergence $R > 0$ and let: $f :]-R, +R[\to \mathbb{R}$ the function defined by:* $f(x) = \sum_{n=0}^{+\infty} a_n x^n$. *Then:*

$$f(x) = \sum_{n=0}^{+\infty} \frac{f^{(n)}(0)}{n!} x^n; \quad \forall x \in]-R, +R[.$$

Power series expansion

Definition 4.3.4. *A function f has a power series expansion if there exists $R > 0$ such that:*

$$f(x) = \sum_{n=0}^{+\infty} a_n x^n; \quad \forall x \in]-R, +R[.$$

Example 4.3.3. *Expand the following functions into power series:*

1) $f(x) = \dfrac{1}{1+x}$, ***2)*** $f(x) = \dfrac{1}{2-3x}$, ***3)*** $f(x) = e^x$,
4) $f(x) = \sin x$, ***5)*** $f(x) = \cos x$, ***6)*** $f(x) = \cosh x$.

Solution. ***1)*** $f(x) = \dfrac{1}{1+x}$*:*
Let $y \in]-R, +R[$. We know that:

$$\frac{1}{1-y} = \sum_{n=0}^{+\infty} y^n, \qquad \text{if } |y| < R.$$

$$\frac{1}{1+x} = \frac{1}{1-(-x)} \overset{y=-x}{=} \sum_{n=0}^{+\infty} (-x)^n \qquad \text{if} \quad |-x| < 1$$

$$\Longrightarrow f(x) = \sum_{n=0}^{+\infty} (-1)^n \cdot x^n \qquad \text{if} \quad |x| < 1.$$

2) $f(x) = \dfrac{1}{2-3x}$*:*

$$\frac{1}{2-3x} = \frac{1}{2}\left(\frac{1}{1-\dfrac{3x}{2}}\right) \overset{y=\frac{3x}{2}}{=} \frac{1}{2}\sum_{n=0}^{+\infty}\left(\frac{3x}{2}\right)^n \qquad \text{if} \quad \left|\frac{3x}{2}\right| < 1$$

$$\Longrightarrow f(x) = \sum_{n=0}^{+\infty} \frac{3^n \cdot x^n}{2^{n+1}} \qquad \text{if} \quad |x| < \frac{2}{3}.$$

3) $f(x) = e^x$*:*
We can apply here the **Proposition** *4.3.2. We obtain the following:*
Since: $f^{(n)}(x) = e^x \Rightarrow f^{(n)}(0) = 1$*. Hence:*

$$e^x = \sum_{n=0}^{+\infty} \frac{x^n}{n!}; \quad \forall x \in \mathbb{R}.$$

3) $\sin x$ *et* $\cos x$*:*
We will use this Euler's formula:

$$e^{ix} = \cos x + i \sin x; \quad \forall x \in \mathbb{R}.$$

Then:

$$\begin{aligned}
e^{ix} &= \sum_{n=0}^{+\infty} \frac{(ix)^n}{n!} = \sum_{n=0}^{+\infty} \frac{i^n}{n!} x^n \\
&= \sum_{p=0}^{+\infty} \frac{i^{2p}}{(2p)!} x^{2p} + \sum_{p=0}^{+\infty} \frac{i^{2p+1}}{(2p+1)!} x^{2p+1} \\
&= \sum_{p=0}^{+\infty} \frac{(-1)^p}{(2p)!} x^{2p} + i \sum_{p=0}^{+\infty} \frac{(-1)^p}{(2p+1)!} x^{2p+1} \\
&= \sum_{n=0}^{+\infty} \frac{(-1)^n}{(2n)!} x^{2n} + i \sum_{n=0}^{+\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}.
\end{aligned}$$

Thus

$$\begin{cases}
\cos x = \displaystyle\sum_{n=0}^{+\infty} \frac{(-1)^n}{(2n)!} x^{2n} \\
\qquad\quad \textit{and} \\
\sin x = \displaystyle\sum_{n=0}^{+\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}.
\end{cases}$$

4) $\sinh x$ *and* $\cosh x$*:*
We can expand these functions using the exponential function:

$$\begin{aligned}
\cosh x &= \frac{1}{2}\left(\sum_{n=0}^{+\infty} \frac{1}{n!} x^n + \sum_{n=0}^{+\infty} \frac{(-1)^n}{n!} x^n \right) = \frac{1}{2} \sum_{n=0}^{+\infty} \frac{1+(-1)^n}{n!} x^n \\
&= \frac{1}{2}\left(\sum_{p=0}^{+\infty} \frac{1+(-1)^{2p}}{(2p)!} x^{2p} + \sum_{p=0}^{+\infty} \frac{1+(-1)^{2p+1}}{(2p+1)!} x^{2p+1} \right) \\
&= \sum_{p=0}^{+\infty} \frac{x^{2p}}{(2p)!} = \sum_{n=0}^{+\infty} \frac{x^{2n}}{(2n)!}.
\end{aligned}$$

$$\begin{aligned}
\sinh x &= \frac{1}{2}\left(\sum_{n=0}^{+\infty} \frac{1}{n!} x^n - \sum_{n=0}^{+\infty} \frac{(-1)^n}{n!} x^n \right) = \frac{1}{2} \sum_{n=0}^{+\infty} \frac{1-(-1)^n}{n!} x^n \\
&= \frac{1}{2}\left(\sum_{p=0}^{+\infty} \frac{1-(-1)^{2p}}{(2p)!} x^{2p} + \sum_{p=0}^{+\infty} \frac{1-(-1)^{2p+1}}{(2p+1)!} x^{2p+1} \right) \\
&= \sum_{p=0}^{+\infty} \frac{x^{2p+1}}{(2p+1)!} = \sum_{n=0}^{+\infty} \frac{x^{2n+1}}{(2n+1)!}.
\end{aligned}$$

Series Solutions of Differential Equations

Power series solutions of differential equations express the solution as an infinite sum, typically around a point. The coefficients of the series are determined by substituting the series into the differential equation and solving for them. For this purpose, we need some background knowledge.

Remark 4.3.1. *Let* $x \in]-R, +R[$.
If $\sum_{n=0}^{+\infty} a_n x^n = \sum_{n=0}^{+\infty} b_n x^n \Longrightarrow a_n = b_n;\ \forall n \in \mathbb{N}$.
In particular, if $\sum_{n=0}^{+\infty} a_n x^n = 0 \Longrightarrow a_n = 0;\ \forall n \in \mathbb{N}$.

Example 4.3.4. (**Change of index**)*: Let* $y(x) = \sum_{n=0}^{+\infty} a_n x^n$ *and* $x \in]-R, +R[$.

1) *Find* $y^{'}$ *and* $y^{''}$*, and express them in terms of* x^n.

2) *Express the following expression in terms of* x^n.

$$A(x) = (1 - x^2)y^{''}.$$

Solution. *We use the properties of power series:*
1) *Finding* $y^{'}$ *and* $y^{''}$*:*
$y^{'}(x) = \sum_{n=1}^{+\infty} n a_n x^{n-1}$ *and* $y^{''}(x) = \sum_{n=2}^{+\infty} n(n-1) a_n x^{n-2}$.
2) *The expression of* $y^{'}$ *and* $y^{''}$ *in terms of* x^n*:*

$$y^{'}(x) = \sum_{n=1}^{+\infty} n a_n x^{n-1} \overset{k=n-1}{=} \sum_{k=0}^{+\infty} (k+1) a_{k+1} x^k = \sum_{n=0}^{+\infty} (n+1) a_{n+1} \cdot x^n,$$

et

$$\begin{aligned} y^{''}(x) &= \sum_{n=2}^{+\infty} n(n-1) a_n x^{n-2} \overset{k=n-2}{=} \sum_{k=0}^{+\infty} (k+2)(k+1) a_{k+2} x^k \\ &= \sum_{n=0}^{+\infty} (n+2)(n+1) a_{n+2} \cdot x^n. \end{aligned}$$

2) The expression of A in terms of x^n:

$$\begin{aligned}
(1-x^2)y''(x) &= \sum_{n=2}^{+\infty} n(n-1)a_n x^{n-2} - \sum_{n=2}^{+\infty} n(n-1)a_n \cdot x^n \\
&= \sum_{n=0}^{+\infty}(n+2)(n+1)a_{n+2}\cdot x^n - \sum_{n=2}^{+\infty} n(n-1)a_n \cdot x^n \\
&= \sum_{n=0}^{+\infty}(n+2)(n+1)a_{n+2}\cdot x^n - \sum_{n=0}^{+\infty} n(n-1)a_n \cdot x^n \\
&= \sum_{n=0}^{+\infty}\left[(n+2)(n+1)a_{n+2} - n(n-1)a_n\right] x^n.
\end{aligned}$$

Example 4.3.5. *Find the power series solution of the equation:*

$$y' - y = 0; \quad y(0) = 1. \tag{4.3}$$

Solution. *Let us set $y(x) = \sum_{n=0}^{+\infty} a_n x^n$, then: $y'(x) = \sum_{n=1}^{+\infty} n a_n x^{n-1}$.*
By substituting into equation (4.3), we obtain:

$$\begin{aligned}
y'(x) - y(x) &= \sum_{n=1}^{+\infty} n a_n x^{n-1} - \sum_{n=0}^{+\infty} a_n x^n \\
&= \sum_{n=0}^{+\infty}(n+1)a_{n+1}\cdot x^n - \sum_{n=0}^{+\infty} a_n x^n \\
&= \sum_{n=0}^{+\infty}\left[(n+1)a_{n+1} - a_n\right] x^n = 0 \\
&\Rightarrow\ (n+1)a_{n+1} - a_n = 0; \quad \forall n \in \mathbb{N} \\
&\Rightarrow\ a_{n+1} = \frac{a_n}{n+1}; \quad \forall n \in \mathbb{N}
\end{aligned}$$

Since, $y(0) = 1 \Rightarrow a_0 = 1 \Rightarrow a_1 = \frac{1}{1} \Rightarrow a_2 = \frac{1}{2} \Rightarrow a_3 = \frac{1}{2\times 3}$. Hence: $a_n = \frac{1}{n!}$; $\forall n \in \mathbb{N}$. This implies that:

$$y(x) = \sum_{n=0}^{+\infty} \frac{x^n}{n!} = e^x; \quad \forall x \in \mathbb{R}$$

the solution of the equation (4.3).

Example 4.3.6. *Determine the power series solution of the equation:*

$$(1-x)y' - y = 0; \quad y(0) = 1. \tag{4.4}$$

Solution. *We have* $y(x) = \sum_{n=0}^{+\infty} a_n x^n$, *so:* $y^{'}(x) = \sum_{n=1}^{+\infty} n a_n x^{n-1}$.
By substituting into equation (4.4), *we get:*

$$\begin{aligned}
(1-x)y^{'}(x) - y(x) &= \sum_{n=1}^{+\infty} n a_n x^{n-1} - \sum_{n=1}^{+\infty} n a_n x^n - \sum_{n=0}^{+\infty} a_n x^n \\
&= \sum_{n=0}^{+\infty} (n+1) a_{n+1} \cdot x^n - \sum_{n=0}^{+\infty} n a_n x^n - \sum_{n=0}^{+\infty} a_n x^n \\
&= \sum_{n=0}^{+\infty} \left[(n+1)a_{n+1} - (n+1)a_n\right] x^n = 0 \\
&\Rightarrow (n+1)a_{n+1} - (n+1)a_n = 0; \quad \forall n \in \mathbb{N} \\
&\Rightarrow a_{n+1} = a_n; \quad \forall n \in \mathbb{N}
\end{aligned}$$

Furthermore, we have: $y(0) = 1 \Rightarrow a_0 = 1 \Rightarrow a_n = 1; \quad \forall n \in \mathbb{N}$. *This implies that:*

$$y(x) = \sum_{n=0}^{+\infty} x^n = \frac{1}{1-x}; \quad \forall |x| < 1$$

the solution of the equation (4.4).

Example 4.3.7. *Find the power series solution of the equation:*

$$y^{''} + xy^{'} - y = 0; \ \ y(0) = 0 \ \ \text{and} \ \ y^{'}(0) = 1. \tag{4.5}$$

Solution. *Let us set* $y(x) = \sum_{n=0}^{+\infty} a_n x^n$, *then:* $y^{'}(x) = \sum_{n=1}^{+\infty} n a_n x^{n-1}$ *and* $y^{''}(x) = \sum_{n=2}^{+\infty} n(n-1) a_n x^{n-2}$. *By substituting into equation* (4.5), *we obtain:*

$$\begin{aligned}
y^{''} + xy^{'} - y &= \sum_{n=0}^{+\infty} (n+2)(n+1) a_{n+2} \cdot x^n + \sum_{n=0}^{+\infty} n a_n x^n - \sum_{n=0}^{+\infty} a_n x^n \\
&= \sum_{n=0}^{+\infty} \left[(n+2)(n+1)a_{n+2} + (n-1)a_n\right] x^n = 0 \\
&\Rightarrow (n+2)(n+1)a_{n+2} + (n-1)a_n = 0; \quad \forall n \in \mathbb{N} \\
&\Rightarrow a_{n+2} = \frac{1-n}{n+2} \cdot a_n; \quad \forall n \in \mathbb{N}
\end{aligned}$$

We have: $y(0) = 0 \Rightarrow a_0 = 0$ *and* $y^{'}(0) = 1 \Rightarrow a_1 = 1$. *Therefore:* $a_2 = a_3 = \cdots = a_n = 0; \ \forall n \geqslant 2$. *This implies that:*
$y(x) = x; \quad x \in \mathbb{R}$ *the solution of the equation* (4.5).

4.3.2 Fourier series

Definition 4.3.5. *A real trigonometric series is called any series of functions of the form:*

$$\frac{a_0}{2} + \sum_{n=1}^{+\infty} [a_n \cos(nx) + b_n \sin(nx)], \tag{4.6}$$

with $x \in \mathbb{R}$ *and* a_0, a_n, $b_n \in \mathbb{R}$, $\forall n \in \mathbb{N}^*$.

Definition 4.3.6. (**Periodic function**)*:*
A function f *is said to be* ***periodic*** *if there exists a number* T *such that:*

$$f(x) = f(x+T); \quad \forall x \in \mathbb{R}.$$

Definition 4.3.7. *We say that* f *is* ***T-periodic*** *if* T *is the smallest* > 0 *that satisfies:*

$$f(x+T) = f(x); \quad \forall x \in \mathbb{R}.$$

Example 4.3.8. $f_1(x) = \cos x$ *and* $f_2(x) = \sin x$ *are* 2π*-periodic functions because:* $\cos(x + 2\pi) = \cos x$ *and* $\sin(x + 2\pi) = \sin x$.

Remark 4.3.2. *Suppose that the series* (4.6) *converges, and let us set:*

$$f(x) \;=\; \frac{a_0}{2} + \sum_{n=1}^{+\infty} [a_n \cos(nx) + b_n \sin(nx)] \Rightarrow f(x+2\pi) = f(x).$$

Then f *is* 2π*-periodic function.*

Proposition 4.3.3. *If the infinite series* $\sum a_n$ *and* $\sum b_n$ *are absolutely convergent, then the series* (4.6) *is normally convergent.*

Calculation of the coefficients of the series (4.6)

We assume that the series (4.6) is uniformly convergent. Therefore:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{+\infty} [a_n \cos(nx) + b_n \sin(nx)].$$

Here f is a continuous anf 2π-periodic function. Then, we can write the coefficients as follows:

$$a_0 = \frac{1}{\pi} \int_0^{2\pi} f(x)\, dx$$

$$a_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \cos(nx)\, dx \quad \text{and} \quad b_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \sin(nx)\, dx \quad \forall n \geq 1.$$

Remark 4.3.3. *If* f *is* T*-periodic and continuous on* $[0, T]$, *so:*

$$\int_0^T f(x)\, dx = \int_\alpha^{\alpha+T} f(x)\, dx \quad \forall \alpha \in \mathbb{R}.$$

Proof.

$$\begin{aligned}
\int_{\alpha}^{\alpha+T} f(x)\,dx &= \int_{\alpha}^{0} f(x)\,dx + \int_{0}^{T} f(x)\,dx + \int_{T}^{\alpha+T} f(x)\,dx \\
&= \int_{\alpha}^{0} f(x)\,dx + \int_{0}^{T} f(x)\,dx + \int_{0}^{\alpha} f(y)\,dy \\
&= \int_{0}^{T} f(x)\,dx.
\end{aligned}$$

In the second line, we made the substitution $y = x - T$. □

Thus, the Fourier coefficients can be written as:

$$a_0 = \frac{1}{\pi}\int_{\alpha}^{\alpha+2\pi} f(x)\,dx$$

$$a_n = \frac{1}{\pi}\int_{\alpha}^{\alpha+2\pi} f(x)\cos(nx)\,dx \ \text{ and } \ b_n = \frac{1}{\pi}\int_{\alpha}^{\alpha+2\pi} f(x)\sin(nx)\,dx \ \ \forall n \geq 1.$$

In particular:

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{+\pi} f(x)\,dx$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{+\pi} f(x)\cos(nx)\,dx \quad \text{and} \quad b_n = \frac{1}{\pi}\int_{-\pi}^{+\pi} f(x)\sin(nx)\,dx \quad \forall n \geq 1.$$

Definition 4.3.8. (**Fourier Series**)*: The* ***Fourier Series*** *associated with f which is 2π-periodic function is the following trigonometric series:*

$$\frac{a_0}{2} + \sum_{n=1}^{+\infty}[a_n\cos(nx) + b_n\sin(nx)]$$

with $$a_0 = \frac{1}{\pi}\int_{-\pi}^{+\pi} f(x)\,dx$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{+\pi} f(x)\cos(nx)\,dx \quad \textit{and} \quad b_n = \frac{1}{\pi}\int_{-\pi}^{+\pi} f(x)\sin(nx)\,dx \quad \forall n \geq 1.$$

Properties

For a function f which is $2L$-periodic defined on the interval $[-L, L]$, the Fourier Series associated with f is given as follows:

$$\frac{a_0}{2} + \sum_{n=1}^{+\infty}\left[a_n\cos\left(\frac{n\pi x}{L}\right) + b_n\sin\left(\frac{n\pi x}{L}\right)\right]$$

with $$a_0 = \frac{1}{L}\int_{-L}^{+L} f(x)\,dx$$

$$a_n = \frac{1}{L}\int_{-L}^{+L} f(x)\cos(nx)\,dx \quad \text{and} \quad b_n = \frac{1}{L}\int_{-L}^{+L} f(x)\sin(nx)\,dx \quad \forall n \geq 1.$$

Theorem 4.3.3. *Let f be a 2π-periodic function satisfying the following conditions:*

1) *There exists $M > 0$ such that: $|f(x)| < M$ $\forall x \in \mathbb{R}$.*

2) *f is monotone piecewise on the interval $[a, b]$ meaning that we can divide $[a, b]$ into subintervals such that the function f is monotone on each subinterval.*

Thus, the Fourier series associated with f is convergent, and we have this identities:

$$\begin{aligned} S(x) &= \frac{a_0}{2} + \sum_{n=0}^{+\infty} [a_n \cos(nx) + b_n \sin(nx)] \\ &= \begin{cases} f(x_0) & \text{if } f \text{ is continuous at } \; x_0 \\ \dfrac{f(x_0+0) + f(x_0-0)}{2} & \text{if } f \text{ is discontinuous at } \; x_0. \end{cases} \end{aligned}$$

The notations $f(x_0+0)$ and $f(x_0-0)$ represent the right-hand and left-hand limits of f at x_0, respectively:

$$f(x_0+0) = \lim_{\substack{x \to x_0 \\ >}} f(x) \quad \text{and} \quad f(x_0-0) = \lim_{\substack{x \to x_0 \\ <}} f(x).$$

Moreover, if the convergence is uniform then f is continuous $\forall x \in \mathbb{R}$. Hence:

$$S(x) = f(x) = \frac{a_0}{2} + \sum_{n=1}^{+\infty} [a_n \cos(nx) + b_n \sin(nx)].$$

This equality is true for all $x \in \mathbb{R}$.

Remark 4.3.4. *We will study some particular cases. First, let us recall a few properties:*
$f : [-k, +k] \to \mathbb{R}$ a continuous function. Therefore:

1) *If f is even, then:*

$$\int_{-k}^{+k} f(x)\, dx = 2 \int_0^{+k} f(x)\, dx.$$

2) *If f is odd, then:*

$$\int_{-k}^{+k} f(x)\, dx = 0.$$

Application: If f is expandable in a Fourier series, then:
First case. If f is even:

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{+\pi} f(x)\, dx = \frac{2}{\pi} \int_{0}^{\pi} f(x)\, dx$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{+\pi} f(x) \cos(nx)\, dx = \frac{2}{\pi} \int_{0}^{\pi} f(x) \cos(nx)\, dx \quad \text{and} \quad b_n = 0 \quad \forall n \geq 1.$$

Second case. If f is odd:

$$a_0 = a_n = 0 \ \ \forall n \geq 1$$

and $b_n = \dfrac{1}{\pi} \displaystyle\int_{-\pi}^{+\pi} f(x) \sin(nx)\, dx = \dfrac{2}{\pi} \displaystyle\int_{0}^{\pi} f(x) \sin(nx)\, dx \quad \forall n \geq 1.$

Parseval's Identity

Theorem 4.3.4. *Let f a function that is expandable in a Fourier series and has a period of 2π (or any other period), then we have:*

$$\frac{1}{\pi} \int_{-\pi}^{+\pi} |f(x)|^2\, dx = \frac{|a_0|^2}{2} + \sum_{n=1}^{+\infty} \left(|a_n|^2 + |b_n|^2 \right).$$

1) *If f is even, then f^2 is also even, so:*

$$\frac{2}{\pi} \int_{0}^{\pi} |f(x)|^2\, dx = \frac{|a_0|^2}{2} + \sum_{n=1}^{+\infty} |a_n|^2.$$

2) *If f is odd, then f^2 is even, then:*

$$\frac{2}{\pi} \int_{0}^{\pi} |f(x)|^2\, dx = \sum_{n=1}^{+\infty} |b_n|^2.$$

Example 4.3.9. *Consider the 2π-periodic function f which is defined by:*

$$f(x) = |x| \quad \textit{with} \quad -\pi \leq x \leq +\pi.$$

1) *Sketch the graph of $f(x)$ over the interval $[-3\pi, 3\pi]$,.*
2) *Calculate the Fourier coefficients of f.*
3) *Obtain a Fourier series expansion of this function f.*
4) *Deduce the sums of the following infinite series:*

a) $A = \displaystyle\sum_{n=0}^{+\infty} \frac{1}{(2n+1)^2}$, ***b)*** $B = \displaystyle\sum_{n=0}^{+\infty} \frac{1}{(2n+1)^4}$.

Solution. *1) The graph of f:*

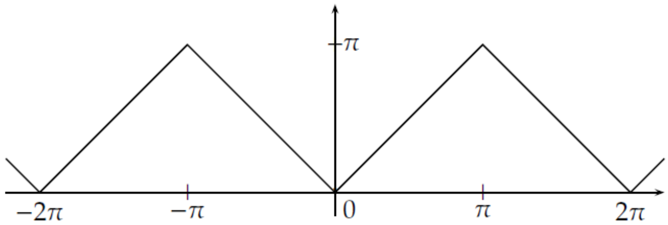


Figure 4.1: Graph of the function $f(x)$.

2) The Fourier coefficients of f:
We have: $f : [-\pi, \pi] \longrightarrow \mathbb{R}$ a 2π-periodic function defined by $f(x) = |x|$.

1) $|f(x)| \leq \pi$, *so f is bounded.*

2) $f_{|[-\pi,0]}$ *is decreasing and continuous, and* $f_{|[0,\pi]}$ *is increasing and continuous.*

Since f satisfies the conditions of Theorem 4.3.3, it can be expanded in a Fourier series. Moreover, f is an even function, so $b_n = 0$; $\forall n \geq 1$.
$a_0 = \dfrac{1}{\pi}\displaystyle\int_{-\pi}^{+\pi} f(x)\, dx = \dfrac{2}{\pi}\int_0^{\pi} x\, dx = \pi.$

$$
\begin{aligned}
a_n &= \frac{1}{\pi}\int_{-\pi}^{+\pi} f(x)\cos(nx)\, dx = \frac{1}{\pi}\int_{-\pi}^{+\pi} |x|\cos(nx)\, dx \\
&= \frac{1}{\pi}\left[\int_{-\pi}^{0} (-x)\cos(nx)\, dx + \int_0^{\pi} x\cos(nx)\, dx\right] \\
&= \frac{2}{\pi}\int_0^{+\pi} x\cos(nx)\, dx = \frac{2}{\pi}\left(\left[x\frac{\sin(nx)}{n}\right]_0^{\pi} - \frac{1}{n}\int_0^{\pi}\sin(nx)\, dx\right) \\
&= \frac{2}{n\pi}\left[\frac{\cos(nx)}{n}\right]_0^{\pi} = \frac{2[(-1)^n - 1]}{n^2\pi}.
\end{aligned}
$$

We notice that the coefficients with even indices are zero. Therefore:

$$
a_n = \begin{cases} 0 & \text{if} \quad n = 2p \\[2ex] -\dfrac{4}{\pi(2p+1)^2} & \text{if} \quad n = 2p+1. \end{cases}
$$

3) The Fourier series expansion of f.

We know that: $S(x) = \dfrac{a_0}{2} + \displaystyle\sum_{n=1}^{+\infty}[a_n\cos(nx) + b_n\sin(nx)].$

According to Figure 4.1, f is continuous on $\mathbb{R}$, thus: $S(x) = f(x)$.

$$f(x) = |x| = \frac{\pi}{2} - \frac{4}{\pi}\sum_{n=0}^{+\infty}\frac{\cos(2n+1)x}{(2n+1)^2}; \qquad \forall x \in \mathbb{R}. \tag{4.7}$$

***4)** Calculation of sums:*
***4-1)** By substituting x by 0 into equation 4.7, we obtain:*
$0 = \frac{\pi}{2} - \frac{4}{\pi}\sum_{n=0}^{+\infty}\frac{1}{(2n+1)^2} \Longrightarrow \sum_{n=0}^{+\infty}\frac{1}{(2n+1)^2} = \frac{\pi^2}{8}.$
***4-2)** We can apply Parseval's identity for an even function. We have:*
$\frac{2}{\pi}\int_0^{+\pi}|f(x)|^2\,dx = \frac{|a_0|^2}{2} + \sum_{n=1}^{+\infty}|a_n|^2$. *Hence:*
$\frac{2}{\pi}\int_0^{\pi}x^2\,dx = \frac{2\pi^2}{3} = \frac{\pi^2}{2} + \frac{16}{\pi^2}\sum_{n=0}^{+\infty}\frac{1}{(2n+1)^4} \Longrightarrow \sum_{n=0}^{+\infty}\frac{1}{(2n+1)^4} = \frac{\pi^4}{96}.$

Example 4.3.10. *Consider the 2π-periodic function f which is defined by:*

$$f(x) = \begin{cases} -1 & \text{if} \quad -\pi < x \le 0 \\ +1 & \text{if} \quad 0 < x < +\pi. \end{cases}$$

***1)** Sketch the graph of $f(x)$ over the interval $[-5\pi, 5\pi]$,.*
***2)** Find the Fourier coefficients of f.*
***3)** Get a Fourier series expansion of this function f.*
***4)** Conclude the sums of the following infinite series:*

***a)** $A = \sum_{n=0}^{+\infty}\frac{(-1)^n}{2n+1}$,* ***b)** $B = \sum_{n=0}^{+\infty}\frac{1}{(2n+1)^2}$,* ***c)** $C = \sum_{n=1}^{+\infty}\frac{1}{n^2}$.*

Solution. ***1)** The graph of f:*

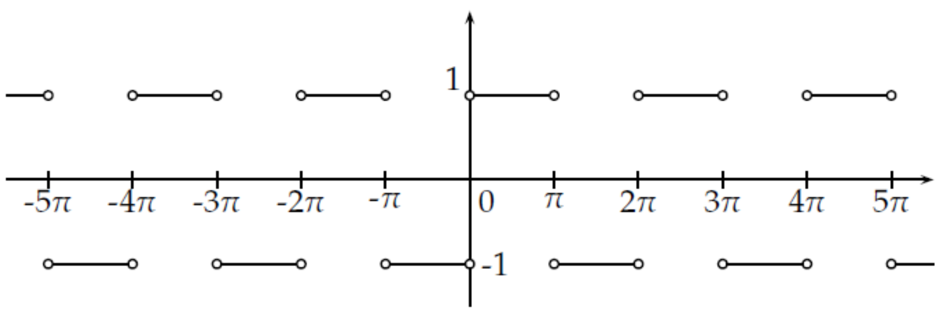


Figure 4.2: Graph of the function $f(x)$.

2) The Fourier coefficients of f:
Since f is an odd function, then $a_n = 0$; $\forall n \in \mathbb{N}$.

$$\begin{aligned} b_n &= \frac{1}{\pi}\int_{-\pi}^{+\pi} f(x)\sin(nx)\,dx = \frac{1}{\pi}\left[\int_{-\pi}^{0}(-1)\sin(nx)\,dx + \int_{0}^{+\pi} 1\sin(nx)\,dx\right] \\ &= \frac{2}{\pi}\int_{0}^{+\pi}\sin(nx)\,dx = \frac{2}{n\pi}\big[-\cos(nx)\big]_0^{\pi} = \frac{2[1-(-1)^n]}{n\pi}. \end{aligned}$$

We notice that the coefficients with even indices are zero. Therefore:

$$b_n = \begin{cases} 0 & \text{if} \quad n = 2p \\ \dfrac{4}{\pi(2p+1)} & \text{if} \quad n = 2p+1 \end{cases}$$

3) The Fourier series expansion of f.

$$S(x) = \frac{4}{\pi}\sum_{n=0}^{+\infty}\frac{\sin(2n+1)x}{2n+1} = \begin{cases} f(x) & \text{if} \quad x \neq k\pi;\ \ k \in \mathbb{Z} \\ 0 & \text{if} \quad x = k\pi;\ \ k \in \mathbb{Z} \end{cases} \tag{4.8}$$

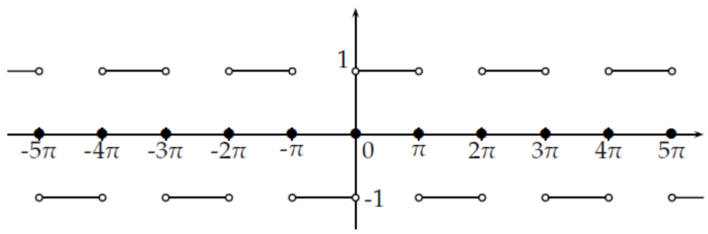


Figure 4.3: Graph of the function $S(x)$.

4) Calculation of sums:
4-1) By substituting x by $\pi/2$ into equation (4.8), we get:

$$S(\pi/2) = f(\pi/2) = 1 = \frac{4}{\pi}\sum_{n=0}^{+\infty}\frac{\sin(2n+1)\pi/2}{2n+1} \Longrightarrow \sum_{n=0}^{+\infty}\frac{(-1)^n}{(2n+1)} = \frac{\pi}{4}.$$

4-2) We will apply Parseval's identity for an odd function. We have:

$$\frac{2}{\pi}\int_0^{\pi} f^2(x)\,dx = 2 = \frac{16}{\pi^2}\sum_{n=0}^{+\infty}\frac{1}{(2n+1)^2} \Longrightarrow \sum_{n=0}^{+\infty}\frac{1}{(2n+1)^2} = \frac{\pi^2}{8}.$$

4-3) We can separate the series into two sums: one for the even indices and one for the odd indices.

$$\sum_{n=1}^{+\infty}\frac{1}{n^2} = \sum_{p=0}^{+\infty}\frac{1}{(2p+1)^2} + \sum_{p=1}^{+\infty}\frac{1}{(2p)^2} = \frac{\pi^2}{8} + \frac{1}{4}\sum_{p=1}^{+\infty}\frac{1}{p^2} \Longrightarrow \sum_{n=1}^{+\infty}\frac{1}{n^2} = \frac{\pi^2}{6}.$$

4.4 Supplementary exercises

Exercise 4.1. *Find the sum of the series:*

1) $\displaystyle\sum_{n=1}^{+\infty} \frac{1}{4n^2-1}$, ***2)*** $\displaystyle\sum_{n=1}^{+\infty} \ln\left[\frac{n(2n+1)}{(n+1)(2n-1)}\right]$, ***3)*** $\displaystyle\sum_{n=1}^{+\infty} \frac{2n+1}{n^2(n+1)^2}$,

4) $\displaystyle\sum_{n=1}^{+\infty} \frac{n-1}{n!}$, ***5)*** $\displaystyle\sum_{n=1}^{+\infty} \arctan\frac{1}{n^2+n+1}$, ***6)*** $\displaystyle\sum_{n=1}^{+\infty} \frac{(-1)^{n-1}}{n}$.

Exercise 4.2. *Study the convergence of the following series:*

1) $\displaystyle\sum_{n=1}^{+\infty} \ln\frac{1}{n}$, ***2)*** $\displaystyle\sum_{n=0}^{+\infty} \cos(n\pi)$, ***3)*** $\displaystyle\sum_{n=1}^{+\infty} \left(1-\frac{2}{n}\right)^n$,

4) $\displaystyle\sum_{n=0}^{+\infty} \frac{4^n}{5^n+3}$, ***5)*** $\displaystyle\sum_{n=1}^{+\infty} \frac{1+\cos n}{n^2}$, ***6)*** $\displaystyle\sum_{n=2}^{+\infty} \frac{\ln n}{n}$.

Exercise 4.3. *Investigate the convergence of the following series:*

1) $\displaystyle\sum_{n=1}^{+\infty} \frac{1}{n}\ln\left(1+\frac{1}{n}\right)$, ***2)*** $\displaystyle\sum_{n=1}^{+\infty} \frac{n+1}{n^2\sqrt{n}}$, ***3)*** $\displaystyle\sum_{n=1}^{+\infty} \sin\frac{1}{2^n}$,

4) $\displaystyle\sum_{n=1}^{+\infty} \left(\frac{3n+2}{2n+1}\right)^{n/2}$, ***5)*** $\displaystyle\sum_{n=1}^{+\infty} n^2 e^{-n}$, ***6)*** $\displaystyle\sum_{n=2}^{+\infty} \frac{\ln^n n}{n^n}$.

Exercise 4.4. *Study the convergence of the following series:*

1) $\displaystyle\sum_{n=1}^{+\infty} \frac{(2n)!}{(n!)^2}$, ***2)*** $\displaystyle\sum_{n=1}^{+\infty} \frac{n^{\sqrt{2}}}{2^n}$, ***3)*** $\displaystyle\sum_{n=2}^{+\infty} \frac{n!\ln n}{n(n+2)!}$,

4) $\displaystyle\sum_{n=1}^{+\infty} \frac{(2n)!}{n^{2n}}$, ***5)*** $\displaystyle\sum_{n=1}^{+\infty} \frac{n^3}{5^n}$, ***6)*** $\displaystyle\sum_{n=1}^{+\infty} \frac{1\times 3\times\cdots\times(2n-1)}{2^n\cdot 4^n\cdot n!}$.

Exercise 4.5. *Investigate the convergence of the following series:*

1) $\displaystyle\sum_{n=1}^{+\infty} \frac{(-1)^n}{n^5+1}$, ***2)*** $\displaystyle\sum_{n=0}^{+\infty} (-1)^{n+1}(0.1)^n$, ***3)*** $\displaystyle\sum_{n=0}^{+\infty} (-1)^n\frac{n!}{2^n}$,

4) $\displaystyle\sum_{n=1}^{+\infty} \frac{(-1)^n}{\sqrt{n}+1}$, ***5)*** $\displaystyle\sum_{n=1}^{+\infty} (-1)^n\frac{\arctan n}{n^2+1}$, ***6)*** $\displaystyle\sum_{n=3}^{+\infty} \frac{(-1)^n}{n\ln n}$.

Exercise 4.6. *Study the pointwise convergence and the uniform convergence of the following sequences of functions:*

1) $f_n(x) = \dfrac{nx}{1+n^2x^2}$; $n \geq 1$ *and* $x \in \mathbb{R}^+$,

2) $f_n(x) = x^n(1-x)$; $n \in \mathbb{N}$ *and* $x \in [0,1]$,

3) $f_n(x) = e^{-nx^2}$; $n \geq 1$ *and* $x \geq 1$.

4) $f_n(x) = nxe^{-nx^2}$; $n \geq 1$ *and* $x \in \mathbb{R}^+$.

Exercise 4.7. *Investigate the pointwise convergence and the uniform convergence of the following series of functions:*

1) $\displaystyle\sum_{n=0}^{+\infty} \frac{x}{(1+x^2)^n};\ x \in \mathbb{R}$ *2)* $\displaystyle\sum_{n=1}^{+\infty} \frac{(-1)^n}{n^2+x};\ x \in \mathbb{R}^+.$

3) $\displaystyle\sum_{n=1}^{+\infty} \frac{x}{1+n^5x^2};\ x \in \mathbb{R}^+$ *4)* $\displaystyle\sum_{n=1}^{+\infty} \frac{\cos(n\pi)}{2^n+x^2};\ x \in \mathbb{R}.$

Exercise 4.8. *We consider the following series of functions:*

$$\sum_{n=1}^{+\infty} \frac{\sin^2(nx)}{n(n+2)}; \qquad \forall x \in \mathbb{R}.$$

1) Study the normal convergence of this series on $\mathbb{R}$*.*
2) Show that the sum of the series, denoted by S *is a continuous function.*
3) Compute the following integral:

$$\int_0^{\pi} S(x)\, dx.$$

Exercise 4.9. *determine the radius of convergence* R *and the domain de convergence* D *for the following power series:*

1) $\displaystyle\sum_{n=0}^{+\infty} \frac{x^n}{3^{n+1}},$ *2)* $\displaystyle\sum_{n=0}^{+\infty} \frac{x^n}{(2n)!},$ *3)* $\displaystyle\sum_{n=0}^{+\infty} \sin(n)\cdot x^n,$

4) $\displaystyle\sum_{n=1}^{+\infty} \frac{\arctan n}{n^2}\cdot x^n,$ *5)* $\displaystyle\sum_{n=2}^{+\infty} \frac{\sqrt{\ln n}}{n^4}\cdot x^n,$ *6)* $\displaystyle\sum_{n=1}^{+\infty} \frac{e^n}{\sqrt{n}}\cdot x^n,$

7) $\displaystyle\sum_{n=1}^{+\infty} \frac{\arcsin(1/n)}{3^n}\cdot x^n,$ *8)* $\displaystyle\sum_{n=3}^{+\infty} \frac{(-1)^n}{n\ln n}\cdot x^n,$ *9)* $\displaystyle\sum_{n=0}^{+\infty} \frac{n!}{e^n}\cdot x^n.$

Exercise 4.10. *Let* f *be a function defined on* $]-1,+1[$ *by:*

$$f(x) = \frac{\arcsin x}{\sqrt{1-x^2}}.$$

1) Justify that f *is expandable in a power series.*
2) Show that f *is a solution to the following differential equation:*

$$(1-x^2)y' - xy = 0.$$

3) Determine the power series expansion of the function f *on the interval* $]-1,+1[$*.*

Exercise 4.11. *Consider the 2π-periodic function f which is defined by:*

$$f(x) = x^2 \quad et \quad -\pi \leq x \leq +\pi.$$

1) *Sketch the graph of $f(x)$ over the interval $[-3\pi, 3\pi]$,.*
2) *Calculate the Fourier coefficients of f.*
3) *Obtain a Fourier series expansion of this function f.*
4) *Deduce the sums of the following infinite series:*

a) $A = \sum_{n=1}^{+\infty} \frac{1}{n^2}$, ***b)*** $B = \sum_{n=1}^{+\infty} \frac{(-1)^{n+1}}{n^2}$, ***c)*** $C = \sum_{n=1}^{+\infty} \frac{1}{n^4}$.

Exercise 4.12. *Let f be a function 2-periodic defined by:*

$$f(x) = \begin{cases} x & \text{if} \quad 0 \leq x < 1 \\ \frac{1}{2} & \text{if} \quad 1 \leq x < 2. \end{cases}$$

1) *Sketch the graph of $f(x)$ over the interval $[-2, 4]$,.*
2) *Find the Fourier coefficients of f.*
3) *Obtain a Fourier series expansion of this function f.*

Chapter 5

Fourier Transform

The Fourier Transform is an extension of the Fourier Series expansion for periodic functions to non-periodic functions.
The Fourier Transform associates a function f (with values in $\mathbb{R}$ or $\mathbb{C}$) with another function denoted by $\mathcal{F}(f)(s)$; where s is an independent variable of t, called the dual variable.

5.1 Definitions and properties

We denote by $L^1(\mathbb{R})$ the set of functions $f : \mathbb{R} \to \mathbb{R}$ that are integrable and for which $\displaystyle\int_{-\infty}^{+\infty} |f(t)|\, dt$ converges.

Definition 5.1.1. *Let $f \in L^1(\mathbb{R})$. The Fourier transform of f denoted by $\mathcal{F}(f)$ is defined as follows:*
$\mathcal{F}(f) : \mathbb{R} \to \mathbb{C}$

$$s \mapsto \mathcal{F}(f)(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(t) e^{-ist}\, dt.$$

Theorem 5.1.1. *Let $f \in L^1(\mathbb{R})$.*
1) *If f is even, then $\mathcal{F}(f)(s) = \dfrac{2}{\sqrt{2\pi}} \displaystyle\int_{0}^{+\infty} f(t)\cos(st)\, dt.$*
2) *If f is odd, then $\mathcal{F}(f)(s) = -\dfrac{2i}{\sqrt{2\pi}} \displaystyle\int_{0}^{+\infty} f(t)\sin(st)\, dt.$*

Example 5.1.1. *Calculate the Fourier transform of f, which is defined by:*
$f : \mathbb{R} \to \mathbb{R}$

$$t \mapsto f(t) = \begin{cases} 1 & \text{if} \quad |t| \leq 1/2 \\ 0 & \text{if} \quad |t| > 1/2. \end{cases}$$

Solution. *Here f, is even according to Figure 5.1.*

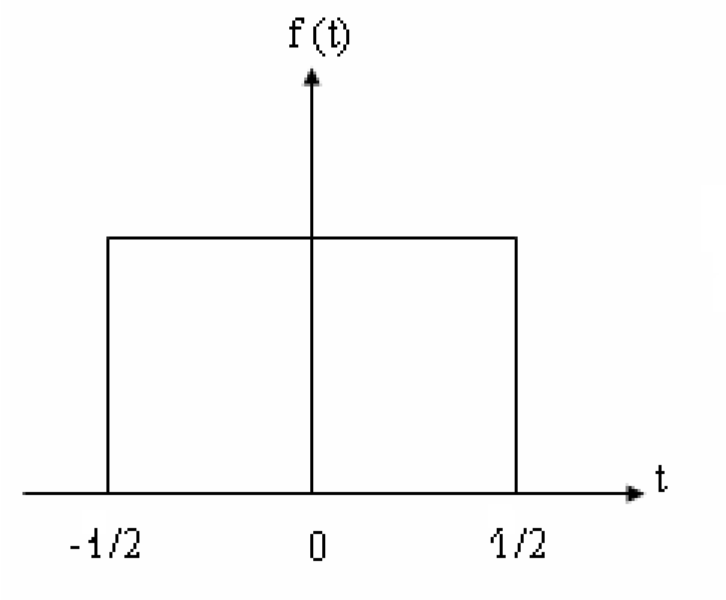


Figure 5.1: The Box function

$$\begin{aligned}
\mathcal{F}(f)(s) &= \frac{2}{\sqrt{2\pi}} \int_0^{+\infty} f(t)\cos(st)\, dt \\
&= \frac{2}{\sqrt{2\pi}} \int_0^{1/2} \cos(st)\, dt \\
&= \frac{2}{\sqrt{2\pi}} \left[\frac{\sin(st)}{s}\right]_0^{1/2}; s \neq 0 \\
&= \frac{2}{\sqrt{2\pi}} \frac{\sin(s/2)}{s}; s \neq 0.
\end{aligned}$$

If $s = 0$ $\mathcal{F}(f)(0) = \dfrac{2}{\sqrt{2\pi}} \displaystyle\int_0^{1/2} 1\, dt = \dfrac{1}{\sqrt{2\pi}}$.

Example 5.1.2. *Let $f(t) = e^{-a|t|}$; $t \in \mathbb{R}$ et $a > 0$.*
The same question.

Solution. *We use integration by parts twice to find the value of the following integral.*

$$\begin{aligned}
\mathcal{F}(f)(s) &= \frac{2}{\sqrt{2\pi}} \int_0^{+\infty} f(t)\cos(st)\, dt \\
&= \frac{2}{\sqrt{2\pi}} \int_0^{+\infty} e^{-at}\cos(st)\, dt \\
&= \frac{2}{\sqrt{2\pi}} \frac{a}{a^2+s^2}; s \in \mathbb{R}.
\end{aligned}$$

Theorem 5.1.2. *Let $f \in L^1(\mathbb{R})$. Then the Fourier transform $\mathcal{F}(f)$ satisfies the following properties:*
1) *$\mathcal{F}(f)$ is continuous on $\mathbb{R}$, meaning that $\lim_{s\to s_0} \mathcal{F}(f)(s) = \mathcal{F}(f)(s_0)$*
2) *$\mathcal{F}(f)$ is bounded on $\mathbb{R}$, meaning that there exists $M > 0$ such that for all $s \in \mathbb{R}$, $|\mathcal{F}(f)(s)| < M$.*

5.1.1 Properties of the Fourier Transform

1) $\mathcal{F}(\alpha f + \beta g)(s) = \alpha\mathcal{F}(f)(s) + \beta\mathcal{F}(g)(s)$.
2) If f is continuous and if $\dfrac{df}{dt} \in L^1(\mathbb{R})$, then we have:

$$\mathcal{F}(f')(s) = is\mathcal{F}(f)(s).$$

5.2 Inverse Fourier Transform

Definition 5.2.1. *Let $f \in L^1(\mathbb{R})$.The inverse Fourier transform of f is defined as the function denoted by $\mathcal{F}^{-1}$, which is given by:*

$$(\mathcal{F}^{-1}f)(t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} f(s)e^{ist}\, ds.$$

Theorem 5.2.1. *Let $f \in L^1(\mathbb{R})$ be a continuous function, and suppose that $\mathcal{F}(f) = \widehat{f} \in L^1(\mathbb{R})$. Therefore: $\mathcal{F}^{-1}\mathcal{F}(f) = f$ and $\mathcal{F}\mathcal{F}^{-1}(f) = f$. Thus:*

$$f(t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} (\mathcal{F}f)(s)e^{ist}\, ds.$$

Example 5.2.1. *Let the function f be defined by: $f(t) = e^{-|t|}$; $t \in \mathbb{R}$.*
1) *Calculate: $(\mathcal{F}f)(s)$.*
2) *Calculate this expression:*

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} \Big(\frac{2}{\sqrt{2\pi}}\frac{1}{1+s^2}\Big)e^{ist}\, ds.$$

3) *Calculate the following improper integral:*

$$\frac{2}{\pi}\int_{0}^{+\infty} \frac{\cos(st)}{1+s^2}\, ds.$$

4) *Calculate $\displaystyle\int_0^{+\infty} \frac{\cos(u)}{t^2+u^2}\, du$; $t > 0$ and deduce the value of the following integral:*

$$\int_0^{+\infty} \frac{\cos(x)}{1+x^2}\, dx$$

Solution. ***1)*** *According to example 5.1.2, we conclude that:*

$$(\mathcal{F}f)(s) = \frac{2}{\sqrt{2\pi}}\frac{1}{1+s^2}.$$

2) *We know that:*

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}\Big(\frac{2}{\sqrt{2\pi}}\frac{1}{1+s^2}\Big)e^{ist}\,ds = \mathcal{F}^{-1}\mathcal{F}(f)(t) = f(t) = e^{-|t|}.$$

3) *We have:*

$$e^{-|t|} = \frac{1}{\pi}\int_{-\infty}^{+\infty}\frac{e^{ist}}{1+s^2}\,ds = \frac{2}{\pi}\int_{0}^{+\infty}\frac{\cos(st)}{1+s^2}\,ds. \tag{5.1}$$

4) *We can make the substitution $st = u$ in equation (5.1), then: $\displaystyle\int_0^{+\infty}\frac{\cos(u)}{t^2+u^2}\,du = \frac{\pi}{2t}e^{-t};\ \ t>0$. Hence:*

$$\int_0^{+\infty}\frac{\cos(x)}{1+x^2}\,dx = \frac{\pi}{2e}.$$

5.3 Applications to differential equations

Example 5.3.1. *Consider the function f defined by:*

$$f(x) = \int_0^{+\infty}\frac{e^{-t}}{\sqrt{t}}e^{itx}\,dt.$$

1) *Show that f is defined for all $x\in\mathbb{R}$.*
2) *Show that f is differentiable and prove that f satisfies a first-order differential equation, which we will solve. Deduce the expression for $f(x)$. We recall that $\displaystyle\int_0^{+\infty}e^{-u^2}\,du = \frac{\sqrt{\pi}}{2}$.*

Solution. *We denote $g(t,x) = \dfrac{e^{-t}}{\sqrt{t}}e^{itx}$ with $|g(t,x)| \le \dfrac{e^{-t}}{\sqrt{t}}$. o prove the existence and differentiability of f, we will use the following theorem.*

Theorem 5.3.1. *Let $g : I\times J \to \mathbb{R}, \mathbb{C}$ be a function such that:*
1) *For all $x\in J$, $g(t,x)$ is continuous and admits an improper integral over the interval I.*
2) *$g(t,x)$ admits a partial derivative $\dfrac{\partial g}{\partial x}(t,x)$.*
3) *For all $t\in I$, $\dfrac{\partial g}{\partial x}(t,x)$ is continuous on J.*
4) *There exists a function $h : I\to\mathbb{R}$ that is continuous and admits an improper integral, such that for all $t\in I$ and $x\in J$:*

$$\Big|\frac{\partial g}{\partial x}(t,x)\Big| \le h(t).$$

Then, the function $f(x) = \displaystyle\int_I g(t,x)\,dt$ *is differentiable on* J*, and its derivative is given by* $f'(x) = \displaystyle\int_I \frac{\partial g}{\partial x}(t,x)\,dt$.
In this case, we have

$$g(t,x) = \frac{e^{-t}}{\sqrt{t}} e^{itx}.$$

Let's check the conditions of the previous theorem.
1) *We first observe that* $g(\cdot, x)$ *is well-defined for all* $x \in \mathbb{R}$*. Indeed*

$$\left| \frac{e^{-t}}{\sqrt{t}} e^{itx} \right| \leq \frac{e^{-t}}{\sqrt{t}}.$$

This function is integrable on $]0,+\infty[$*, because near* 0 *it is equivalent to* $\dfrac{1}{\sqrt{t}}$*, which is integrable (Riemann integral), and near* $+\infty$*, it satisfies:*

$$\left| \frac{e^{-t}}{\sqrt{t}} \right| \leq \frac{1}{t^2}.$$

Then, $g(t,x)$ *is continuous on* $]0,+\infty[$ *and admits an improper integral.*
2) $\dfrac{\partial g}{\partial x}(t,x) = ie^{-t}\sqrt{t}e^{itx}$ *exists and is continuous* $\forall x \in J$.
3) $\left|\dfrac{\partial g}{\partial x}(t,x)\right| \leq \dfrac{\sqrt{t}}{e^t} = h(t)$*;* h *verifies the conditions of the Theorem 5.3.1.*

Consequence*:*
$f(x) = \displaystyle\int_0^{+\infty} g(t,x)\,dt$ *is differentiable on* J *and*

$$\begin{aligned}
f'(x) &= \int_0^{+\infty} ie^{-t}\sqrt{t}e^{itx}\,dt = i\int_0^{+\infty} \sqrt{t}e^{(ix-1)t}\,dt \\
&\overset{I.B.P}{=} -\frac{i}{2(ix-1)}\int_0^{+\infty} \frac{e^{-t}}{\sqrt{t}}e^{itx}\,dt \\
&= -\frac{i}{2(ix-1)}f(x) = \frac{-x+i}{2(x^2+1)}f(x).
\end{aligned}$$

Thus, we obtain the following differential equation:

$$y' + \frac{-x+i}{2(x^2+1)}y = 0; \quad y = f(x). \tag{5.2}$$

Therefore, the solution to equation (5.2), is the function:

$$f(x) = \lambda(1+x^2)^{1/4}\exp\left[\frac{i}{2}\arctan x\right]$$

with $\lambda = \sqrt{\pi}$*, by using the following identity:* $\displaystyle\int_0^{+\infty} e^{-u^2}\,du = \frac{\sqrt{\pi}}{2}$.

5.4 Supplementary exercises

Exercise 5.1. *Determine the Fourier transform of the following functions:*

1) $f_1(t)$ *equals to* 1 *on* $[-1,1]$ *and* 0 *elsewhere.*

2) $f_2(t)$ *equals* 1 *on* $[-T,T]$ *and* 0 *elsewhere* $(T>0)$.

3) $f_3(t)=e^{-\frac{|t|}{T}}$ $(T>0)$.

4) $f_4(t)=\dfrac{\sin t}{t}$.

5) $f_5(t)=\dfrac{1}{1+t^2}$.

Exercise 5.2. *Solve the following equations:*

$$\int_0^{+\infty} f(t)\cos(st)\,dt \;=\; \frac{1}{1+s^2}. \tag{5.3}$$

$$y(t)+\int_{-\infty}^{+\infty} y(t-u)e^{-a|u|}\,du \;=\; e^{-a|t|};\quad a>0. \tag{5.4}$$

Exercise 5.3. *Let the function* f *which is defined by:*

$$f(t)=\begin{cases} e^t & \textit{if} \quad t<0 \\ 0 & \textit{if} \quad t\geq 0. \end{cases}$$

Consider the following differential equation:

$$y''(t)+2y'(t)+y(t)=f(t). \tag{5.5}$$

1) *Calculate the Fourier transform of* f.

2) *Find the function* g *such that*

$$(\mathcal{F}g)(s)=\frac{1}{(s+i)(s-i)^2}.$$

3) *Determine the solution of the equation* (5.5), *using the Fourier transform, such that* $y,y'\in L^1(\mathbb{R})$.

Chapter 6

Laplace Transform

6.1 Definitions and properties

Definition 6.1.1. *The Laplace transform of a function f is given by the following expression:*

$$\mathcal{L}\big(f(t)\big) = F(s) = \int_0^{+\infty} f(t)e^{-st}\,dt. \tag{6.1}$$

Where the symbol $\mathcal{L}\big(f(t)\big)$ means the Laplace transform of $f(t)$.
We also use the notation $F(s)$ to represent the Laplace transform.
In this course, we assume that the functions are zero for $t < 0$.

Example 6.1.1. *Find the Laplace transform of the following functions:*
1) $f(t) = 1$; $t \geq 0$.
$\mathcal{L}\big(f(t)\big) = \displaystyle\int_0^{+\infty} e^{-st}\,dt = \left[-\frac{e^{-st}}{-s}\right]_0^{+\infty} = \frac{1}{s} \qquad \big(Re(s) > 0\big).$
2) $f(t) = e^{at}$; $t \geq 0$.
$\mathcal{L}\big(f(t)\big) = \displaystyle\int_0^{+\infty} e^{at}e^{-st}\,dt = \left[-\frac{e^{-(s-a)t}}{-(s-a)}\right]_0^{+\infty} = \frac{1}{s-a} \qquad \big(Re(s) > a\big).$
The same calculation holds for a complexe and $Re(s) > Re(a)$.
3) $f(t) = e^{i\omega t}$; $t \geq 0$ *et* $\omega \in \mathbb{R}$.
$\mathcal{L}\big(f(t)\big) = \displaystyle\int_0^{+\infty} e^{i\omega t}e^{-st}\,dt = \frac{1}{s-i\omega} \qquad \big(Re(s) > 0\big).$
4) $f(t) = t$; $t \geq 0$.

$$\begin{aligned}\mathcal{L}\big(f(t)\big) &= \int_0^{+\infty} te^{-st}\,dt = \left[-\frac{te^{-st}}{-s}\right]_0^{+\infty} + \frac{1}{s}\int_0^{+\infty} e^{-st}\,dt \\ &= \frac{1}{s}\mathcal{L}\big(1\big) = \frac{1}{s^2} \qquad \big(Re(s) > 0\big).\end{aligned}$$

By performing integration by parts twice as above, we find that:
$\mathcal{L}\big(t^2\big) = \displaystyle\int_0^{+\infty} t^2e^{-st}\,dt = \frac{2}{s^3} \qquad \big(Re(s) > 0\big).$

In the general case, we have:

$$\mathcal{L}\big(t^n\big) = \int_0^{+\infty} t^n e^{-st}\,dt = \frac{n!}{s^{n+1}} \qquad \big(Re(s) > 0\big).$$

6.1.1 Properties of the Laplace Transform

Let f and g be functions such that $\mathcal{L}\big(f(t)\big)$ and $\mathcal{L}\big(g(t)\big)$ exist. Then, we have the following properties:

$$\mathcal{L}\big(af(t) + bg(t)\big) = a\mathcal{L}\big(f(t)\big) + b\mathcal{L}\big(g(t)\big).$$

a and b are two arbitrary constants.

Example 6.1.2. *Find the Laplace transform of the following functions:*
1) $f_1(t) = \cos(\omega t)$; $\omega \in \mathbb{R}$, ***2)*** $f_2(t) = \sin(\omega t)$; $\omega \in \mathbb{R}$,
3) $f_3(t) = \cosh(\omega t)$; $\omega \in \mathbb{R}$, ***4)*** $f_4(t) = \sinh(\omega t)$; $\omega \in \mathbb{R}$.

Solution. ***1)*** $f_1(t) = \cos(\omega t)$; $t \geq 0$ *with* $\omega \in \mathbb{R}$.

$$\begin{aligned}
\mathcal{L}\big(f(t)\big) &= \mathcal{L}\bigg(\frac{1}{2}[e^{i\omega t} + e^{-i\omega t}]\bigg) = \frac{1}{2}\mathcal{L}\big(e^{i\omega t}\big) + \frac{1}{2}\mathcal{L}\big(e^{-i\omega t}\big) \\
&= \frac{1}{2}\bigg(\frac{1}{s - i\omega} + \frac{1}{s + i\omega}\bigg) = \frac{s}{s^2 + \omega^2} \qquad \big(Re(s) > 0\big).
\end{aligned}$$

By the same method, we find: $\mathcal{L}\big(\sin(\omega t)\big) = \dfrac{\omega}{s^2 + \omega^2} \qquad \big(Re(s) > 0\big).$
3) $f_3(t) = \cosh(\omega t)$; $t \geq 0$ *with* $\omega \in \mathbb{R}$.

$$\begin{aligned}
\mathcal{L}\big(f(t)\big) &= \mathcal{L}\bigg(\frac{1}{2}[e^{\omega t} + e^{-\omega t}]\bigg) = \frac{1}{2}\mathcal{L}\big(e^{\omega t}\big) + \frac{1}{2}\mathcal{L}\big(e^{-\omega t}\big) \\
&= \frac{1}{2}\bigg(\frac{1}{s - \omega} + \frac{1}{s + \omega}\bigg) = \frac{s}{s^2 - \omega^2} \qquad \big(Re(s) > \omega\big).
\end{aligned}$$

By the same method, we find: $\mathcal{L}\big(\sinh(\omega t)\big) = \dfrac{\omega}{s^2 - \omega^2} \qquad \big(Re(s) > \omega\big).$

Theorem 6.1.1. *Let f, $f^{'}$ and $f^{''}$ are continuous functions and supposing that $\mathcal{L}\big(f(t)\big) = F(s)$ exists. Thus:*

$$\mathcal{L}\big(f^{'}(t)\big) = sF(s) - f(0) \quad \textit{and} \quad \mathcal{L}\big(f^{''}(t)\big) = s^2F(s) - sf(0) - f^{'}(0).$$

6.2 Inverse Laplace Transform

Definition 6.2.1. *If we know the Laplace transform $F(s)$ of a certain continuous function f, we can always determine the expression of the function by using partial fraction decomposition and applying the linearity property of the Laplace transform.*

$$\mathcal{L}\big(f(t)\big) = F(s) \Rightarrow \mathcal{L}^{-1}\big(F(s)\big) = f(t).$$

Example 6.2.1. *We have:*
1) $\mathcal{L}^{-1}\left(\dfrac{1}{s-1}\right) = e^t;\ t \geq 0.$
2) $\mathcal{L}^{-1}\left(\dfrac{s}{s^2+1}\right) = \cos t;\ t \geq 0.$

Theorem 6.2.1. *If $F(s) = \mathcal{L}\big(f(t)\big)$; $Re(s) > 0$. Then:*
$F(s-a) = \mathcal{L}\big(e^{at}f(t)\big)$*; $Re(s) > a$ such that $a \in \mathbb{R}$.*

Example 6.2.2. *Since $\mathcal{L}\big(t\big) = \dfrac{1}{s^2}$, $Re(s) > 0$. So:*
$\mathcal{L}\big(te^{at}\big) = \dfrac{1}{(s-a)^2}$*; $Re(s) > a$.*
We also know that: $\mathcal{L}\big(t^n\big) = \dfrac{n!}{s^{n+1}}$; $Re(s) > 0$. Therefore:
$\mathcal{L}\big(t^n e^{at}\big) = \dfrac{n!}{(s-a)^{n+1}}$*; $Re(s) > a$. Finally:*
$\mathcal{L}^{-1}\left(\dfrac{1}{(s-a)^{n+1}}\right) = \dfrac{t^n e^{at}}{n!}$*; $t \geq 0$ and $n \in \mathbb{N}$.*

6.3 Applications to differential equations

The Theorem 6.1.1 opens the possibility of using the Laplace transform to solve ordinary differential equations.

Example 6.3.1. *Give the solution to this differential equation:*

$$y'' + y = 1; \quad y(0) = y'(0) = 0. \tag{6.2}$$

Solution. $\mathcal{L}\big(y''\big) + \mathcal{L}\big(y\big) = \mathcal{L}\big(1\big) \Rightarrow s^2\mathcal{L}\big(y\big) - sy(0) - y'(0) + \mathcal{L}\big(y\big) = \dfrac{1}{s}.$
Then: $\mathcal{L}\big(y\big) = \dfrac{1}{s(s^2+1)} = \dfrac{1}{s} - \dfrac{s}{s^2+1}.$
By applying the inverse Laplace transform, we obtain:
$y(t) = \mathcal{L}^{-1}\left(\dfrac{1}{s} - \dfrac{s}{s^2+1}\right) = 1 - \cos t;\ t \geq 0.$

Example 6.3.2. *Find the solution to this differential equation:*

$$y'' - 2y' + y = e^{2t}; \quad y(0) = 0 \ \textit{and} \ y'(0) = 1. \tag{6.3}$$

Solution. $\mathcal{L}(y'') - 2\mathcal{L}(y') + \mathcal{L}(y) = \mathcal{L}(e^{2t})$. *Therefore:*
$s^2\mathcal{L}(y) - sy(0) - y'(0) - 2s\mathcal{L}(y) - 2y(0) + \mathcal{L}(y) = \dfrac{1}{s-2}$. *Hence:*

$$\mathcal{L}(y) = \frac{s-1}{(s-2)(s^2-2s+1)} = \frac{1}{s-2} - \frac{1}{s-1}.$$

By applying the inverse Laplace transform, we get:

$$y(t) = \mathcal{L}^{-1}\left(\frac{1}{s-2} - \frac{1}{s-1}\right) = e^{2t} - e^t; \ \ t \geq 0.$$

6.4 Supplementary exercises

Exercise 6.1. *Consider the following differential equation:*

$$y'' + 2y' + y = te^t; \quad y(0) = 1 \ \ \textit{et} \ \ y'(0) = 0. \tag{6.4}$$

1) *Determine the Laplace transform of the solution to the differential equation (6.4).*

2) *Deduce, by applying the inverse Laplace transform, the explicit solution of (6.4).*

Exercise 6.2. *Consider the following differential equation:*

$$y'' - 4y = 3e^{-t} - t^2; \quad y(0) = 0 \ \ \textit{et} \ \ y'(0) = 0. \tag{6.5}$$

1) *Calculate the Laplace transform of the solution to the differential equation (6.5).*

2) *Conclude, by applying the inverse Laplace transform, the explicit solution of (6.5).*

Bibliography

[1] K. Allab, Éléments d'Analyse: fonction d'une variable réelle. *O.P.U*, 2002.

[2] A. Angot, Compléments de mathématiques. *Maisons Masson*, 1982.

[3] J. Arnaudiès, Séries entières, séries de Puiseux, séries de Fourier-Et compléments sur les fonctions presque-périodiques, 2e cycle universitaire, agrégations de mathématiques. *Ellipses Marketing*, 1999.

[4] S. Balac, L. Chupin, Analyse et algèbre: Cours de mathématiques de deuxième année avec exercices corrigés et illustrations avec Maple. *PPUR presses polytechniques*, 2008.

[5] P. Dyke, An introduction to Laplace transforms and Fourier series. *Springer*, 2014.

[6] M. El-Amrani, Suites et séries numériques, Suites et séries de fonctions. *Ellipses*, 2011.

[7] D. Fredon, M. Bridier, Mathématiques pour les sciences de l'ingénieur. *Dunod*, 2003.

[8] S. Guerre-Delabrière, Suites, séries, intégrales: Cours et exercices corrigés niveau L2. *Ellipses*, 2009.

[9] B. Malgrange, Équations Différentielles Linéaires et Transformation de Fourier: Une Introduction . *Sociedade Brasileira de Mathematica*, 1989.

[10] J. P .Marco, P. Thieullen, J. A. Weil, Mathématiques L2: Cours complet avec 700 tests et exercices corrigés. *Pearson*, 2007.

[11] N. Piskounov, Calcul différentiel et intégral, Tome 1 et 2. *Éditions Mir, Moscou 1980 ou Édition Ellipses*, 1993.

[12] I. Podlubny, Fractional differential equations. An introduction to fractional derivatives, fractional differential equations, to methods of their solution and some of their applications. *Mathematics in Science and Engineering*, **198**. *Academic Press, Inc.*, San Diego, CA, 1999.

[13] M.D. Segonzac, G. Monna, R. Morvan, Séries numériques: Exercices corrigés avec rappels de cours. *Cépaduès*, 2010.