PEOPLE'S DEMOCRATIC REPUBLIC OF ALGERIA

MINISTRY OF HIGHER EDUCATION AND SCIENTIFIC RESEARCH

IBN-KHALDOUN UNIVERSITY IN TIARET

FACULTY OF APPLIED SCIENCES

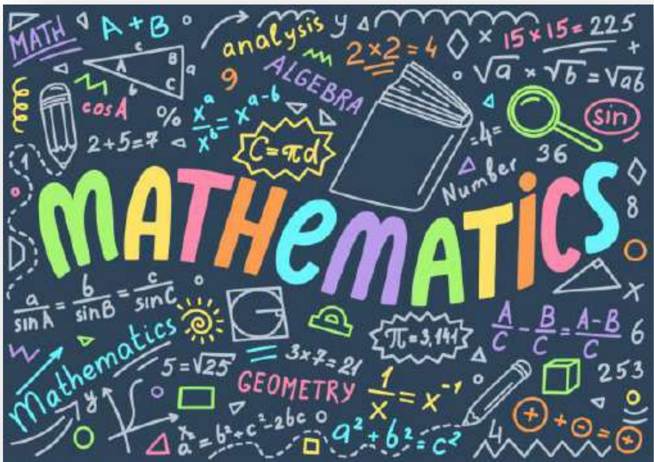
DEPARTMENT OF SCIENCE AND TECHNOLOGY

COURSE HANDOUT

ALGEBRA 1

(Course)

Prepared by: **Mostefa KOULALI**



Promotion : Common core engineering technology, 1st year

Semester: 01

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Academic year: 2024/2025

1) *"Pure mathematics is, in its way, the poetry of logical ideas."*	[Albert Einstein]
2) *"Mathematics is the music of reason."*	[James Joseph Sylvester]
3) *"Mathematics is a place where you can do things which you can't do in the real world."*	[Marcus du Sautoy]
4) *"The mathematics of life is not the multiplication of wealth, but is the division of tasks, subtraction of greed and the addition of humility."*	[Galileo Galilei]
5) *"Mathematics is the language with which God has written the universe."*	[Ian Stewart]
6) *"Mathematics is the science of patterns, and nature exploits just about every pattern that there is."*	
7) *""The only way to learn mathematics is to do mathematics."*	[Paul Halmos]
8) *Mathematics is not about numbers, equations, computations, or algorithms: it is about understanding."*	[William Paul Thurston]
9) *Sometimes the questions are complicated and the answers are simple."*	[Dr. Seuss]
10) *Mathematics consists of proving the most obvious thing in the least obvious way."*	[George Póly]
11) *"A mathematician is a blind man in a dark room looking for a black cat which isn't there."*	[Charles Darwin (humoristique)]
12) *"Mathematics is the supreme judge; from its decisions there is no appeal."*	[Tobias Dantzig]
13) *"The essence of math is not to make simple things complicated, but to make complicated things simple."*	[Stan Gudder]
14) *"Give me a lever long enough and a fulcrum on which to place it, and I shall move the world."*	[Archimedes]

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GENERAL INTRODUCTION

I. General Introduction:

Algebra is a fundamental branch of mathematics concerned with the study of operations, equations and algebraic structures. Historically, it emerged with the introduction of the concept of the equation, making it possible to translate concrete problems into manipulable mathematical expressions. The term 'algebra' comes from the Arabic al-jabr, meaning "restoration" or 'reunion', and was popularised by the mathematician Al-Khwarizmi in the 9th century. Today, the discipline extends from the solution of classical equations to the abstract study of structures such as groups, rings and bodies.

Algebra plays a central role in many fields of science and technology. It provides a universal language for modelling, analysing and solving complex problems in fields as diverse as physics, chemistry, biology, engineering, computer science and the social sciences. For example, in physics, the theory of Lie groups, an advanced branch of algebra, is essential for describing the symmetry of space-time and fundamental interactions. In technology, algebra underpins the development of algorithms, cryptography and artificial intelligence, contributing to the evolution of computers, smartphones and automated systems..

In addition to its theoretical importance, algebra makes it easier to solve practical problems quickly and efficiently, such as calculating the quantities needed in various industrial applications or personal financial management. It also develops the ability to reason in a logical and structured way, a skill that is indispensable in many scientific and technical professions.

In short, algebra is a pillar of mathematics which, through its many applications, supports and advances modern science and technology, while providing a rigorous framework for understanding and manipulating the relationships between quantities and mathematical objects.

The aims of teaching this subject are:

- To ensure a gradual transition to higher education, taking into account the lycée syllabus, consolidating and extending what has already been learnt;
- To consolidate students' training in the areas of logic, reasoning and calculation techniques, which are essential tools in both mathematics and the scientific disciplines, and to provide an introduction to algebraic structures.
- To present rich new concepts in a way that {suscitates students' interest Recommended prerequisites Basic mathematical knowledge

This subject contains 05 chapters, as follows:

Chapter 0: A reminder chapter on mathematical equations and inequations:

Chapter 1: A reminder chapter on reasoning methods.

Chapter 2: Binary relations and applications

Chapter 3: Algebraic structures

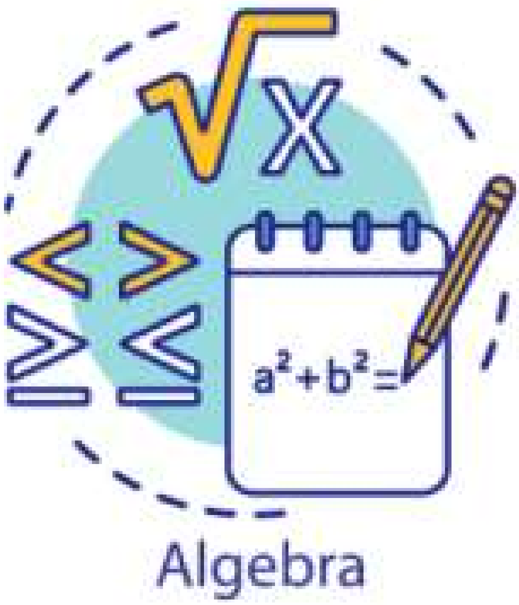
Chapter 4: Bodies of complex numbers.

الجَبْر هو فرع من علم الرياضيات وجاء اسم الجبر من كتاب عالم الرياضيات والفلكي والرحالة مُحَمَّد بن موسى الخوارزمي (الكتاب المختصر في حساب الجبر والمقابلة) الذي قدم العمليات الجبرية التي تنظم إيجاد حلول للمعادلات الخطية والتربيعية. والكلمة (الجبر) مأخوذة من اللغة العربية، ومعنى علم الجبر في قاموس المعاني: (فَرْعٌ مِنَ الرِّيَاضِيَّاتِ يَقُومُ عَلَى إِحْلاَلِ الرُّمُوزِ مَحَلَّ الأَعْدَادِ المَجْهُولَةِ أَوِ الْمَعْلُومَةِ)

أبو عبد الله مُحَمَّد بن موسى الخوارزمي عالم مسلم يكنى باسم الخوارزمي ولد حوالي164 هـ (781 م) و توفي حوالي 232 هـ أي (847 م). يعتبر من أوائل علماء الرياضيات المسلمين حيث ساهمت أعماله بدور كبير في تقدم الرياضيات في عصره. وقد عرض في كتابه (حساب الجبر والمقابلة) أول حل منهجي للمعادلات الخطية والتربيعية. ويعتبر مؤسس علم الجبر، (اللقب الذي يتقاسمه مع ديوفانتوس (في القرن الثاني عشر)، ولقد قدمت ترجمات اللاتينية عن حسابه على الأرقام الهندية، النظام العشري إلى العالم الغربي.

CHAPTER 0 :

MATHEMATICAL REVIEW



I. Equations:

Definition: An equation is an equality between two algebraic quantities, which expresses a situation in which a value.

The solution to an equation: Solving an equation means finding all the values of the unknown that satisfy it. The values calculated are called the solutions of the equation

I.1 Equations of second degree 2:

Equations of second degree, also known as quadratic equations: These are equations which, after transformations, take the following form: a^2 +bx+c=0............(I).

where a, b and c are the coefficients of equation (I) and x is the unknown.

Complete equations. If $b \neq 0$ and $c \neq 0$, the equation is said to be complete.

- **Solving method:**
1) Calculate the discriminate $\Delta = b^2 - 4a.c$
2) Depending on the sign of Δ, determine the solutions of (I)

Δ >0	Δ=0	Δ < 0
(I) Has two real roots: $x_1 = \frac{-b-\sqrt{\Delta}}{2a}$, $x_1 = \frac{-b+\sqrt{\Delta}}{2a}$	(I) Has a double root in R : $x_0 = \frac{-b}{2a}$	(I) does not have solutions in R

I.2 Equations of degree 3:

These are equations of the form: $Q(x) = ax^3 + bx^2 + cx + d = 0$............(II).

Where a, b, c and d are the coefficients of equation (II) and x is the unknown.

- **Solving method:**

In order to solve (II) we need to :
1) Find a particular solution x_0 of (II) in R.
2) Using the method of identification or Euclidean division, find a polynomial of degree 2 P(x) /{ Q(x) = (x-x0).P(x).
3) Solve in R the equation P(x) = 0.
4) Derive the solutions of (II).

I.3 Bisquare equations:

These are equations which, after transformations, can be reduced to the form $ax^4 + bx^2 + c = 0$.......(III), where a, b and c are known numbers and x is the unknown. A two-sided equation is of degree 4 and has no terms of degree 3 or 1.

❖ **Solving method:**

In order to solve (III), we need to make the following change of variable: $X = x^2$, so (III) becomes a second-degree equation and is very simple to solve.

I.4 Irrational equations (equations with radicals)

These are equations where the unknown is in a radical.

❖ **Solving method:**

To solve these equations you must follow the following steps:

1) We determine the domain of study of the equation (dominance of existence of DE solutions)
2) We leave the root alone in one of the members and then we square the whole, thus we find the solutions
3) We have verified that these solutions belong to the domain of existence of solutions From DE

I.5 Trigonometric equations:

A trigonometric equation is an equation which involves at least one of the following functions: a trigonometric function, such as sine, cosine or tangent, a reciprocal trigonometric function, such as cosecant, secant or cotangent, or the inverse of these functions.

❖ **Solving method:**

To solve these equations, apply the following rules:

- $\cos a = \cos b \quad \Rightarrow \quad a = \pm b + 2k\pi$
- $\sin a = \sin b \Rightarrow \begin{cases} a = b + 2k\pi \\ a = \pi - b + 2k\pi \end{cases}$

$tga = tgb \quad \Rightarrow a = b + k\pi$

To solve this type of equation, we use the transformation rules shown in the following table

Circular (trigonometric) functions

Remarkable values, periodicity :

$\cos(-a) = \cos(a)$ $\sin(-a) = -\sin(a)$

$\cos(p + a) = -\cos(a)$ $\sin(p + a) = -\sin(a)$

$\cos(p - a) = -\cos(a)$ $\sin(p - a) = \sin(a)$

$\cos\left(\frac{p}{2} - a\right) = \sin(a)$ $\sin\left(\frac{p}{2} - a\right) = \cos(a)$

$\cos\left(\frac{p}{2} + a\right) = -\sin(a)$ $\sin\left(\frac{p}{2} + a\right) = \cos(a)$

$\tan(p + a) = \tan(a)$

Basic formulas :

$\cos^2(a) + \sin^2(a) = 1$ $\tan(a) = \frac{\sin(a)}{\cos(a)}$

$1 + \tan^2(a) = \frac{1}{\cos^2(a)}$ $\sin = \frac{\text{côté} - \text{opposé}}{\text{hypothénuse}}$

$\cos = \frac{\text{côté} - \text{adjacent}}{\text{hypothénuse}}$ $\tan = \frac{\text{côté} - \text{opposé}}{\text{côté} - \text{adjacent}}$

Addition formulas :

$\cos(a + b) = \cos(a)\cos(b) - \sin(a)\sin(b)$

$\cos(a - b) = \cos(a)\cos(b) + \sin(a)\sin(b)$

$\sin(a + b) = \sin(a)\cos(b) + \sin(b)\cos(a)$

$\sin(a - b) = \sin(a)\cos(b) - \sin(b)\cos(a)$

$\tan(a + b) = \frac{\tan(a) + \tan(b)}{1 - \tan(a)\tan(b)}$

$\tan(a - b) = \frac{\tan(a) - \tan(b)}{1 + \tan(a)\tan(b)}$

Duplication formulas :

$\cos(2a) = \cos^2(a) - \sin^2(a) = 2\cos^2(a) - 1$

$= 1 - 2\sin^2(a)$

$\sin(2a) = 2\sin(a)\cos(a)$

$\tan(2a) = \frac{2\tan(a)}{1-\tan^2(a)}$

Expression as a function of tan(a/2) :

$\cos(a) = \frac{1-\tan^2\left(\frac{a}{2}\right)}{1+\tan^2\left(\frac{a}{2}\right)}$ sin? $(a) = \frac{2\tan\left(\frac{a}{2}\right)}{1+\tan^2\left(\frac{a}{2}\right)}$

$\tan(a) = \frac{2\tan\left(\frac{a}{2}\right)}{1-\tan^2\left(\frac{a}{2}\right)}$

Linearisation of squares, Euler, Moivre :

$\cos^2(a) = \frac{1+\cos(2a)}{2}$ $\sin^2(a) = \frac{1-\cos(2a)}{2}$

$\cos(x) = \frac{e^{ix} + e^{-ix}}{2}$

$[\cos(x) + i * \sin(x)]^n = \cos(nx) + i * \sin(nx)$

Product → sum transformation:

$\cos(a)\cos(b) = \frac{1}{2}[\cos(a - b) + \cos(a + b)]$

$\sin(a)\sin(b) = \frac{1}{2}[\cos(a - b) - \cos(a + b)]$

$\sin(a)\cos(b) = \frac{1}{2}[\sin(a + b) + \sin(a + b)]$

$\cos(a)\sin(b) = \frac{1}{2}[\sin(a + b) + \sin(a - b)]$

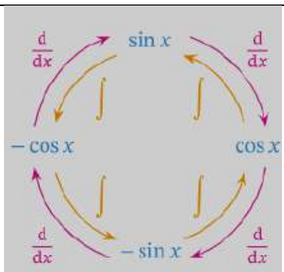
Sum → product transformation :

$\cos(p) + \cos(q) = 2\cos\left(\frac{p+q}{2}\right)\cos\left(\frac{p-q}{2}\right)$

$\cos(p) - \cos(q) = -2\sin\left(\frac{p+q}{2}\right)\sin\left(\frac{p-q}{2}\right)$

$\sin(p) + \sin(q) = 2\sin\left(\frac{p+q}{2}\right)\cos\left(\frac{p-q}{2}\right)$

$\sin(p) - \sin(q) = 2\cos\left(\frac{p+q}{2}\right)\sin\left(\frac{p-q}{2}\right)$



Functions and derivatives :

cos(x) de R dans $[-1; 1]$ $\Rightarrow$ $\cos'(x) = -\sin(x)$

sin(x) de R dans $[-1; 1]$ $\Rightarrow$ $\sin'(x) = \cos(x)$

tan(x) de $R\backslash\left\{\frac{kp}{2}, k \in Z\right\}$ dans R $\Rightarrow$ $\tan'(x) = 1 + \tan^2(x) = \frac{1}{\cos^2(x)}$

Table1: Circular (trigonometric) functions

I.6 Systems of linear equations

A linear equation with two unknowns is an algebraic identity of the type ax+by = c

The set of points (x,y) in the plane verifying ax+by = c is a straight line a,b,c are real numbers x, y are two unknowns

A system of two linear equations with two unknowns is a set of equations: $\begin{cases} ax + by = c \\ a'x + b'y = c' \end{cases}$

Solving this system means finding all the pairs of values (x,y) for which both equations are true simultaneously. It therefore means finding all the common solutions to the equations.

- **Solving method:**

To solve these equations we use several types of method, such as:

1) Solving by the substitution method.
2) Solving by the comparison method.
3) Solving by the combinations method

I.7 Non-Linear Equation Systems:

These are systems in which there are one or more non-linear equations (of degree greater than 1, with algebraic fractions, with radicals).

II. Inequations

An inequation consists of two members separated by one of the signs <, >,≤ ,≥. Solving an inequation means finding all the values of the unknown that satisfy it. These values are the solutions to the inequation. They often form an interval or a combination of intervals.

- $a < b$: [a is less than b or a is smaller than b]
- $a > b$: [a is greater than b or a is larger than b]
- $a \leq b$: [a is less than or equal to b]
- $a \geq b$: [a is greater than or equal to b]

To solve an inequation with one unknown, we transform the way it is written, leaving the algebraic expression in one member and a zero in the other. Then we solve it as an equation and the solution is an interval or a combination of intervals.

❖ **Properties of inequations:**

To solve an inequation, we transform the way it is written. We can :

- Add the same algebraic sum to both members
- Multiply or divide both members by the same strictly positive number, keeping the direction of the inequation
- Multiply or divide both members by the same strictly negative number, changing the direction of the inequation.

II.1 Second-degree equations 2:

Second-degree equations again: These are inequations in the following form $ax^2 + bx + c$ (<, >,≤ ,≥) 0.............(*).

❖ **Solving method:**

In order to solve (*) we need to :

1) Solve the second-degree equation appropriate to this inequation (associated 2nd-degree trinomial).
2) Calculate the discriminant $\Delta = b2 - 4a.c$ and calculate the roots of the trinomial and, depending on the sign of a, determine the sign of the trinomial using the following theorem:

Theorem :

A quadratic trinomial $P(x) = ax^2 + bx + c$, with $a \neq 0$ is :

❖ For $\Delta > 0$, always has the same sign as a outside the roots (when they exist) and the oppo sign between the roots.

❖ In particular, if $\Delta < 0$, the trinominal maintains a constant sign, the sign of a, for all x in R.

II.2 Inequation of degree 3:

These are inequations which take the following form: $ax3 + bx2 + cx + d$ (<, >,≤ ,≥) 0.............(**).

❖ **Method of solving:**

In order to solve (**) we need to:

1) Solve the equation of degree 3 appropriate to this inequation (associated 3rd degree trinomial).
2) Find the sign of the trinomial using the sign table.
3) Determine the interval or the combination of intervals of solutions.

II.3 Irrational inequations (equations with radicals)

These are inequations where the unknown is in a radical.

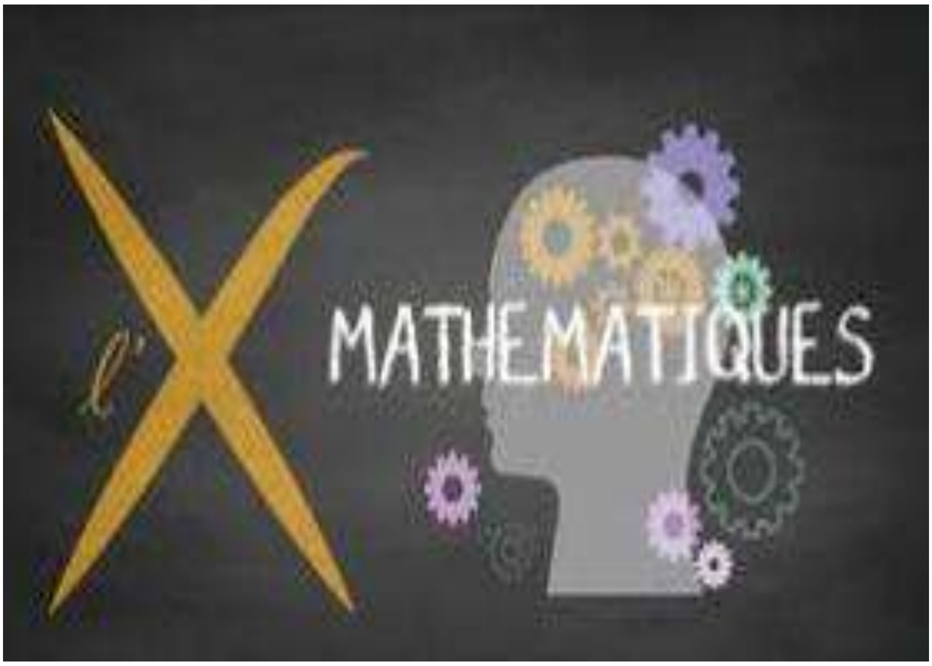
❖ **Method of solving:**

To solve these inequations, the following steps must be followed:

1) Determine the domain of the appropriate equation (domain existence of DE solutions)
2) Leave only the root in one of the members and then to the power of 2of the whole equation, thus finding the resolution domain DR.
3) The solution domain S is the intersection of the two domains obtained.

CHAPTER 1:

REMINDERS ON: CONCEPTS OF LOGIC, METHODS OF REASONING AND BINARY RELATIONS



I. Help with reasoning

I.1. Analysis of the problem

The first thing to do is to distinguish between the hypotheses (= true propositions) and the question (= proposition to be demonstrated): you need to be able to distinguish clearly between what is known or accepted and what needs to be demonstrated.

Once all the hypotheses have been recognized, they need to be made explicit, and possibly written down in a synonymous form: introduce notations (you need to name the objects so that you can talk about them), and remember the properties that the hypotheses may imply.

Note: Hypotheses generally come into play during reasoning; they are rarely the starting point. You need to find the right place to use them.

- We then look at the problem to be shown:
- Try to relate it to a problem that has already been solved
- Drawing a figure can be helpful (but not a demonstration)
- Look at special cases (to get an idea)
- Do not forget any hypotheses and do not weaken them.

I.2 Writing a demonstration

- Announce what you are going to do and give your conclusions: structure your demonstrations
- Fix your notations: if you introduce a new notation, define it clearly: ‘let x be a positive real number’
- Justify the stages of your demonstration: cite the theorems with their hypotheses and check these hypotheses (multiplication of an inequality by a positive term)
- Reread the demonstration to see if it is clear and check that you have not forgotten any cases.

I.3. Some mistakes to avoid

- Beware of negations
- Some examples do not make a demonstration
- Beware of notations: do not give the same name to two different objects and do not assume that two objects with two different names are distinct.

II. Notions of Logic:

Logic defines the formal rules that any correct reasoning must adhere to. The concept of proof is essential in mathematics and, more generally, in all scientific activities. It is not just about determining what is true or false but about being able to provide explanations that justify that the announced result is valid.

II.1 Assertion :

- **Definition 1:** A proposition (or assertion) is a mathematical statement that can take two values: true (T or 1) or false (F or 0).
- **Definition 2:** Within the framework of a given mathematical theory, an assertion is a mathematical sentence to which one, and only one, boolean value can be assigned, namely true (T for short) or false (F for short). Some assertions are declared true a priori: these are the axioms; otherwise, the truth of an assertion must result from a proof.

II.2 Operations on assertions (logical connectors):

1) **Negation :** Let P be a proposition. We call the negation of P and we note not P or $\overline{P}$, the proposition defined by: $\overline{P}$ is true when P is false; $\overline{P}$ is false when P is true.
2) **Conjunction:** Let P and Q be two propositions. We call the conjunction of P and Q the proposition denoted (P and Q) or (P.Q) or ($P \wedge Q$) and defined as follows: P and Q is false when at least one of the two propositions is false.
3) Disjunction: Let P and Q be two propositions. We call disjunction of P and Q the proposition denoted (P or Q) or (P + Q) or ($P \vee Q$) and defined as follows: P or Q is false when P and Q are false.
4) **Implication:** Let P and Q be two propositions. We call implication of Q by P the proposition not P or Q. ($\overline{P} \vee Q$) .This proposition is noted $P \Rightarrow Q$.

- Vocabulary: the proposition $P \Rightarrow Q$ reads ≪ P implies Q ≫ or even ≪ if P then Q ≫
- Note: when $P \Rightarrow Q$ is true, we say that P is a sufficient condition for having Q, or that Q is a necessary condition for having P.
- Reciprocal:. Let P and Q be two propositions. We call the reciprocal of $P \Rightarrow Q$ the implication $Q \Rightarrow P$

5) Equivalence: Let P and Q be two propositions. We call equivalence of P and Q the proposition $P \Rightarrow Q$ and $Q \Rightarrow P$. This proposition is denoted $P \Leftrightarrow Q$.

- ❖ Vocabulary: the proposition $P \Leftrightarrow Q$ reads ≪ P if and only if Q ≫.
- ❖ Note: When $P \Leftrightarrow Q$ is true, P is a necessary and sufficient condition for having Q. Thus, equivalences are the necessary and sufficient conditions

II.3 Truth table of logical connectors

Let P and Q be two propositions

P	**Q**	$\overline{P}$	$\overline{Q}$	**P∧Q**	**P ∨ Q**	**P ⇒ Q**	$\overline{P} \vee Q$	**Q ⇒ P**	$\overline{Q} \vee P$	**P ⇔ Q**
0	0	1	1	0	0	1		1		1
0	1	1	0	0	1	1		0		0
1	0	0	1	0	1	0		1		0
1	1	0	0	1	1	1		1		1

II.4 Quantifiers :

Quantifiers are symbols used to write statements. A quantified sentence is a mathematical assertion containing one or more quantifiers

- The symbol ∃ designates the existential quantifier. Thus, ∃x, reads "there exists at least one element x" and ∃!x means "there exists one and only one element x."
- The symbol designates the universal quantifier, and ∀x means "for every element x."

The letter affected by a quantifier is silent and can be replaced by any other letter that does not already have meaning in the statement.

Note: The negation of a quantified sentence is defined as follows:

- ❖ (no ($\forall x \in E$, P(x))),($\exists x \in E$, no P(x)),
- ❖ (no ($\exists x \in E$, P(x))),($\forall x \in E$, no P(x)).

III. Reasoning methods:

Reasoning is the means of validating or invalidating a hypothesis and explaining it to others. The question is what type of reasoning is needed to arrive at the expected result. Here are some classic methods of reasoning

III.1 Direct reasoning:

We want to show that the assertion "$P \Rightarrow Q$" is true. We assume that P is true and we show that then Q is true. This is the method you are most accustomed to.

Example :

Show that the sum of two rational numbers is rational:

Let $a = \frac{p}{q}$ et $b = \frac{r}{s}$ be two rational numbers with: $p,q, r, s \in Z$ and $q,s \neq 0$.

The numerator $ps+qs$ is an whole number, , the denominator is a whole number non-zero , so a+b is rational ⇒ This is direct reasoning because we start from the definition and conclude directly.

III.2 Contra positive reasoning:

Reasoning by contraposition is based on the following equivalence:
The assertion "P ⇒ Q" is equivalent to "$\overline{Q} \Rightarrow \overline{P}$"
So if we wish to show the assertion "P ⇒ Q", we actually show that if non(Q) is true then non(P) is true

Example:

Show that if n^2 is even then n is even:

We prove the contrapositive: if n is odd then n^2 is odd

Since n is odd then $n= 2k+1$ with $k \in Z \Rightarrow n^2 = (2k+1)^2 = 4k2+ 4k+1 = 2(2k2+2k)+1 = 2k'+1$ □

n^2 is odd ⇒ So the contrapositive is true ⇒ the original statement is true

III.3 Reasoning by the Absurd:

Reasoning by the absurd to show "P ⇒ Q" is based on the following principle: we assume both that P is true and that Q is false, and we look for a contradiction. So if P is true then Q must be true and therefore "P ⇒ Q" is true.

Example:

Show that if a and b are positive real numbers and: $\frac{a}{1+b} = \frac{b}{1+a}$ then a=b.

Suppose by contradiction that a ≠ b : ⇒ a(1+a)=b(1+b) ⇒□ $a+a^2=b+b^2$

Therefore: $a^2-b^2 = b-a$ ⇒□ (a-b)(a+b)=-(a-b), Since a ≠ b we can divide by a-b □⇒ a+b = -1

Now a,b=0, so a+b=0, contradiction. So a=b.

II-4 Reasoning by counterexample

If we want to show that an assertion of the type « ∀ x ∈ E, P(x) » is true, then for each x of E we must show that P(x) is true. On the other hand, to show that this assertion is false, all we need to do is find x in E such that P(x) is false (remember that the negation of « ∀ x ∈ E, P(x)» is « ∃ x ∈ E, P(x) non P(x) »). To find such an x is to find a counterexample to the statement « ∀ x ∈ E, P(x) »

Example :

Show that the following implication is false: $\forall x \in \mathbb{Z}, x < 4 \Rightarrow x^2 < 16$

$\exists x = -6$ telque $x = -6 < 4$ et $x^2 = (-6)^2 = 36 > 16$

II.5 Reasoning by disjunction of cases (or case by case) :

If we wish to verify an assertion P(x) for all x in a set E, we show the assertion for x in a part A of E, then for x not belonging to A. This is the disjunction or case-by-case method.

Example:

Show that for all $x \in \mathbb{R}, |x-1| \leq x^2 - x + 1$

1) 1st case : $|x-1| = -x+1$ for $x < 1 \Rightarrow -x+1 \leq x^2 - x + 1 \Rightarrow 0 \leq x^2$ is still true $\forall x \in R$.
2) 2st case : $|x-1| = x-1$ for $x > 1 \Rightarrow 0 \leq x^2 - 2x + 2 \Rightarrow 0 \leq (x-1)^2 + 1$ is still true $\forall x \in R$.

From both cases, concluded that: $|x-1| \leq x^2 - x + 1 \quad \forall x \in R$

II-6 Reasoning by recurrence:

The principle of recurrence is used to show that an assertion P(n), depending on n, is true for all $n \in N$. Demonstration by recurrence takes place in three stages:

1) **Stage 1 Initialisation:** during initialisation we prove $P(n_0)$.
2) **Step 2 Heredity:** Assume $n \geq 0$ given with P(n) true and show that the assertion P(n + 1) is true.
3) **Step 3 Conclusion:** Finally, in the conclusion, recall that by the principle of recurrence P(n) is true for all $n \in N$.

Example:

Show that for any natural number $n \geq 1$, $1 + 2 + \cdots + n = \frac{n(n+1)}{2}$

Initialization: for n = 1, $1 = \frac{1\times 2}{2} = 1$, true.

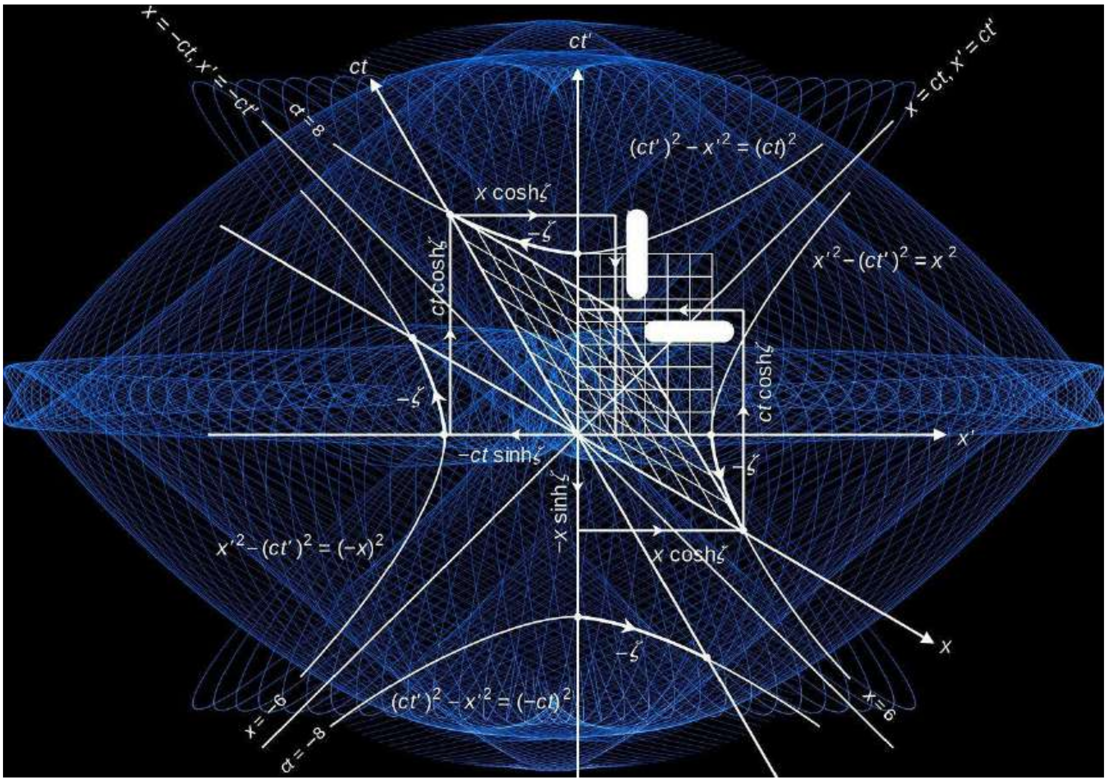
Heredity: Let's assume that this is true for n, then:

$$1 + 2 + \cdots + n + (n+1) = \frac{n(n+1)}{2} + (n+1) = \frac{n(n+1) + 2(n+1)}{2} = \frac{(n+1)(n+2)}{2}$$

So this is true for n + 1. By induction, the formula is true for all n.

CHAPTER II

BINARY RELATIONS AND APPLICATIONS



I. Binary relations:

A binary relation on a set E is given by a part Γ of E×E. We write x $\mathcal{R}$y if (x,y)∈ Γ.

The relation $\mathcal{R}$ is:

- **Reflexive:** if, for all x∈E, x $\mathcal{R}$x.
- **Symmetric:** if, for all x,y∈E, if x $\mathcal{R}$y, then y $\mathcal{R}$x.
- **Antisymmetric:** if, for all x,y∈E, if x$\mathcal{R}$y and y$\mathcal{R}$x, then x=y.
- **Transitive:** if, for all x,y,z ∈E, if x $\mathcal{R}$y and y$\mathcal{R}$z, then x$\mathcal{R}$z.
- **An equivalence relation** is a reflexive, symmetric, transitive relation.
- **An order relation** is a reflexive, antisymmetric, transitive relation.

Remarks :

- If $\mathcal{R}$ is an equivalence relation and x is an element of E, then the set of elements equivalent (in relation) to a is called the equivalence class of a: $\bar{a} = \dot{a} = \{x \in E \ / x\Re a\}$.
- If $\mathcal{R}$ is an order relation on E, then we say that the order is total if we can always compare two elements of E: for all x,y∈E, we have x$\mathcal{R}$y or y$\mathcal{R}$x. Otherwise, the order is said to be partial.

Example:

1. Equivalence relation

Example: Let E = ℝ \ {0} be the set of non-zero real numbers. A relation $\mathcal{R}$ is defined by :

$x\boldsymbol{R}\, y \Leftrightarrow \frac{x}{y} > 0$ In other words, x and y have the same sign.

Show that $\mathcal{R}$ is an equivalence relation

1. Reflexive: For all $x \neq 0$, $\frac{x}{x} = 1 > 0$ x therefore x$\mathcal{R}$x

2. Symmetric: If x$\mathcal{R}$y, then $\frac{x}{y} > 0$ *so* $\frac{y}{x} > 0$ y > 0 → y$\mathcal{R}$x

3. Transitive: If x$\mathcal{R}$y and y$\mathcal{R}$z, then $\frac{x}{y} > 0$ and $\frac{y}{z} > 0$ So $\frac{x}{z} = \frac{x}{y} \cdot \frac{y}{z} > 0 \Rightarrow x\boldsymbol{R}\, z$

Conclusion: $\mathcal{R}$ is an equivalence relation.

2. Total order relationship

Example: Let E = ℝ, the relation $\mathcal{R}$ is defined by : x$\mathcal{R}$y ⇔ x≤ y

Show that $\mathcal{R}$ is a relation of order

1. Reflexive: x ≤ x

2. Antisymmetric: if x ≤ y and y ≤ x, then x = y

3. Transitive: if $x \leq y$ and $y \leq z$, then $x \leq z$

4. Totality: For all $x, y \in \mathbb{R}$, either $x \leq y$, or $y \leq x$

Conclusion: this is a relation of total order in $\mathbb{R}$.

II. Applications:

II.1 Definition:

Let E and F be two sets, We call an application of E in F a mathematical object f which associates with any element x of E an element f(x) of F. Such an application is denoted f

$$\begin{cases} E \to F \\ x \to f(x) \end{cases}$$

E and F are called the start set and end set of f respectively.

II.2 Image and antecedent :

Let f be an application from E into F

- If $x \in E$, f(x) is called the image of x by f.
- Let $y \in F$, if there exists x such that $y = f(x)$, x is called a antecedent of y by f.

Remarks:

- An element of E always has a unique image by f. An element of F can have zero, one or more antecedents by f.
- Do not confuse f and f(x). f is an application whereas f(x) is an element.
- Let $f : E \to F$ be an application. The set of elements of F which have an antecedent of f in E is called the image of f and is denoted by Imf. More formally
  $\text{Imf} = \{ y \in F / \exists x \in E, y = f(x)\} = \{f(x), x \in E\}$

Example:

1) The correspondence $f : Z \to N$ such that $f(n) = n^2$ is an application of Z in N. We have $f(1) = 1$, $f(-2) = 4$ and $f(3) = 9$.
2) The mapping $h : R \to R$, defined by $h(x) = 1/x-1$ is not an application, because 1 has no image by h.
3) The correspondence that associates with each month the possible number of days in the month is not an application of the set M of months in N, because it associates with February the two elements 28 and 29.

Properties

Let: f: A → B be a function, A1 and A2 be subsets of A, and B1 and B2 be subsets of B. The following properties hold :

- $f(A_1 \cup A_2) = f(A_1) \cup f(A_2)$
- $f(A_1 \cap A_2) = f(A_1) \cap f(A_2)$
- $f^{-1}(B_1 \cup B_2) = f^{-1}(B_1) \cup f^{-1}(B_2)$
- $f^{-1}(B_1 \cap B_2) = f^{-1}(B_1) \cap f^{-1}(B_2)$

II.3 Representations of applications

The representation of an application f : A → B depends on the nature of the sets A and B: The most commonly used representations are the following:

a) Representation by means of a formula:

Example: Let be the function f : Z→ N such that f (n) = n^2

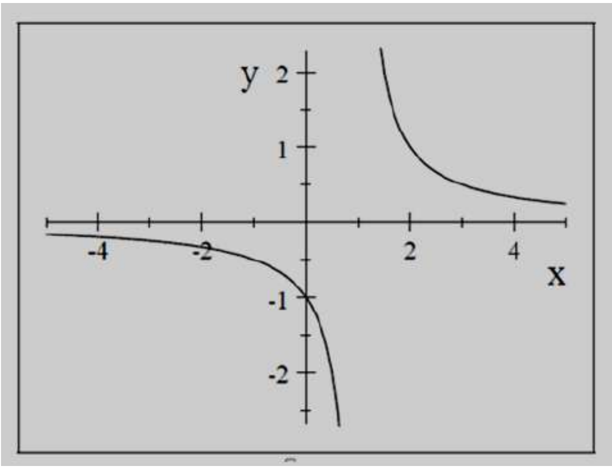
b) Representation using a table of values (useful when A is finite).

Example: Let be the function g_1 : {-2;-1; 0; 1; 2; 3}→N such that:

n	-2	-1	0	1	2	3
g_1 (n)	4	1	0	1	4	9

c) Representation by means of a graph:

example: or the function g_2 : R-{1]→ R



II.4 Composed of applications

Let f : E → F and g : F → G be two applications. The application $\begin{cases} E \to G \\ x \to g(f(x)) \end{cases}$ is called the composite of f followed by g and is denoted g o f.

Notes:

In the notation gof, g precedes f but we perform f first and then g. This notation convention is due to the fact that (gof)(x) = g(f(x)).

- If f and g are two applications of E in E, in general fog ≠ gof.
- If fo g = go f, f and g are said to commute.

Example :

Let the applications f : Z→Z and g: Z→N be defined by f(n) = n + $(-1)^n$ and g(n) = n^2

The composite of f and g is the function gof : : Z→N such that: gof(n) = $(n + (-1)^n)^2$

II.5 Direct image, reciprocal image :

II.5.1 Direct image :

Let f : E → F be an application and A be a part of E. We call the direct image of A by f, denoted f(A) , the set of elements of F which are images of elements of A which have an antecedent in A, in other words: $f(A) = \{ y \in F / \exists x \in A, y = f(x)\} = \{f(x), x \in A\}$

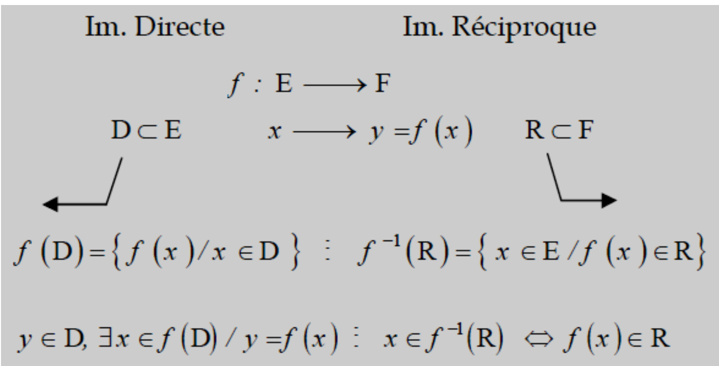
II.5.2 Reciprocal image :

Let f : E → F be an application and B a part of F. We call the reciprocal image of B by f, denoted $f^{-1}(B)$, the set of elements of E which are antecedents of elements of B which have an image in B, in other words : $f^{-1}(B) = \{ x \in E / f(x) \in B\}$

Exemple:

Let *f* be defined in R as f(x)=$2x^2$-1 and let A=[-1,3].

Determine the direct and reciprocal images of A by *f*.



1) The direct image of A by f :

$f(A) = \{f(x)/x \in A\} = \{f(x)/x \in [-1,3]\} = [-1,17]$

2) The reciprocal image of A by f :

$f^{-1}(A) = \{x \in \mathbb{R}/f(x) \in A\} = \{x \in \mathbb{R}/(2x^2 - 1) \in [-1,3]\} = [-\sqrt{2}, \sqrt{2}]$

II.6 Injectivity, surjectivity and bijectivity:

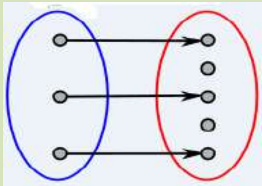
II.6.1 Injectivity:

An application $f : E \to F$ is said to be injective or an injection if one of the following statements is true:

$\forall x, x' \in E, f(x) = f(x') \Rightarrow x = x'$

$\forall x, x' \in E, x \neq x' \Rightarrow f(x) \neq f(x')$

Any element of F has at most one antecedent by f



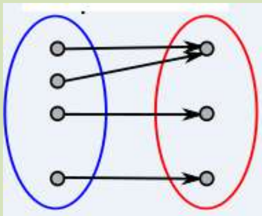
II.6.2 Surjectivity :

We say that an application f : E → F is surjective or that it is a surjection if one of the following propositions is true: :

$\forall y \in F, \exists x \in E,\ y = f(x)$

Imf = F

Every element of F has at least one antecedent by f



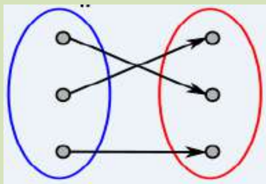
II.6.3 bijectivity :

We say that an application f : E → F is bijective or that it is a bijection if one of the following propositions is true:

$\forall y \in F, \exists\ x \in E,\ y = f(x)$

Injective and surjective *f*

Every element of f has a unique antecedent by *f*



Note: To show that f : E → F is bijective, we give y ∈ F and we show that the equation y = f(x) with unknown x ∈ E has a unique solution.

Example:

We give the function *f* defined from]1 , +∞[to]0, +∞[by: $f(x) = \frac{1}{x-1}$

Let us show that *f* is bijective:

f is bijective ⇔ *f* is injective and *f* is surjective

Injectivity of *f*:

$\forall x_1, x_2 \in]1, +\infty[\quad f(x_1) = f(x_2) \Leftrightarrow \frac{1}{x_1-1} = \frac{1}{x_2-1} \Rightarrow x_2 - 1 = x_1 - 1 \Rightarrow x_1 = x_2; f$ is injective

Surjectivity of *f*:

$\forall y \in]0,+\infty[, \exists x \in]1,+\infty[; y = f(x) \quad \Rightarrow \quad y = f(x) \Leftrightarrow y = \frac{1}{x-1} \Leftrightarrow \ xy - y = 1 \ \Rightarrow \ xy = 1 + y$

$\Rightarrow x = \frac{1+y}{y} \in \left]1,+\infty\right[; f$ is surjective $\Rightarrow$ f is bijective

II.6.4 Monotonicity bijection theorem:

Let I be an interval of R and f a continuous and strictly monotonic function on the interval I. Then f establishes a bijection from I to the interval J=f(I). Moreover, if I=[a,b], we have:

- If f is increasing, f(I)=[f(a),f(b)]
- If f isdecreasing, f(I)=[f(b),f(a)].

Analogous results hold if I is an open or semi-open interval (where a and b may be equal respectively at +∞ , -∞

II.6.5 Inverse Bijection Theorem:

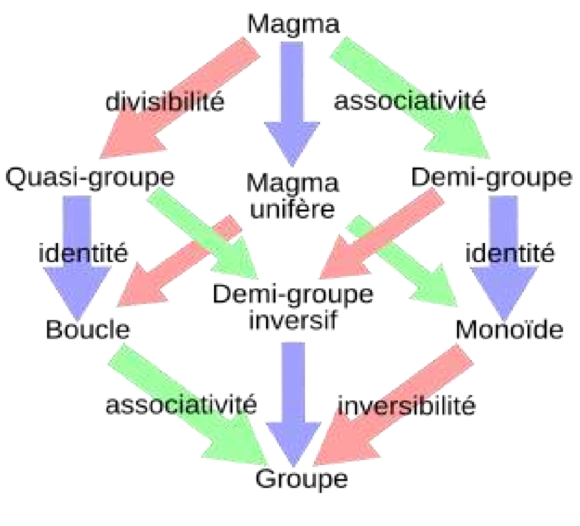
Let f : E → F be a bijection. The inverse bijection of f is the map f^{-1} : F → E that associates its unique antecedent by f with every element of F.

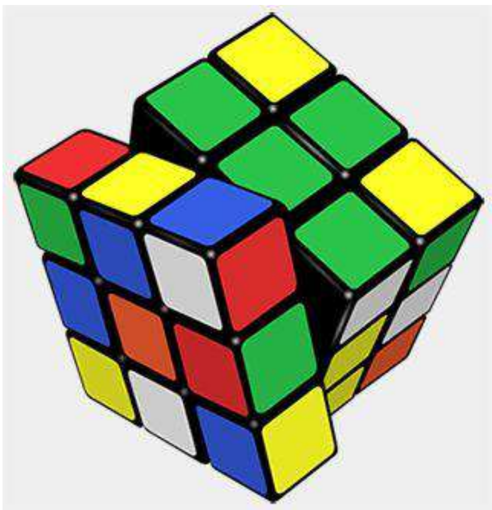
II.6.6 Bijectivity and Composition:

Let f : E → F and g : F → G be two maps. If f and g are bijective, then gof is bijective and $(gof)^{-1} = f^{-1} \circ g^{-1}$.

CHAPTER III

ALGEBRAIC STRUCTURES





I. Laws of internal composition:

I.1. Definition:

Let E be a set. An internal composition law (ICL) on E is an application (*) of E × E in E, which to any pair (x, y) we make correspond a unique element z of E generally noted in an infix way: $z = x * y$.

x: the first term, y: the second term, and z the composite of x and y]

Example: (N , +) , (Z, -), (R, ∗)

Remarks:

1) A set equipped with an internal composition law (E,) is called MAGMA
2) If $z \notin E$ or (x or y) are from two different sets then (∗) and an external composition law (an ECL)]

I.2 Properties of internal composition laws:

Let E be a set equipped with an internal composition law (∗):

I.2.1 Commutativity

$$(*) \text{ commutative} \Leftrightarrow \forall (x,y) \in E^2, x * y = y * x$$

I.2.2 Associativity :

$$(*) \text{ est associative} \Leftrightarrow \forall (x,y,z) \in E^3, (x * y) * z = x * (y * z)$$

I.2.3 Neutral element:

Let E be a non-empty set and (∗) an internal law on E.

Let $e \in E$, e is a neutral element for (∗) ⇔ $\forall x \in E$, $e * x = x * e = x$

Theorem: If ∗ admits a neutral element, it is unique.

Remark:

For a commutative law, a neutral element on one side is a neutral element.

Examples:

- The internal composition laws "+" and "x" on ℂ are associative and commutative and admit as respective neutral elements 0 and 1.
- The internal composition law "-" on ℝ is neither associative nor commutative on ℝ, moreover it does not have a neutral element.]

Soient E un ensemble non vide et (∗) et (T) deux lois interne sur E.
(T) est distributive sur (∗)
Soit e∈E, e est élément neutre pour (∗) ⇔ ∀x∈E, e∗x = x∗ e = x

I.2.4 Distributivity of one law over another:

Let E be a non-empty set and (∗) and (T) be two internal laws on E. (T) is distributive on (*) $\Leftrightarrow \forall (x,y,z)\in E^3, xT(y*z) = (xTy)*(xTz)$ et $(y*z)Tx = (yTx)*(zTx)$

I.2.5 Absorbent element :

Let E be a non-empty set and (∗) an internal law on E. let a∈E. a is an absorbing element for (∗) $\Leftrightarrow \forall x\in E, a*x = x*a = a$

Note: For a commutative law, an element absorbing on one side is absorbing.

I.2.6 Symmetrizable element:

Let E be a non-empty set and (*) an internal law on E with a neutral element e, let x∈E:

- x admits a left symmetric for ∗ $\Leftrightarrow \exists x' \in E / x' * x = e$ (x is left symmetrizable)
- x admits a right symmetric for∗ $\Leftrightarrow \exists x' \in E / x * x' = e$ (x is right symmetrizable)
- x admits a symmetric for ∗ $\Leftrightarrow \exists x' \in E / x' * x = x * x' = e$ (x is symmetrizable)

Note: For a commutative law, an element symmetrizable on one side is symmetrizable

Examples:

In ℝ with the internal composition law "+", the inverse of any element x∈ ℝ is denoted –x.

In ℝ* with the internal composition law "x", the inverse of any element x∈ ℝ* is denoted $\frac{1}{x}$

I.2.7 Simplifiable Element

Let E be a nonempty set and (*) an internal law on E with a neutral element, let x?E:

1) x is left-simplifiable for ∗ $\Leftrightarrow \forall (y,z)\in E^2, x*y = x*z \Rightarrow y = z$

2) x is right-simplifiable for ∗ $\Leftrightarrow \forall (y,z)\in E^2, y*x = z*x \Rightarrow y = z$

3) x is simplifiable ⇔ x is left-simplifiable and x is right-simplifiable

Note: For a commutative law, an element that is simplifiable on one side is simplifiable.

I.2.8 Stable parts :

Soit E un ensemble non vide et (∗)une loi de composition interne sur E.Soit F une partie non vide de E. Alors F est stable pour (∗) ⇔ $\forall (x , y) \in F^2$, x∗ y∈F

I.3 MONOID:

Definition: An associative uniferous MAGMA is called a **MONOID**

Properties:

1) In a monoid any symmetrical and simplifiable element

2) Every symmetrical element admits a unique symmetry

3) Let (E, $*$) be a monoid, for any pair (x , y) of invertible elements of E the composite $x*y$ is invertible and; $(x*y)^{-1} = y^{-1} * x^{-1}$.

II. Group

II.1. Definition:

Let G be a non-empty set with an internal composition law then (G,) is a group $\Leftrightarrow$:

1) $*$ is associative
2) $*$ Has a neutral element in G
3) Every element of G has a symmetry for $*$ in G]

If moreover is commutative, the group (G, $*$) is said to be commutative or abelian.

Examples :

- ($\mathbb{Z}$,+), ($\mathbb{Q}$, +), ($\mathbb{R}$,+) and ($\mathbb{C}$, +) are abelian groups.
- ($\mathbb{N}$, +) is not a group.

II-2 Subgroup: Let (G,) be a group and H a subset of G $(H \subset G)$.

$$\text{H issubgroup of de } (G, *) \Leftrightarrow \begin{cases} e \in H \quad \text{ou} \quad H \neq \varphi \\ \forall (x, y) \in H^2, x*y \in H \\ \forall x \in H, x' \in H \quad \text{(x': symmetric element for x by } *) \end{cases}$$

Theorem: If H and K are subgroups of (G,$*$), H∩K is a subgroup of (G,$*$). Thus, an intersection of subgroups is a subgroup

II.3 Group Morphisms:

Let (G,) and (H,T) be two groups. A map from G to H is a "group morphism" when:

$\forall x, y \in G, f(x*y) = f(x)Tf(y)$

1) If G = H and *=T, we speak of an endomorphism.
2) If f is bijective, we speak of an isomorphism.
3) If f is a bijective endomorphism, we speak of an automorphism.

Theorem: Let (G_1,T_1), (G_2, T_2) and (G_3, T_3) be three groups and $f : G_1 \rightarrow G_2$ and $g : G_2 \rightarrow G_3$ be two group morphisms. Then (gof) is a group morphism. If f and g are group isomorphisms, then (gof) is also.

III. Ring :

III.1. Definition:

Let A be a non-empty set having at least two elements equipped with two internal composition laws noted $(+ , *)$

$$(A, +, *) \text{ is a ring } \Leftrightarrow \begin{cases} (A,+) \text{ group commutative or abelian} \\ * \text{ associativ and et has a natural element in A} \\ * \text{ distributived by } + \end{cases}$$

ring is commutatif $\Leftrightarrow *$ are commutative.

Examples :

- $\mathbb{Z}$,+, *) is a ring for the usual laws « + » et « x »
- ($\mathbb{Q}$,+, *) is a ring for the usual laws « + » et « x »]
- $\mathbb{R}$,+, *) is a ring for the usual laws « + » et « x »]
- ($\mathbb{C}$,+, *) is a ring for the usual laws « + » et « x »]

III-2 Subring

Let (A, + , x) be a ring and B a subset of A, then B is a subring of A if:

- ✓ (B , +) is an abelian subgroup of (A,+)]
- ✓ $1_A \in B$
- ✓ $\forall x, y \in B \Rightarrow x \text{ x } y \in B$

Example : $\mathbb{Z}$ is subring for $\mathbb{Q}$.

Theorems:

1) Let A and B be two rings, an application f from A to B is a ring morphism ⇔ $\forall x, y \in A$, $f(x+y) = f(x) + f(y)$; $f(x \text{ x } y) = f(x) \text{ x } f(y)$; $f(1_A) = 1_B$
2) When there exist in a ring A elements a and b such that $a \neq 0_A$, $b \neq 0_A$ and $a \text{ x } b = 0_A$, we say that a and b are divisors of zero. An integral ring is a commutative ring, not reduced to $\{0_A\}$ and without divisor of zero.

IV. Field :

IV.1. Definition:

[Let (K, +, x) be a ring. (K, +, x) is a field ⇔ every non-zero element of K has an inverse for x in k plus if (x) is commutative then (K, +, x) is a commutative field]

Examples: (Q,+,×), (R,+,×) et (Care commutative field

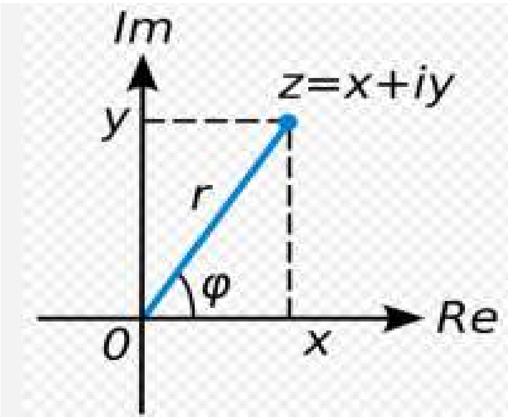
IV.2 Subfield :

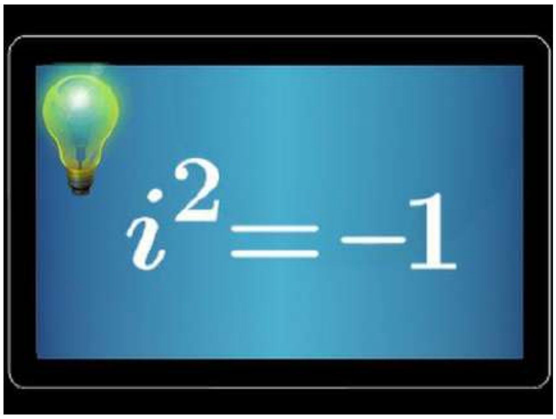
Let (K, +, x) be a field and L a part of K. (L, +, x) is a subfield of K if] :

- ❖ (L,+, x) is a subring of K , not reduced to $\{0_K\}$]
- ❖ $\forall x \in$ L with $x \neq 0_K$, the opposite $\frac{1}{x} \in$ L

CHAPTER IV

COMPLEX NUMBERS





I. Definition:

Complex numbers are expressions of the form $z = a + bi$, with $i^2 = -1$. The set of complex numbers is denoted by $\mathbb{C}$.

We call a the real part and b the imaginary part of the complex number z. We denote them $a = \text{Re}(z)$ and $b = \text{Im}(z)$ so a complex number is equal to zero if and only if $a = 0$ and $b = 0$.

Then the set $\mathbb{C}$ of complex numbers is the set which:

- Contains all real numbers
- Is equipped with an addition and a multiplication verifying the same properties as the corresponding operations of $\mathbb{R}$.
- Contains a number i such that $i^2 = -1$ and made up of all numbers $z = a + ib$, with a and b in $\mathbb{R}$.

II. Real part and imaginary part:

The algebraic form of a complex number is x + iy where x and y are two real numbers.

If $z = x + iy$ where $x \in \mathbb{R}$ and $y \in \mathbb{R}$, x is the real part of z, denoted $Re(z)$, and y is the imaginary part of z, denoted $Im(z)$.

- The real part and the imaginary part of a complex number are real numbers. Example: the imaginary part of $3 + 2i$ is 2 and not $2i$.
- Real numbers are complex numbers whose imaginary part is zero.
- Pure imaginaries are complex numbers whose real part is zero
- Two complex numbers are equal if they have the same real part and the same imaginary part.
- A complex number is zero if its real part and its imaginary part are zero.
- Any complex number of the form $z = yi$ (where $y \in \mathbb{R}$) is called a pure imaginary.
- The set of pure imaginaries is denoted i_R

Note: We consider two complexes z and z' of respective algebraic forms $x+iy$ *et* $x'+iy'$.

- The sum of z and z' is the complex $z+z' = x+x'+i(y+y')$.
- If k is a real number, then the product of k and z is the complex number $kz = kx + iky$.
- The product of z and z' is the complex number $zz' = xx'-yy'+i(xy'+yx')$.

III. Conjugate of a complex number:

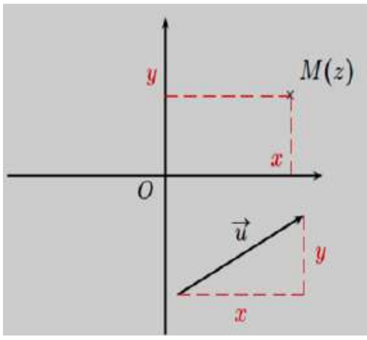
Consider a complex number z of algebraic form $x+iy$. The complex number $x-iy$, denoted $\bar{z}$ is the conjugate of z

Properties :

For all complexes z and z' of algebraic forms z = x+iy et z′ = x′+iy′ :

- $\bar{\bar{z}} = z$; $z+\bar{z} = 2\operatorname{Re}(z) = 2x$; $z-\bar{z} = 2i\operatorname{Im}(z) = 2iy$; $z\bar{z} = x^2+y^2$
- $\overline{z+z'} = \bar{z}+\bar{z}'$; $\overline{zz'} = \bar{z}.\bar{z}'$; $if\ z'\ no$ zero $\overline{\left(\frac{z}{z'}\right)} = \frac{\bar{z}}{\bar{z}'}$
- z *is real if* $z = \bar{z}$; z *is pure immaginary if* $z = -\bar{z}$

IV. Affix of a point, affix of a vector. Point image, vector image of a complex number:

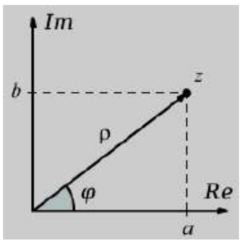


If M is the point with coordinates *(x,y)*, the affix of M is the number $z_M = x+iy$
If u is the vector with coordinates *(x,y)*, the affix of u is the number $z_u = x+iy$.

If $z = x+iy$ or x and y are two real numbers then :

- ❖ The point image of z is the point $M(x, y)$
- ❖ The vector image of z is the vector $u(x, y)$

- The point M is called the image of the complex number z
- The complex number $z= a+bi$ is often represented by the vector OM . The angle that this vector makes with the horizontal axis is called the argument of z. The length of the vector $\rho=|z|$ is called the modulus



- The vector OM is called the image vector of the complex number z
- The complex number z is called the affix of the point M
- The plane, considered as the set of points M(x, y) is called a complex plane, or Cauchy plane

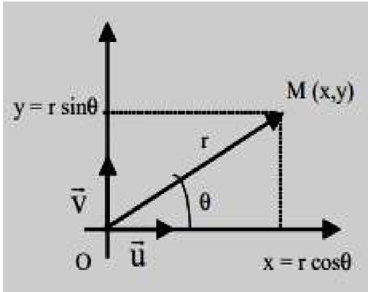
- The Ox axis which corresponds to the points such that $y = 0$, $z = x$, is the real axis (Re); the Oy axis which corresponds to the points such that $x = 0$, $z = iy$ is the pure imaginary axis (Im)

Application: if Z_A is the affix of A and Z_B is the affix of B, then the affix of the vector $\vec{AB}$ are $Z_B - Z_A$: $\text{aff}(\vec{AB}) = Z_B - Z_A$

V. Module and argument :

Consider a non-zero complex number z, the affix of a point M in a plane with a direct orthonormal reference point $(O,\vec{u},\vec{v})$.

If M has polar coordinates (r,θ) , then r is the modulus of z, noted ρ and θ is an argument of z, noted arg z.



Note : On a $|z| = \rho_z = = r = OM$ et $arg\ z = \theta = (\vec{u},\overrightarrow{OM})\ (2\pi)$.

Theorem :

For any complex number z whose image M has Cartesian coordinates (x; y) and polar coordinates (r ; θ), we have

$$\begin{cases} x = r.\cos\theta \\ y = r.\sin\theta \end{cases} \Leftrightarrow \begin{cases} r = \sqrt{x^2 + y^2} \\ \cos\theta = \dfrac{x}{r}\ et\ \sin\theta = \dfrac{y}{r} \end{cases}$$

Note : $\text{tg}\theta = \dfrac{x}{y}$, taking into account the signs of $\left\{ \cos\theta = \dfrac{x}{r} \quad et \quad \sin\theta = \dfrac{y}{r} \right.$

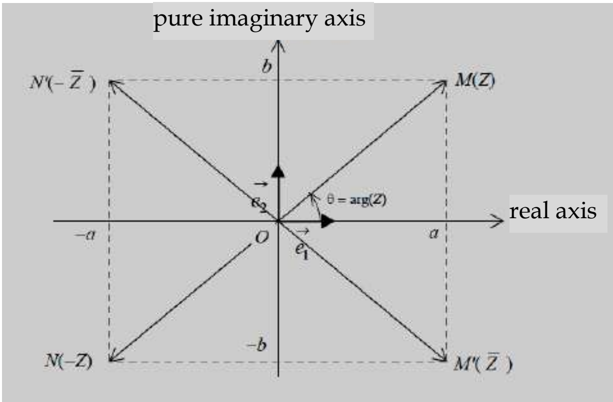
Properties:

For all non-zero complexes z and z′:

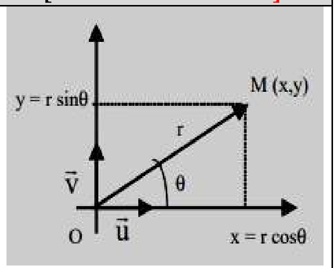
- $|zz'| = |z|.|z'| \quad et \quad \arg(zz') = \arg z + \arg z'\ (2\pi)$
- Pour tout entier naturel n : $|z^n| = |z|^n \quad et \quad \arg(z^n) = n.\arg z\ (2\pi)$
- $\left|\dfrac{1}{z}\right| = \dfrac{1}{|z|} \quad et \quad \arg\left(\dfrac{1}{z}\right) = -\arg z\ (2\pi)$
- $\left|\dfrac{z}{z'}\right| = \dfrac{|z|}{|z'|} \quad et \quad \arg\left(\dfrac{z}{z'}\right) = \arg z - \arg z'\ (2\pi)$
- For all $Z \in \mathbb{C}^*$:

$\arg(\bar{z}) = -\arg(z)\ [2\pi]$; $\arg(-z) = \arg(z) + \pi[2\pi]$; $\arg(-\bar{z}) = \pi - \arg(z)[2\pi]$

This property is illustrated in the following figure



VI. The different forms of a complex number:

Algebraic form	Trigonometric form	Exponential form	Polar form	Forme géométrique [Geometric form]
The algebraic (or Cartesian) form of a complex number z = (x, y) is called the expression *z = x +iy.*	$z = r(\cos\theta + i.\sin\theta)$ $\begin{cases} r = \sqrt{x^2+y^2} \\ \cos\theta = \frac{x}{r} \; et \; \sin\theta = \frac{y}{r} \end{cases}$	For any non-zero complex number z of modulus r and argument , we set : $z = r.e^{i\theta}$	For any non-zero complex number z of modulus r and argument θ, we set: $z = r^{\;\theta^0}$ With ϴ in degrees	 $r = OM$ *et* $\arg z = \theta$

Tab 1 : The different forms of a complex number

Morality: to multiply two non-zero complex numbers, multiply the modules and add the arguments. To divide two non-zero complex numbers, divide the modules and subtract the arguments.

Properties:

Trigonometric form	Exponential form	Polar form
For all non-zero complex numbers z and z′:	Or : $z_1 = e^{i\theta_1}$ et $z_2 = e^{i\theta_2}$	Or : $z_1 = \rho_1^{\theta_1}$ et $z_2 = \rho_2^{\theta_2}$
❖ arg(z.z') = arg(z) + arg(z') (2π)	❖ $e^{i(\theta+\theta')} = e^{i\theta} \times e^{i\theta'}$	❖ $z_1.z_2 = \rho_1.\,\rho_2^{\;\theta_1+\theta_2}$
❖ $\arg\left(\frac{1}{z}\right) = \arg(\bar{z}) = -\arg(z)\ (2\pi)$	❖ $e^{-i\theta} = \frac{1}{e^{i\theta}}$	❖ $\frac{z_1}{z_2} = \left(\frac{\rho_1}{\rho_2}\right)^{\theta_1-\theta_2}$
❖ $\arg\left(\frac{z}{z'}\right) = \arg(z) = -\arg(z')\ (2\pi)$	❖ $e^{i(\theta-\theta')} = \frac{e^{i\theta}}{e^{i\theta'}}$	❖ $\frac{1}{z_1} = \left(\frac{1}{\rho_1}\right)^{-\theta_1}$
❖ Pour tout entier relatif n , $\arg(z^n) = n.\arg(z)\ (2\pi)$	❖ $e^{in\theta} = \left(e^{i\theta}\right)^n$	❖ $z_1^{\;n} = \left(\rho_1^{\;n}\right)^{n\theta_1}$
for any real θ, $e^{i\theta} = \cos(\theta) + i\sin(\theta)$		

Tab 2 : The properties of a complex number

VII. EULER formula - MOIVRE formula :

VII.1 EULER formula:

Leonard Euler , (1707 -1783), was a Swiss engineer who founded the study of graph theory and topology and made pioneering and influential discoveries in many other branches of mathematics such as analytic number theory, complex analysis and infinitesimal calculus. He introduced much of the terminology and notation, including the notion of a mathematical function. He is also known for his work in mechanics, fluid dynamics, optics, astronomy and music theory.

Let a complex number have modulus 1: $\begin{cases} z = e^{i\theta} = \cos\theta + i\sin\theta \\ \bar{z} = e^{-i\theta} = \cos\theta - i\sin\theta \end{cases}$

By expressing the sum and the difference: $\begin{cases} z+\bar{z} = 2.\cos\theta \\ z-\bar{z} = 2\cos\theta = 2.i.\sin\theta \end{cases}$

This leads to the D'EULER formulae : $\cos\theta = \dfrac{e^{i\theta}+e^{-i\theta}}{2}$ et $\sin\theta = \dfrac{e^{i\theta}-e^{-i\theta}}{2.i}$

- Generalisation to complex numbers of any modulus:

For any complex number whose modulus is different from unity, the cosine and sine of the argument can be obtained as follows:

$$\begin{cases} z = r.e^{i\theta} \Rightarrow \cos\theta + i\sin\theta = \dfrac{z}{r} \\ \bar{z} = r.e^{-i\theta} \Rightarrow \cos\theta - i\sin\theta = \dfrac{\bar{z}}{r} \end{cases} \Rightarrow \begin{cases} \cos\theta = \dfrac{z+\bar{z}}{2.r} \\ \sin\theta = \dfrac{z-\bar{z}}{2.i.r} \end{cases}$$

VII.2 MOIVRE formula :

Abraham de Moivre (1667-1754) was a French mathematician renowned for his important contributions to probability theory and mathematical analysis. He laid solid foundations for the calculus of probability, in particular by introducing the approximation of the binomial distribution by the normal distribution. He is also known for his use of complex numbers in trigonometric calculus. His formula, which expresses sine and cosine using complex numbers, was a major breakthrough in this field.

Let be a complex number of unit modulus $z = e^{i\theta}$.

The increase in power n gives: $z^n = (e^{i\theta})^n = e^{in\theta} = \cos n\theta + i.\sin n\theta$

Hence the MOIVRE formula:

$$(con\theta + i.\sin\theta)^n = \cos n\theta + i.\sin n\theta$$

This relationship remains valid when the exponent n is negative.

VIII. Second degree equations:

VIII.1 Second degree equations with real coefficients:

Any 2nd degree equation with real coefficients has two distinct or coincident solutions in $\mathbb{C}$: $az^2 + bz + c = 0$ with $a \in \mathbb{R}^*. b \in \mathbb{R}. c \in \mathbb{R}$

First we calculate the discriminant : **Δ = b² -4ac**

$\Delta > 0$	$\Delta = 0$	$\Delta < 0$
the equation admits two distinct real solutions $z_1 = \frac{-b-\sqrt{\Delta}}{2a}$ et $z_2 = \frac{-b+\sqrt{\Delta}}{2a}$	the equation admits two real solutions combined $z_1 = z_2 = -\frac{b}{2a}$	the equation admits two complex conjugate solutions $z_1 = \frac{-b-i\sqrt{\Delta}}{2a}$ et $z_2 = \frac{-b+i\sqrt{\Delta}}{2a}$ $z_2 = \overline{z}_1$

VIII.2. Square roots of a complex number:

Let the complex number $z = x+iy$ and the complex number $\omega = a+i\beta$; then ω is the square root of z

$$\omega^2 = z \Leftrightarrow \begin{cases} \alpha^2 + \beta^2 = |z| \\ \alpha^2 - \beta^2 = x \\ 2\alpha\beta = y \end{cases} \quad \text{we find:} \quad \begin{cases} \omega_1 = \alpha + i\beta \text{ et } \omega_2 = -\alpha - i\beta \text{ pour y} > 0 \\ \omega_1 = -\alpha + i\beta \text{ et } \omega_2 = \alpha - i\beta \text{ pour y} < 0 \end{cases}$$

VIII.3. Second degree equations with complex coefficients:

$az^2 + bz + c = 0$ avec $a \in \mathbb{C}^*. b \in \mathbb{C}. c \in \mathbb{C}$

We first calculate the discriminant **Δ = b² -4ac.**

- If Δ is a real, in this case we are in the cases of paragraph 8.1
- If Δ is a complex, in this case we seek the roots of Δ according to paragraph 8.2.

is the solutions of the equation are :$z_1 = \frac{-b+\omega 1}{2a}$ et $z_2 = \frac{-b-\omega 2}{2a}$

VIII.4. Third degree equations :

In order to solve the equation] : P(z) = $az^3 + bz^2 + cz + d = 0$ avec $a \in \mathbb{C}^*. b \in \mathbb{C}. c \in \mathbb{C}, d \in \mathbb{C}$ we must follow the following steps:

1) We seek a particular solution z1 such that P(z1) = 0.
2) Using the identification method (or Euclidean division or HORNER's method) we seek a polynomial Q(z) of degree 2, such that P(z) = (z-z1).Q(z) with: Q(z) = αz2 + βz + γ
3) We solve the system : $\begin{cases} z = z_1 \\ Q(z) = \alpha z^2 + \beta z + \gamma = 0 \end{cases}$
4) Finally we find all the solutions S= {z_1 , z_2 , z_3 }.

IX. Complex numbers and geometry

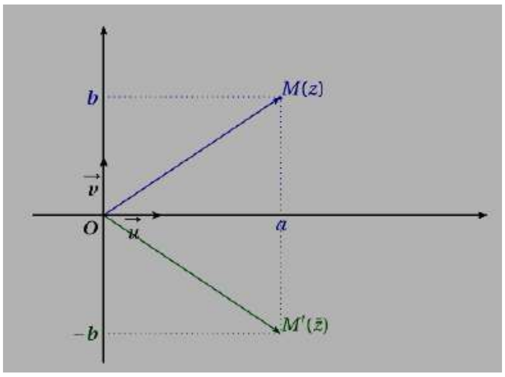
IX.1. Distance calculations

If ZA and ZB are the respective affixes of two points A and B then :

$$AB = |Z_B - Z_A|$$

IX.2. Two conjugate complex numbers:

Two conjugate complex numbers have affixes symmetrical about the x-axis



IX.3. Affix of a vector :

The affix of the vector $\vec{AB}$ is equal to: $z_{\vec{AB}} = z_B - z_A$

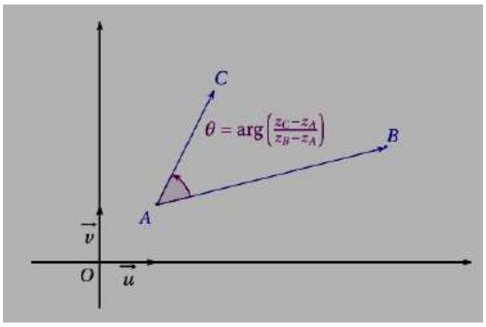
IX.4. Affix of Mid-segment affix [A ,B] :

The affix of the midpoint M of the segment [A ,B] is equal to : $z_M = \frac{z_A + z_B}{2}$

IX.5. Angle of vectors and arguments:

Let A, B and C be three points in the plane with affixes z_A, z_B and z_C respectively, where

$A \neq B$ et $A \neq C$: $(\vec{AB};\vec{AC}) = \arg\left(\frac{z_C - z_A}{z_B - z_A}\right)$



Propriétés :

a) The first important special case:

A, B and C are aligned:

$\arg\left(\frac{z_C - z_A}{z_B - z_A}\right) = 0 \text{ ou } \pi [mod. 2\pi] \Leftrightarrow \frac{z_C - z_A}{z_B - z_A} \in \mathbb{R}$

b) Second important special case:

$\widehat{BAC}$ est un angle droit

$\arg\left(\frac{z_C - z_A}{z_B - z_A}\right) = \pm\frac{\pi}{2} [\text{mod}. 2\pi] \Leftrightarrow \frac{z_C - z_A}{z_B - z_A}$

is pure imagination

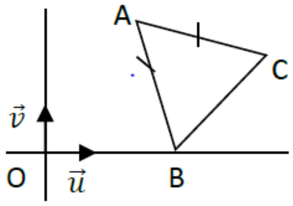
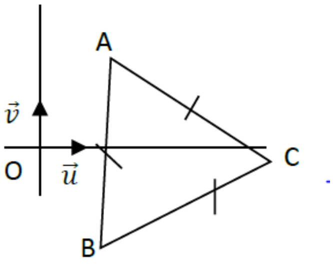
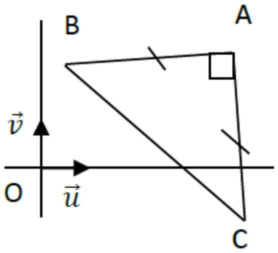
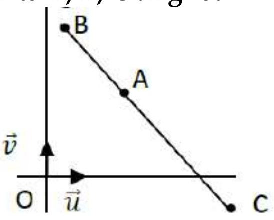
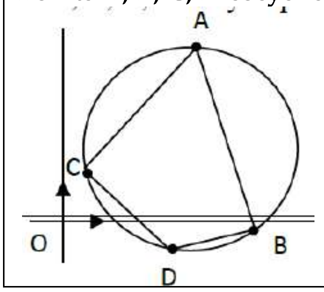
IX.6 Summary table

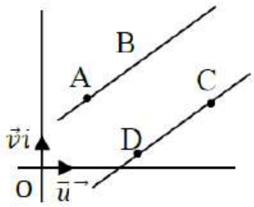
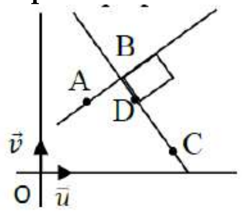
The complex plane is given by a direct orthonormal coordinate system (O, e_1, e_2).

- A and B are two distinct points in the plane with affixes z_A and z_B respectively.
- M is any point in the plane with affix z and $\vec{u}$ is the image vector of z.

Complex characterisation	Geometric characterisation	Set of points
$\lvert z - z_A \rvert = r, r \in \mathbb{R}_+^*$	$AM = r$	Circle with centre A and radius r.
$\lvert z - z_A \rvert = \lambda . \lvert z - z_B \rvert, \lambda \in \mathbb{R}_+^*$	$AM = \lambda BM \Rightarrow \frac{AM}{BM} = \lambda$	the bisector of segment [AB] when λ=1 the circle of diameter [G_1 G_2] when λ≠1, where G_1=bar{(A;1),(B;λ)} and G_2=bar{(A;1),(B;-λ)}
$\arg\left(\frac{z_B - z}{z_A - z}\right) = k\pi, k \in \mathbb{Z}$	$\text{mes}(\vec{MA};\vec{MB}) = k\pi$, $k \in \mathbb{Z}$	The line (AB) deprived of points A and B.
$\arg\left(\frac{z_B - z}{z_A - z}\right) = \frac{p}{2} + k\pi, k \in \mathbb{Z}$	$\text{mes}(\vec{MA};\vec{MB}) = \frac{\pi}{2} + k\pi$, $k \in \mathbb{Z}$	The circle of diameter [AB] deprived of points A and B .
$\arg(z - z_A) = a + k\pi, a \in \mathbb{R}$	$\text{mes}\left(\widehat{e_1, AM}\right) = \alpha + k\pi$, $k \in \mathbb{Z}$	the reference line $((A, \vec{u})$, deprived of A, where $\text{mes}\left(\widehat{e_1, \vec{u}}\right) = \alpha$
$\arg(z - z_A) = a + k2\pi, a \in \mathbb{R}$	$\text{mes}\left(\widehat{\vec{e_1}, \vec{AM}}\right) = \alpha + 2k\pi$, $k \in \mathbb{Z}$	the reference half-line $(A, \vec{u})$, deprived of A, where $\text{mes}\left(\widehat{e_1, \vec{u}}\right) = \alpha$

IX.7. Configurations du Plan et Nombres Complexes :

Configurations	Geometric characterisation	Complex characterisation
Triangle ABC isosceles at A 	AB = AC and $mes\widehat{A} = \alpha$ $(0 < \alpha < \pi)$.	$\frac{z_C - z_A}{z_B - z_A} = e^{i\alpha}$ or $\frac{z_C - z_A}{z_B - z_A} = e^{-i\alpha}$
Equilateral triangle ABC 	AB = AC and $mes\,\widehat{A} = \frac{\pi}{3}$	$\frac{z_C - z_A}{z_B - z_A} = e^{i\frac{\pi}{3}}$ or $\frac{z_C - z_A}{z_B - z_A} = e^{-i\frac{\pi}{3}}$
Triangle ABC is right-angled and isosceles at A 	$mes\widehat{A} = \frac{\pi}{2}$	$\frac{z_C - z_A}{z_B - z_A} = bi$, with $b \in \mathbb{R}^*$
Points A, B, C aligned 	$mes\left(\widehat{\overrightarrow{AB}, \overrightarrow{AC}}\right) = k\pi,\ k \in \mathbb{Z}$	$\frac{z_C - z_A}{z_B - z_A} \in \mathbb{R}^*$
Points A, B, C, D cocyclic 	$mes\left(\widehat{\overrightarrow{CA}, \overrightarrow{CB}}\right) = mes\left(\widehat{\overrightarrow{DA}, \overrightarrow{DB}}\right) + k\pi,\ \ k \in \mathbb{Z}$ et $mes\left(\widehat{\overrightarrow{CA}, \overrightarrow{CB}}\right) \neq k\pi,\ k \in \mathbb{Z}$	$\frac{z_B - z_C}{z_A - z_C} : \frac{z_B - z_D}{z_A - z_D} \in \mathbb{R}^*$

Parallel lines 	There is a non-zero real number λ such that : $\overrightarrow{CD} = \lambda\overrightarrow{AB}$. Or : $mes(\overrightarrow{AB;CD}) = k\pi; k \in Z$	$\arg\left(\frac{z_D - z_C}{z_B - z_A}\right) = k\pi$; $k \in \mathbb{Z}$ ou $\frac{z_D - z_C}{z_B - z_A} \in \mathbb{R}^*$
Perpendicular lines 	$\overrightarrow{AB} \cdot \overrightarrow{CD} = 0$ or $mes(\overrightarrow{AB};\overrightarrow{CD}) = \frac{\pi}{2} + k\pi; k \in \mathbb{Z}$	$\arg\left(\frac{z_D - z_C}{z_B - z_A}\right) = \frac{\pi}{2} +$ $k\pi$; $k \in \mathbb{Z}$ ou $\frac{z_D - z_C}{z_B - z_A} \in i\,\mathbb{R}^*$

IX.8. Complex numbers and barycenter :

Let G be the barycenter of n weighted points

$(A_1, \alpha_1), (A_2, \alpha_2), \dots, (A_n, \alpha_n)$ avec $\sum_{p=1}^{n} \alpha_p \neq 0$ Let zp be the affixes of the points $A_p(1 \leqslant$ $p \leqslant n$. Then the affix zGof G is given by

$$z_G = \frac{\sum_{p=1}^{n} \alpha_p z_p}{\sum_{p=1}^{n} \alpha_p}$$

X. n^{th} root of unity

We seek to determine the set of complex numbers z verifying the equality $z^n = 1$ with $n \in \mathbb{N}^*$. The set $\mathbb{U}n$ of roots of unity has exactly n roots : $w_k = e^{i.(2k\pi/n)}$, with k an integer between 0 and $n - 1$

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