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Department of Science and Technology

Handout of Mathematics 4

lessons and exercises

For Second-Year LMD Students in the Science and Technology Domain.

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Contents

1 Holomorphic functions. Cauchy-Riemann equations **5**

1.1 Some Definitions and Properties 5

1.1.1 Complex Numbers 5

1.1.2 The modulus and argument 6

1.1.3 The trigonometric and exponential form 7

1.1.4 Roots of Complex Numbers 9

1.2 Elementary functions 9

1.2.1 The exponential function 9

1.2.2 Trigonometric and Hyperbolic functions 10

1.2.3 The Logarithm function 11

1.3 Holomorphic functions. Cauchy-Riemann equations 12

1.4 EXERCISES 16

2 Power series. Radius of convergence. Disc of convergence. Power series expansion. Analytic functions. **18**

2.1 Power series 18

2.2 Radius of convergence 18

2.3 Disc of convergence 20

2.4 Power series expansion 21

2.5 Analytic functions 26

2.5.1 Properties of analytic functions 27

2.5.2 Identity theorem for analytic functions 27

2.6 EXERCISES 28

3 Cauchy's theory **29**

3.1 Curvilinear Integral 29

3.2 Cauchy's Theorem 34

3.3 Cauchy's Integral Formula 35

3.4 EXERCISES 38

4 Applications **40**

4.1 Equivalence between holomorphy and analyticity. 40

4.2 Maximum theorem. 41

4.3 Liouville's theorem. 43

4.4 Rouche's theorem. 43

4.5 Residue theorem. 44

4.5.1 The Poles 44

4.5.2 The residues 44

4.6 Calculation of integrals using the residue method. 46

4.6.1 Integrals of the form $\int_0^{2\pi} R(\cos\theta, \sin\theta)\, d\theta$ 46

4.6.2 Integrals of the form $\int_{-\infty}^{+\infty} R(x)\, dx$ 49

4.6.3 Integrals of the form $\int_{-\infty}^{+\infty} R(x)\, e^{i\alpha x} dx$ 50

4.7 EXERCISES 52

5 Harmonic Functions **53**

5.1 Harmonic Functions 53

5.2 EXERCISES 56

6 Exams **57**

6.1 Exam 2022 57

6.2 Exam 2023 61

6.3 Make-up Exam 2023 65

Introduction

Complex analysis is a profound and elegant branch of mathematics that explores functions of complex variables. At its heart lies the idea that functions can behave in strikingly rich ways when their inputs are allowed to move beyond the real number line and into the complex plane.

From its origins in solving polynomial equations to its applications in physics, engineering, and number theory, complex analysis offers tools that are both theoretical and practical. Concepts such as holomorphic functions, contour integration, and conformal mappings reveal not just new methods of solving problems, but also new ways of seeing mathematical structures.

This handout provides a structured introduction to the fundamental concepts of complex analysis. Beginning with a review of complex numbers, it gradually builds up to core topics such as differentiability, analyticity, and the foundational Cauchy-Riemann equations. Through these themes, students will gain insight into the elegant structure and far-reaching applications that make complex analysis a central pillar of higher mathematics.

The course is designed for students in mathematics and the sciences who are being introduced to complex variable theory for the first time. It is particularly well-suited for second-year undergraduate students enrolled in the fourth semester of a Bachelor of Science and Technology (L2S4) program. Students from other disciplines or preparatory classes will also find the content good, with clear phrasing.

To keep the focus clear and practical, we present key results and computational techniques without formal proofs. The goal is to offer an efficient pathway to understanding and application. The material is organized as follows:

In **Chapter 1**, we introduce fundamental definitions and properties of complex numbers and functions. This includes an overview of complex arithmetic and the basic concepts that underpin complex analysis. We then explore the notion of holomorphic functions, highlighting their key properties and significance.

Chapter 2 focuses on power series and their convergence. We examine how holomorphic functions can be expressed as power series within their radius of convergence, establishing a deep connection between analyticity and holomorphy.

In **Chapter 3**, we study Cauchy's theorem and its powerful implications for complex integration. This includes applications to curvilinear integrals and the formulation of Cauchy's integral formula.

Chapter 4 is dedicated to important theorems and applications that stem from Cauchy's theory. These include Liouville's theorem, Rouche's theorem, and the residue theorem essential tools for evaluating complex integrals and understanding the behavior of functions in the complex plane.

The final chapter, **Chapter 5**, introduces harmonic functions and explores their close relationship with holomorphic functions. We investigate how these real-valued functions arise naturally from the real and imaginary parts of complex analytic functions.

By the end of this handout, students will have acquired both a theoretical foundation and practical tools for further exploration in complex analysis and its applications.

Chapter 1

Holomorphic functions. Cauchy-Riemann equations

1.1 Some Definitions and Properties

1.1.1 Complex Numbers

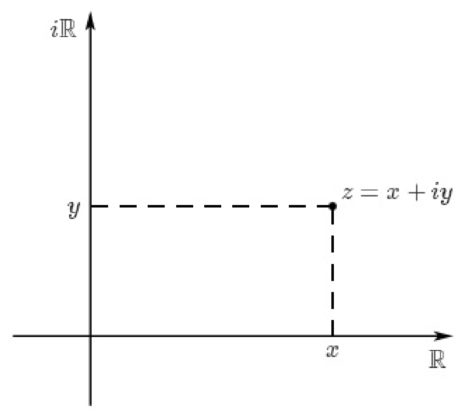
Definition 1.1. A complex number is any expression of the form

$$z = x + iy, \quad x, y \in \mathbb{R} \tag{1.1}$$

with i defined by the relation $i^2 = -1$.

Notation:

- We denote by $\mathbb{C}$ the set of complex numbers.
- x is called the real part of z and is denoted by $Re(z)$
- y is called the imaginary part of z and is denoted by $Im(z)$.



Remark 1.1. This representation (1.1) is known as the algebraic form.

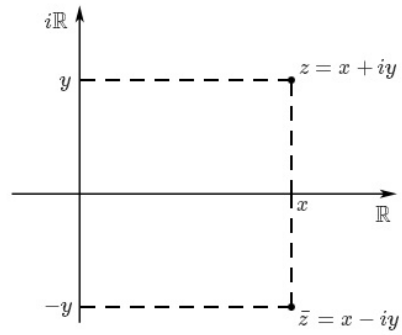
Example 1.1. We have

- $z_1 = 1 + i \in \mathbb{C}$, then we have $Re(z_1) = Im(z_1) = 1$
- $z_2 = \cos\alpha + i\sin\alpha \in \mathbb{C}$, with $\alpha \in \mathbb{R}$, we have $Re(z_2) = \cos\alpha$ and $Im(z_2) = \sin\alpha$

Definition 1.2. (**The conjugate**)
The complex conjugate of $z = x + iy$ is defined by

$$\overline{z} = x - iy,$$

It is easy to verify that, for all complex numbers z and $w \neq 0$



$$\overline{\overline{z}} = z, \quad \overline{z + w} = \overline{z} + \bar{w}, \quad \overline{zw} = \overline{z} \times \overline{w}, \qquad \overline{\left(\frac{z}{w}\right)} = \frac{\overline{z}}{\overline{w}}. \tag{1.2}$$

and

$$z + \overline{z} = 2Re\,(z)\,, \quad z - \overline{z} = 2iIm\,(z)\,, \quad z\overline{z} = x^2 + y^2. \tag{1.3}$$

Note also that

- If $Im\,(z) = 0 \Longrightarrow \overline{z} = z$, then z is real.
- If $Re\,(z) = 0 \Longrightarrow \overline{z} = -z$, then z is pure imaginary.

Exercise. Prove the previous properties (1.2) and (1.3).

1.1.2 The modulus and argument

Definition 1.3. (**The modulus**)
Let $z = x + iy$, the non-negative number noted by $|z|$ with

$$|z| = \sqrt{x^2 + y^2}$$

is called the absolute value or modulus of z.

Example 1.2. Let $z = 1 - i$, then we have $|1 - i| = \sqrt{1^2 + (-1)^2} = \sqrt{2}$.

We have the following properties, for all complex numbers z and $w \neq 0$,

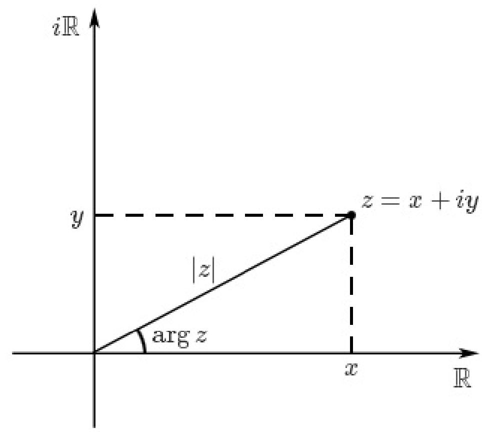
$$|z| = |-z| = |\overline{z}|\,, \quad |zw| = |z| \times |w| \quad \left|\frac{z}{w}\right| = \frac{|z|}{|w|}. \tag{1.4}$$

and

$$z\overline{z} = |z|^2\,, \qquad |z + w| \leq |z| + |w| \qquad \text{(called the triangle inequality)} \tag{1.5}$$

Exercise. Prove the previous properties (1.4) and (1.5).

The plane with a direct orthogonal coordinate system $\left(O, \overrightarrow{i}, \overrightarrow{j}\right)$ is called a complex plane. The complex number $z = x + iy$ is represented by the point M with coordinates (x, y) as shown in the figure. We say that M is the image of z , or that z is the affix of M. In addition, the length OM is the modulus of z ($|z| = OM$).



Definition 1.4. (**The argument**)
Let z be a nonzero complex number. The measure θ of the oriented angle $\left(\overrightarrow{i}, \overrightarrow{OM}\right)$ is called an argument of z, denoted by $\arg(z)$

Remark 1.2. If θ is a an argument of z then $\theta + 2k\pi$ likewise an argument of z with $k \in \mathbb{Z}$. Therefore, the argument of a complex number is not unique.Denoted by $Arg\,(z)$ the principal value of the $\arg(z)$ function to take values in the interval $]-\pi, \pi]$, so

$$\arg(z) = Arg\,(z) + 2k\pi, \qquad k \in \mathbb{Z}.$$

Example 1.3. For example $\arg(i) = \dfrac{\pi}{2} + 2k\pi$, generally we have

$$\arg(iy) = \frac{\pi}{2} + 2k\pi \quad \text{If} \quad y > 0$$

and

$$\arg(iy) = -\frac{\pi}{2} + 2k\pi \quad \text{If} \quad y < 0$$

For all complex numbers z and $w \neq 0$, we have the following properties

$$\arg(zw) = \arg(z) + \arg(w), \quad \arg\left(\frac{z}{w}\right) = \arg(z) - \arg(w)$$

1.1.3 The trigonometric and exponential form

Let z be a complex number represented by the point M with coordinates (x, y) and the polar coordinates (r, θ), we know that

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$$

Hence we can write $z = x + iy = r(\cos\theta + i\sin\theta)$.

Definition 1.5. The trigonometric form of a complex number z is

$$z = r(\cos\theta + i\sin\theta)$$

where r is the modulus of z, and θ the argument of z with $-\pi < \theta \leq \pi$.

Definition 1.6. The exponential form (polar form) of a complex number z is

$$z = re^{i\theta}$$

where r is the modulus of z, and θ the argument of z with $-\pi < \theta \leq \pi$.

Example 1.4. Write the following complex numbers in trigonometric and exponential form:

a) $z = i$ b) $z = -\sqrt{3} + i$

Solution. We have

a) we know that $r = |i| = 1$ and $\arg(i) = \dfrac{\pi}{2} + 2k\pi$, so we have

$z = i = \cos\dfrac{\pi}{2} + i\sin\dfrac{\pi}{2} = e^{i\dfrac{\pi}{2}}$

b) To write the number in trigonometric form, we need r and θ

$r = \left|-\sqrt{3} + i\right| = \sqrt{4} = 2,$

$$\begin{cases} \cos\theta = \dfrac{x}{r} = \dfrac{-\sqrt{3}}{2} \\ \sin\theta = \dfrac{y}{r} = \dfrac{1}{2} \end{cases} \Longrightarrow \theta = \frac{5\pi}{6}$$

Then $z = -\sqrt{3} + i = 2\left(\cos\dfrac{5\pi}{6} + i\sin\dfrac{5\pi}{6}\right) = 2e^{i\dfrac{5\pi}{6}}$.

Euler's formula

Let $\alpha \in \mathbb{R}$, we have

$$\begin{cases} e^{i\alpha} = \cos\alpha + i\sin\alpha \\ e^{-i\alpha} = \cos\alpha - i\sin\alpha \end{cases} \tag{1.6}$$

1.1.4 Roots of Complex Numbers

Let $a = r(\cos\theta + i\sin\theta)$. The n^{th} root of a is the complex number z solution to the equation $z^n = a$. They are:

$$z_k = \sqrt[n]{r}\left(\cos\frac{\theta + 2k\pi}{n} + i\sin\frac{\theta + 2k\pi}{n}\right) \tag{1.7}$$

where $k = 0, 1, ..., n-1$.

Example 1.5. Find all square roots of i

Solution. The trigonometric form of i is $i = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2}$. Then, we use the formula (1.7) where $r = 1$, $\theta = \frac{\pi}{2}$, $n = 2$, with $k \in \{0,\ 1\}$. So the square roots of i are:

$$z_0 = \sqrt{1}\left(\cos\frac{\pi/2}{2} + i\sin\frac{\pi/2}{2}\right) = \cos\frac{\pi}{4} + i\sin\frac{\pi}{4} = \frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}$$

$$z_1 = \sqrt{1}\left(\cos\frac{\pi/2 + 2\pi}{2} + i\sin\frac{\pi/2 + 2\pi}{2}\right) = \cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4} = -\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}$$

1.2 Elementary functions

1.2.1 The exponential function

Definition 1.7. We define the exponential of a complex number z ; $z = x+iy$; $x, y \in \mathbb{R}$, by

$$\begin{array}{rcll} f: & \mathbb{C} & \longrightarrow & \mathbb{C}^* \\ & z & \longmapsto & f(z) = \ e^z = e^x\cos y + ie^x\sin y; \end{array}$$

It results from the definition

Proposition 1.1. *We have*

1. $Re(e^z) = e^x\cos y$ *and* $Im(e^z) = e^x\sin y$
2. $|e^z| = e^x$, *and* $\arg(e^z) = y + 2k\pi$; $k \in \mathbb{Z}$.
3. $e^{z_1+z_2} = e^{z_1}e^{z_2}$; $\quad \forall z_1,\ z_2 \in \mathbb{C}$

1.2.2 Trigonometric and Hyperbolic functions

From Euler's formula, we define the cosine, sine and tangent functions as follows:

$\cos z = \dfrac{e^{iz} + e^{-iz}}{2}, \quad \sin z = \dfrac{e^{iz} - e^{-iz}}{2i}$

And $\tan z = \dfrac{\sin z}{\cos z}$ with $z \neq \dfrac{\pi}{2} + k\pi;\ \ k \in \mathbb{Z}$.

Remark 1.3. Most of the properties of trigonometric functions in the real case remain valid in the complex case. Such as:

1. $\cos^2 z + \sin^2 z = 1$
2. $\cos(z \pm \omega) = \cos z \cos \omega \mp \sin z \sin \omega$
3. $\sin(z \pm \omega) = \sin z \cos \omega \pm \cos z \sin \omega$

Hyperbolic functions are also defined from e^z as

$\cosh z = \dfrac{e^{z} + e^{-z}}{2}, \quad \sinh z = \dfrac{e^{z} - e^{-z}}{2}$

$\tanh z = \dfrac{\sinh z}{\cosh z}$ with $z \neq (\frac{\pi}{2} + k\pi)i;\ k \in \mathbb{Z}$.

Remark 1.4. Hyperbolic functions in the complex case has a same properties in the real case. Such as:

1. $\cosh^2 z - \sinh^2 z = 1$
2. $\cosh(z \pm \omega) = \coth z \coth \omega \pm \sinh z \sinh \omega$
3. $\sinh(z \pm \omega) = \sinh z \cosh \omega \pm \cosh z \sinh \omega$

Proposition 1.2. *Let $z \in \mathbb{C}$, we have*

$$\cos(iz) = \cosh z, \qquad \sin(iz) = i \sinh z$$

$$\cosh(iz) = \cos z, \qquad \sinh(iz) = i \sin z$$

Example 1.6. Solve the following equations:

1) $2\cos z + 3e^{-iz} = 2\sqrt{3}$ 2) $2\sinh z - 5e^{-z} = -1$

Solution. We have

1) $2\cos z + 3e^{-iz} = 2\sqrt{3} \Longrightarrow 2\left(\dfrac{e^{iz} + e^{-iz}}{2}\right) + 3e^{-iz} = 2\sqrt{3} \Longrightarrow e^{iz} + 4e^{-iz} = 2\sqrt{3}$

$$\xRightarrow{\times e^{iz}} e^{2iz} - 2\sqrt{3}e^{iz} + 4 = 0$$

Let's ask $e^{iz} = M$;we obtain:

$M^2 - 2\sqrt{3}M + 4 = 0 \Longrightarrow \Delta = 12 - 16 = -4 = (2i)^2$.

$M_1 = \sqrt{3} + i \Longrightarrow e^{iz} = \sqrt{3} + i \Longrightarrow iz = \log(\sqrt{3} + i) = \ln 2 + i\Big(\frac{\pi}{6} + 2k\pi\Big)$,

$M_2 = \sqrt{3} - i \Longrightarrow e^{iz} = \sqrt{3} - i \Longrightarrow iz = \log\Big(\sqrt{3} - i\Big) = \ln 2 + i\Big(- \frac{\pi}{6} + 2k\pi\Big)$.

Then:

$iz = \ln 2 + i\Big(\pm \frac{\pi}{6} + 2k\pi\Big) \Longrightarrow z_k = \pm\frac{\pi}{6} + 2k\pi - i\ln 2;\;\; k \in \mathbb{Z}$.

2) $2\sinh z - 5e^{-z} = -1 \Longrightarrow 2\left(\dfrac{e^z - e^{-z}}{2}\right) - 5e^{-z} = -1 \Longrightarrow e^z - 6e^{-z} = -1$ Let's ask $\overset{\times e^{iz}}{\Longrightarrow} e^{2z} + e^z - 6 = 0$

$e^{iz} = t$;we obtain:

$t^2 + t - 6 = 0 \Longrightarrow \Delta = 1 + 24 = 25 = (5)^2$.

$t_1 = 2 \Longrightarrow e^z = 2 \Longrightarrow z = \log(2) = \ln 2 + i\Big(0 + 2k\pi\Big)$,

$t_2 = -3 \Longrightarrow e^z = -3 \Longrightarrow iz = \log\Big(- 3\Big) = \ln 3 + i\Big(\pi + 2k\pi\Big)$.

So:

$z_1 = \ln 2 + i2k\pi$ and $z_2 = \ln 3 + i\Big(2k + 1\Big)\pi.;\;\; k \in \mathbb{Z}$.

1.2.3 The Logarithm function

Definition 1.8. Let $z \in \mathbb{C}^*$, The complex logarithm of a complex number z is given by:

$$\log z = \ln|z| + i \arg z = \ln r + i\,(\theta + 2k\pi)\,;\; k \in \mathbb{Z}$$

where $|z| = r$ and $\arg z = \theta + 2k\pi$; $k \in \mathbb{Z}$, with $-\pi < \theta \leq \pi$.

Example 1.7. Calculate the following complex numbers:

a) $\log(1 + i)$ b) $\log(-1)$

Solution. We have

a)We find that $|1 + i| = \sqrt{2}$, and $\arg(1 + i) = \dfrac{\pi}{4} + 2k\pi$, then

$\log(1 + i) = \ln\sqrt{2} + i\left(\dfrac{\pi}{4} + 2k\pi\right);\; k \in \mathbb{Z}$.

b) $\log(-1) = \ln|-1| + i\arg(-1) = \ln 1$+i$(\pi + 2k\pi) = i\pi\,(2k + 1)\,;\; k \in \mathbb{Z}$.

1.3 Holomorphic functions. Cauchy-Riemann equations

Let $z_0 \in \mathbb{C}$, and $r > 0$. The open disc $D(z_0, r)$ of radius r centered at z_0 is the set of all complex numbers defined as

$$D(z_0, r) = \{z \in \mathbb{C} : \ |z - z_0| < r\}$$

The closed disc $\overline{D}(z_0, r)$ of radius r centered at z_0 is defined by

$$\overline{D}(z_0, r) = \{z \in \mathbb{C} : \ |z - z_0| \leq r\}$$

Let $\Omega \subset \mathbb{C}$ be a set, a point z_0 is an interior point of Ω if there exists $r > 0$ such that $D(z_0, r) \subset \Omega$.

A set Ω is open if every point in it is an interior point.

Definition 1.9. Let Ω be an open set in $\mathbb{C}$ and $f : \Omega \longrightarrow \mathbb{C}$. The function f is a holomorphic at the point $z_0 \in \Omega$ if

$$\frac{f(z_0 + h) - f(z_0)}{h}$$

has a limit when $h \longrightarrow 0$.

The limit of the quotient, when it exists, is denoted by $f'(z_0)$, and is called the derivative of f at z_0:

$$f'(z_0) = \lim_{h \longrightarrow 0} \frac{f(z_0 + h) - f(z_0)}{h}$$

Definition 1.10. The function f is said to be holomorphic on Ω if f is holomorphic at every point of Ω.

Example 1.8. Let $f(z) = z^2$, and $z_0 \in \mathbb{C}$. We have

$$\begin{aligned}\lim_{h \longrightarrow 0} \frac{f(z_0 + h) - f(z_0)}{h} &= \lim_{h \longrightarrow 0} \frac{(z_0 + h)^2 - z_0^2}{h} \\ &= \lim_{h \longrightarrow 0} (2z_0 + h) = 2z_0\end{aligned}$$

Therefore, f is holomorphic at every point of $\mathbb{C}$, with $f'(z) = 2z$

Proposition 1.3. *If f and g are holomorphic in Ω, then:*

1. $\alpha f+\beta g$ is holomorphic in Ω and $(\alpha f+\beta g)^{'}=\alpha f^{'}+\beta g^{'}$, where α, $\beta\in\mathbb{C}$.
2. fg is holomorphic in Ω and $(fg)^{'}=f^{'}g+fg^{'}$.
3. If $g(z_0)\neq 0$, then f/g is holomorphic at z_0 and

$$\left(\frac{f}{g}\right)^{'}=\frac{f^{'}g-fg^{'}}{g^2}$$

Remark 1.5. The functions e^z, $\cos z$, $\sin z$,$\cosh z$, $\sinh z$ and any polynomial are holomorphic in $\mathbb{C}$.

Theorem 1.1. *Let $f:\Omega\longrightarrow\mathbb{C}$ where $f(z)=P(x,y)+iQ(x,y)$*
with $Re(f)=P(x,y)$ and $Im(f)=Q(x;y)$. Then:
f is holomorphic in Ω if and only if the following equations are satisfied

$$\frac{\partial P}{\partial x}=\frac{\partial Q}{\partial y};\quad \textit{and}\quad \frac{\partial P}{\partial y}=-\frac{\partial Q}{\partial x} \tag{1.8}$$

These equations (1.8) are called the Cauchy-Riemann equations.

Proof. By hypothesis, f is differentiable on z_0 , so $f'(z_0)$ exists. To prove Cauchy-Riemann equations. We will consider two different directions for towards z_0 and we will use the fact that the limit of

$$\lim_{\Delta z\to 0}\frac{f(z_0+\Delta z)-f(z_0)}{\Delta z}$$

is the same in all directions.

The following approach the real axis ($\Delta y=0$):

$$\begin{aligned}\lim_{\Delta z\to 0}\frac{f(z_0+\Delta z)-f(z_0)}{\Delta z}&=\lim_{\Delta x\to 0}\frac{f(z_0+\Delta x)-f(z_0)}{\Delta x}\\&=\lim_{\Delta x\to 0}\frac{P(x_0+\Delta x,y_0)-P(x_0,y_0)}{\Delta x}+i\lim_{\Delta x\to 0}\frac{Q(x_0+\Delta x,y_0)-Q(x_0,y_0)}{\Delta x}\\&==\frac{\partial P}{\partial x}(x_0,y_0)+i\frac{\partial Q}{\partial x}(x_0,y_0)\end{aligned}$$

The following approach to the imaginary axis gives ($\Delta x=0$):

$$
\begin{aligned}
\lim_{\Delta z\to 0}\frac{f(z_0+\Delta z)-f(z_0)}{\Delta z} &= \lim_{\Delta y\to 0}\frac{f(z_0+i\Delta y)-f(z_0)}{i\Delta y}\\
&= \lim_{\Delta y\to 0}\frac{P(x_0,y_0+\Delta y)-P(x_0,y_0)}{i\Delta y}+i\lim_{\Delta y\to 0}\frac{Q(x_0,y_0+\Delta y)-Q(x_0,y_0)}{i\Delta y}\\
&= = \frac{\partial Q}{\partial y}(x_0,y_0)-i\frac{\partial P}{\partial y}(x_0,y_0)
\end{aligned}
$$

Since the function is differentiable, the two expressions must be equal:

$$\frac{\partial P}{\partial x}+i\frac{\partial Q}{\partial x}=\frac{\partial Q}{\partial y}-i\frac{\partial P}{\partial y}$$

Then

$$\frac{\partial P}{\partial x}=\frac{\partial Q}{\partial y};\quad \text{and}\quad \frac{\partial P}{\partial y}=-\frac{\partial Q}{\partial x}$$

Hence the Cauchy-Riemann conditions. □

Example 1.9. Show that the Cauchy-Riemann equations are satisfied for the functions

a) $f(z)=e^z$

Let $z=x+iy$, we have

$f(z)=e^z=e^{x+iy}=e^x\cos y+ie^x\sin y\Longrightarrow P(x,y)=e^x\cos y$, and $Q(x,y)=e^x\sin y$

$$\begin{cases}\dfrac{\partial P}{\partial x}=e^x\cos y\\ \dfrac{\partial Q}{\partial y}=e^x\cos y\end{cases}\Rightarrow \frac{\partial P}{\partial x}=\frac{\partial Q}{\partial y}$$

$$\begin{cases}\dfrac{\partial P}{\partial y}=-e^z\sin y\\ \dfrac{\partial Q}{\partial x}=e^z\sin y\end{cases}\Rightarrow \frac{\partial P}{\partial y}=-\frac{\partial Q}{\partial x}$$

The Cauchy-Riemann equations are satisfied then f is holomorphic in $\mathbb{C}$.

b) $f(z)=\overline{z}$

we have $f(z)=\overline{z}=x-iy\Longrightarrow P(x,y)=x$, and $Q(x,y)=-y$

$$\begin{cases}\dfrac{\partial P}{\partial x}=1\\ \dfrac{\partial Q}{\partial y}=-1\end{cases}\Rightarrow \frac{\partial P}{\partial x}\neq\frac{\partial Q}{\partial y}$$

The Cauchy-Riemann equations are not satisfied then f is not holomorphic in $\mathbb{C}$.

Proposition 1.4. *If $f = P + iQ$ is holomorphic in Ω then we have*

$$f^{'}(z) = \frac{\partial P}{\partial x} + i\frac{\partial Q}{\partial x} = \frac{\partial Q}{\partial y} - i\frac{\partial P}{\partial y}$$

Example 1.10. a) Verify the Cauchy-Riemann equations for the function

$$f(z) = \frac{1}{z};\ z \neq 0.$$

b) Calculate $f^{'}(z)$

Solution. a) we have

$$P(x, y) = \frac{x}{x^2 + y^2}; Q(x, y) = \frac{-y}{x^2 + y^2}$$

and

$$\frac{\partial P}{\partial x} = \frac{\partial Q}{\partial y} = \frac{y^2 - x^2}{\left(x^2 + y^2\right)^2};$$

$$\frac{\partial P}{\partial y} = -\frac{\partial Q}{\partial x} = -\frac{2xy}{\left(x^2 + y^2\right)^2};$$

Then f is holomorphic in $\mathbb{C}^*$

b) The derivative is

$$\begin{aligned} f^{'}(z) = \frac{\partial P}{\partial x} + i\frac{\partial Q}{\partial x} &= \frac{y^2 - x^2 + i2xy}{\left(x^2 + y^2\right)^2} \\ &= -\frac{x^2 - 2ixy - y^2}{\left(z\overline{z}\right)^2} \\ &= \frac{\overline{z}^2}{\left(z\overline{z}\right)^2} = -\frac{1}{z^2} \end{aligned}$$

1.4 EXERCISES

Exercise 1.1. Find the values of

$$(1+2i)^2\,, \quad \frac{5}{-3+4i}, \qquad \left(\frac{2+i}{3-2i}\right)^2, \quad (1+i)^n + (1-i)^n$$

Exercise 1.2. Write the following complex numbers in algebraic form.

$$1)\ \frac{(1+i)^9}{(1-i)^7} \qquad 2)\ \frac{1+\alpha i}{2\alpha + (\alpha^2-1)\, i}; \quad \alpha \in \mathbb{R}$$

Exercise 1.3. If $z = x+iy$ (x and y real), find the real and imaginary parts of

$$z^4, \quad \frac{1}{z}, \ \frac{1}{z^2}, \quad \frac{z-1}{z+1}$$

Exercise 1.4. Verify by calculation that the values of $\dfrac{z}{z^2+1}$ for $z = x+iy$ and $z = x-iy$ are conjugate.

Exercise 1.5. Write the following complex numbers in trigonometric, and exponential form.

$$1-i\sqrt{3}, \qquad \frac{1+i\sqrt{3}}{1-i}; \qquad \left(\sqrt{3}+i\right)^6$$

Exercise 1.6. Compute

$$\sqrt{-i}, \quad \sqrt{1+i}, \quad \sqrt[4]{-1}$$

Exercise 1.7. Show that $\overline{e^z} = e^{\overline{z}}$ for all z in $\mathbb{C}$, and deduce that

$$\overline{\cos z} = \cos \overline{z}, \qquad \overline{\sin z} = \sin \overline{z}.$$

Exercise 1.8. Find the values of

$$\log\left(1-i\sqrt{2}\right)\,, \ \sin i, \ \cos i, \quad \tan(1+i), \quad 2^i, \ i^i.$$

Exercise 1.9. Solve in $\mathbb{C}$; the following equations:

1) $2\cos z - e^{-iz} = 1+2i$ 2) $\cos z = 3 + e^{iz}$

3) $\cos z = i \sin z$ 4) $\sin z = i \sinh z$

Exercise 1.10. Solve in $\mathbb{C}$; the following equations:

1) $2\cos z + i\sin z = 1 - i$ 2) $\sinh z = (1+i)\cosh z$

3) $\cos z = \cosh z$ 4) $\sin z = \cosh z$

Exercise 1.11. Let $z = x+iy$ where x and y are two real numbers and let the function

$$f(z) = z^2 + \sin(iz)$$

1) Find the real and imaginary parts of the function f

2) Show that f is holomorphic in $\mathbb{C}$

Exercise 1.12. Let $z = x + iy$ where x et y are two reals and let the function

$$f(z) = ax + iy + ie^z$$

1) Write $f(z)$ in the form $P(x;y) + iQ(x;y)$

2) Determine the constant a so that the function $f(z)$ is holomorphic.

Exercise 1.13. I) Verify the Cauchy-Riemann equations for the functions

a) $f(z) = iz^2 + 2z$;

b) $f(z) = \dfrac{z+i}{2z-3i}$. when $z \neq \dfrac{3i}{2}$

II) Calculate $f^{'}(z)$ by two methods.

Chapter 2

Power series. Radius of convergence. Disc of convergence. Power series expansion. Analytic functions.

2.1 Power series

Definition 2.1. A power series in the complex variable z is a series of the form

$$\sum_{n=0}^{\infty} a_n z^n = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n + \cdots$$

$a_0, a_1, a_2, \ldots a_n, \ldots$ called coefficients of the series.

Example 2.1. The geometric power series

$$\sum_{n=0}^{\infty} z^n = 1 + z + z^2 + \cdots + z^n + \cdots$$

is a power series, with $a_n = 1$ for all $n \in \mathbb{N}$. It converges when $|z| < 1$ and diverges when $|z| > 1$.

In fact, we have

$$\sum_{n=0}^{\infty} z^n = \frac{1}{1-z}; \quad \text{when} \quad |z| < 1.$$

2.2 Radius of convergence

Theorem 2.1. *Given a power series* $\sum_{n=0}^{\infty} a_n z^n$*, there exists* $0 \leq R \leq +\infty$ *such that:*

1. If $|z| < R$ the series converges absolutely.

2. If $|z| > R$ the series diverges.

The number R is called the **radius of convergence** of the power series.

Proof. Let $z \in \mathbb{C}$ be such that $|z| < R$. Choose $w \in \mathbb{C}$ such that $|z| < |w| < R$ and such that $\sum_{n=0}^{\infty} a_n w^n$ converges.

It follows that $|a_n w^n| \longrightarrow 0$ as $n \longrightarrow \infty$ Thus $|a_n w^n|$ is a bounded sequence; that is, there exists $K > 0$ such that $|a_n w^n| < K$ for all n Let $q = |z| \,/\, |w|$ As $|z| < |w|$ we have that $q < 1$ Now

$$|a_n z^n| = |a_n w^n| \left|\frac{z}{w}\right|^n < Kq^n$$

for all $n \in \mathbb{N}$ Hence $\sum\limits_{n=0}^{\infty} |a_n z^n|$ onverges by comparison with the geometric series. Since absolute convergence implies convergence, we are done. It follows immediately from the definition of R that the series diverges whenever $|z| > R$. Indeed, if $|z| < R$ the series converges.

□

Remark 2.1. In the case $|z| = R$ there is no general statement about the series, as one can have either convergence or divergence

Proposition 2.1. *Let* $\sum_{n=0}^{\infty} a_n z^n$ *be a power series with radius of convergence* R.

$$\textit{If} \quad \begin{cases} \lim\limits_{n \to +\infty} \left|\dfrac{a_{n+1}}{a_n}\right| = l \\ \lim\limits_{n \to +\infty} \sqrt[n]{a_n} = l \end{cases} \quad \textit{then} \quad R = \frac{1}{l}$$

Example 2.2. Determine the radius of convergence R of the power series

$$1) \sum_{n=1}^{\infty} \frac{z^n}{n} \qquad 2) \sum_{n=0}^{\infty} \frac{nz^n}{2^n}$$

Solution. 1) $\sum\limits_{n=1}^{\infty} \dfrac{z^n}{n}$

we have $a_n = \dfrac{1}{n} \Longrightarrow a_{n+1} = \dfrac{1}{n+1}$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{n}{n+1} = 1$$

Then the radius of convergence of the power series is $R = 1$

2) $\displaystyle\sum_{n=0}^{\infty}\frac{nz^n}{2^n}$

we have $a_n = \dfrac{n}{2^n} \Longrightarrow a_{n+1} = \dfrac{n+1}{2^{n+1}}$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{1}{2}\left(\frac{n+1}{n}\right) = \frac{1}{2}$$

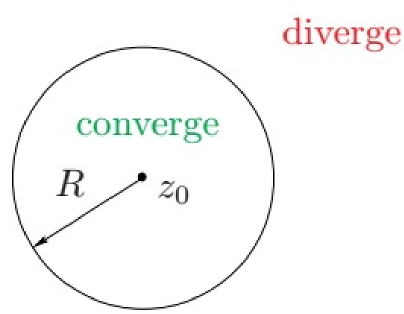
Then the radius of convergence of the power series is $R = 2$

2.3 Disc of convergence

Definition 2.2. The set

$$D = \left\{ z \in \mathbb{C} : \text{the power series } \sum_{n=0}^{\infty} a_n z^n \text{ converges} \right\}$$

is called the **disc of convergence** of the series.



Convergence of a power series.

Remark 2.2. In particular, we have:

$R = 0 \Longrightarrow D = \{0\}$

$R = \infty \Longrightarrow D = \mathbb{C}$

Example 2.3. Determine the radius of convergence R and the disc of convergence D of the power series

$1)\sum_{n=1}^{\infty} \frac{z^n}{n^2} \qquad\qquad 2) \sum_{n=0}^{\infty} \frac{z^n}{n!}$

Solution. 1) $\sum_{n=1}^{\infty} \frac{z^n}{n^2}$

- Radius of convergence R:

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \frac{n^2}{(n+1)^2} = 1 \Longrightarrow R = 1$$

- Disc of convergence D:

As $R = 1$ then, the series is converges when $|z| < 1$, and diverges when $|z| > 1$.

If $|z| = 1$, with $u_n = \dfrac{z^n}{n^2}$ we have:

$|z| = 1 \Longrightarrow |u_n| = \left| \dfrac{z^n}{n^2} \right| = \dfrac{1}{n^2}$ is converges (Riemann series with $\alpha = 2 > 1$)

then the series is converges absolutely therefore, they are converges.

So the disc of convergence is $D = \overline{D}\,(0, 1) = \{z \in \mathbb{C} : \text{with } \ |z| \leq 1\}$

2) $\sum_{n=0}^{\infty} \frac{z^n}{n!}$

- Radius of convergence R:

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \frac{n!}{(n+1)!} = \lim_{n\to\infty} \frac{1}{n+1} = 0 \Longrightarrow R = \infty$$

- Disc of convergence D: as $R = \infty$ then $D = \mathbb{C}$.

2.4 Power series expansion

Definition 2.3. Let $f : \Omega \longrightarrow \mathbb{C}$, we say that f has a power series expansion at a point $z_0 \in \Omega$ if there exists $R > 0$ such that:

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n \qquad \forall\, z \in D\,(z_0, R)$$

;

Example 2.4. We know That $\sum\limits_{n=0}^{\infty} z^n = \dfrac{1}{1-z}$; when $|z| < 1$.

Then the function $f(z) = \dfrac{1}{1-z}$ has a power series expansion $\forall \in D(0,1)$

Theorem 2.2. *The power series* $f(z) = \sum\limits_{n=0}^{\infty} a_n z^n$ *defines a holomorphic function inside its disc of convergence.*

- *The derivative of f is also a power series obtained by differentiating term by term the series for f, that is*
$$f'(z) = \sum_{n=1}^{\infty} n a_n z^{n-1}$$
Moreover, f' has the same radius of convergence as f.

- *The integration of f is also a power series has the same radius of convergence as f, that is*
$$\int_0^z f(\omega)\,d\omega = \int_0^z \sum_{n=0}^{\infty} a_n \omega^n d\omega = \sum_{n=0}^{\infty} \frac{a_n}{n+1} z^{n+1}$$

Example 2.5. Write the power series expansion for the following functions:

1) $f(z) = \dfrac{1}{1+z}$ 2) $g(z) = \dfrac{1}{1+z^2}$ 3) $h(z) = \dfrac{1}{(1-z)^2}$ 4) $k(z) = \ln(1+z)$

Solution. We have
$$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n; \text{ when } |z| < 1 \tag{2.1}$$

1) $$f(z) = \frac{1}{1+z} = \frac{1}{1-(-z)} = \sum_{n=0}^{\infty} (-z)^n; \text{ when } |-z| < 1$$
$$= \sum_{n=0}^{\infty} (-1)^n z^n; \text{ when } |z| < 1$$

2) $$g(z) = \frac{1}{1+z^2} = \frac{1}{1-(-z^2)} = \sum_{n=0}^{\infty} \left(-z^2\right)^n; \text{ when } \left|-z^2\right| < 1$$
$$= \sum_{n=0}^{\infty} (-1)^n z^{2n}; \text{ when } |z| < 1$$

3) $h(z) = \dfrac{1}{(1-z)^2}$

we deduce, by derivative term by term in (2.1), that

$$h(z) = \frac{1}{(1-z)^2} = \frac{d}{dz}\left(\frac{1}{1-z}\right) = \frac{d}{dz}\left(\sum_{n=0}^{\infty} z^n\right);\ \text{when } |z|<1.$$
$$= \sum_{n=1}^{\infty} n z^{n-1} \text{ when } |z|<1.$$

4) $k(z) = \ln(1+z) == \displaystyle\int_0^z \frac{1}{1+\omega} d\omega = \int_0^z \left(\sum_{n=0}^{\infty} (-1)^n \omega^n\right) d\omega$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} z^{n+1}, \quad \text{when } |z|<1.$$

Proposition 2.2. *Let $f(z) = \displaystyle\sum_{n=0}^{\infty} a_n z^n$, be a power series whose convergence radius is R. Then the function $f(z)$ has derivatives of all orders, and these derivatives can be obtained by differentiating the series term by term. The derivative series all have the same convergence radius R, and a*

$$a_n = \frac{f^{(n)}(0)}{n!};\ n = 0, 1, 2, \ldots \tag{2.2}$$

where $f^{(n)}(z)$ is the derivative of order n of $f(z)$.

Remark 2.3. The formula (2.2) is called the Taylor series coefficients.

Example 2.6. Write the power series expansion for the following functions:

1) $f(z) = e^z$ 2) $g(z) = (1+z)^{\alpha}$; $\alpha \in \mathbb{R}$

Solution. We have

1) $f(z) = e^z$

Here we can apply Proposition 2.5. We obtain the following

we have $f^{(n)}(z) = e^z \Longrightarrow f^{(n)}(0) = 1$ then

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!} \qquad \forall z \in \mathbb{C} \tag{2.3}$$

2) $g(z) = (1+z)^\alpha$

The Taylor series coefficients are given by:

$$a_n = \frac{g^{(n)}(0)}{n!}$$

Let's compute the derivatives of $g(z)$:

1. $$g'(z) = \alpha(1+z)^{\alpha-1} \Longrightarrow g'(0) = \alpha$$
2. $$g''(z) = \alpha(\alpha-1)(1+z)^{\alpha-2} \Longrightarrow g''(0) = \alpha(\alpha-1)$$
3. **n-th derivative:**

$$g^{(n)}(z) = \alpha(\alpha-1)(\alpha-2)\cdots(\alpha-n+1)(1+z)^{\alpha-n}$$

$$g^{(n)}(0) = \alpha(\alpha-1)(\alpha-2)\cdots(\alpha-n+1)$$

The coefficient a_n is:

$$a_n = \frac{g^{(n)}(0)}{n!} = \frac{\alpha(\alpha-1)(\alpha-2)\cdots(\alpha-n+1)}{n!}$$

Thus, the Taylor series expansion is:

$$(1+z)^\alpha = \sum_{n=0}^{\infty} \frac{\alpha(\alpha-1)(\alpha-2)\cdots(\alpha-n+1)}{n!} z^n$$

Convergence (Radius of Convergence)

We notice that:

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{\alpha-n}{n+1}\right|$$

has a limit of 1 when n tends towards $+\infty$. Therefore the radius of convergence of the series is $R = 1$.

Special Cases:

1. **When $\alpha = -1$:**

$$a_n = (-1)^n$$

$$\frac{1}{1+z} = \sum_{n=0}^{\infty} (-1)^n z^n \quad \text{(Geometric Series)}$$

2. **When $\alpha = \frac{1}{2}$ (Square Root):**

$$a_1 = \frac{1}{2}, \quad a_2 = \frac{\frac{1}{2} \cdot \left(-\frac{1}{2}\right)}{2!} = -\frac{1}{8}, \quad \text{etc.}$$

$$\sqrt{1+z} = 1 + \frac{1}{2}z - \frac{1}{8}z^2 + \frac{1}{16}z^3 - \cdots$$

Example 2.7. Write the power series expansion for the following functions:

1) $f(z) = \cos z$ 2) $g(z) = \sin z$ 3) $h(z) = \cosh z$ 4) $k(z) = \sinh z$

Solution. 1) $g(z) = \cos z$ 2) $h(z) = \sin z$

By using Euler's formula: $e^{i\alpha} = \cos\alpha + i\sin\alpha$ and (2.3) we have:

$$\begin{aligned} e^{i\alpha} =& \sum_{n=0}^{\infty} \frac{(i\alpha)^n}{n!} = \sum_{n=0}^{\infty} \frac{i^n}{n!}\alpha^n \\ &= \sum_{k=0}^{\infty} \frac{i^{2k}}{(2k)!}\alpha^{2k} + \sum_{k=0}^{\infty} \frac{i^{2k+1}}{(2k+1)!}\alpha^{2k+1} \\ &= \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!}\alpha^{2k} + i\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!}\alpha^{2k+1} \end{aligned}$$

Then

$$\begin{cases} \cos\alpha = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!}\alpha^{2k} \\ \sin\alpha = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!}\alpha^{2k+1} \end{cases}$$

In the general case where $z \in \mathbb{C}$, we find

$$\begin{cases} \cos z = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}z^{2n} \\ \sin z = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}z^{2n+1} \end{cases}$$

3) $h(z) = \cosh z$ 4) $k(z) = \sinh z$

We know that: $\cos(iz) = \cosh z$, then

$$\begin{aligned}\cosh z = \cos iz &= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}(iz)^{2n} \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}(i)^{2n}(z)^{2n} \\ &= \sum_{n=0}^{\infty} \frac{z^{2n}}{(2n)!}\end{aligned}$$

In the same way and by using the formula $\sin iz = i \sinh z$, we find

$$\begin{aligned}\sinh z = \frac{\sin iz}{i} &= \frac{1}{i}\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}(iz)^{2n+1} \\ &= \frac{1}{i}\sum_{n=0}^{\infty} \frac{(-1)^n (i)^{2n+1}}{(2n+1)!} z^{2n+1} \\ &= \sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n+1)!}\end{aligned}$$

2.5 Analytic functions

Definition 2.4. A function f defined on an open set Ω is said to be analytic at a point $z_0 \in \Omega$ if there exists a power series $\sum a_n(z-z_0)^n$ centered at z_0, with positive radius of convergence R, such that

$$f(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n \quad \forall\, z \in D(z_0, R)$$

If f has a power series expansion at every point in Ω, we say that f is analytic on Ω.

Example 2.8. We have

1. Any polynomial of degree n,
$$f(z) = a_0 + a_1 z + ... + a_n z^n$$
where $a_0, a_1, ..., a_n \in \mathbb{C}$, is analytic at all points $z \in \mathbb{C}$.
2. A rational function $f(z) = P(z)/Q(z)$ where $P(z)$ and $Q(z)$ are polynomials of degrees p and q, is analytic everywhere, except when $Q(z) = 0$.
3. The functions e^z , $\cos(z)$ and $\sin(z)$ are analytic everywhere.
4. The function $\log(z)$ is analytic everywhere except at $z = 0$.

2.5.1 Properties of analytic functions

Proposition 2.3. *If the functions f and g are analytic at point z_0, then*

1. the functions $f(z)+g(z)$ and $f(z)g(z)$ are analytic at point z_0;
2. the function $f(z)/g(z)$ is analytic at point z_0 provided $g(z_0)\neq 0$.
3. the function $(f\circ g)(z)$ is analytic at point z_0.

Example 2.9. the functions:

1. $f(z)=e^z+2z^2-1$ is analytic in $\mathbb{C}$. (the sum of two analytical functions)
2. $g(z)=\tan z$ is analytic in $\mathbb{C}-\left\{\frac{\pi}{2}+k\pi\right\}$ where $k\in\mathbb{Z}$.
3. $h(z)=e^{\cos z}$ is analytic in $\mathbb{C}$. (The decomposition of two analytical functions)

Proposition 2.4. *If*

$$f(z)=\sum_{n=0}^{\infty}a_n(z-z_0)^n$$

is convergent in $D(z_0,R)$ for some $R>0$, then $f:D(z_0,R)\longrightarrow\mathbb{C}$ is continuous. In particular, analytic functions are continuous.

2.5.2 Identity theorem for analytic functions

Corollary 2.1. *Suppose $f(z)=\sum\limits_{n=0}^{\infty}a_nz^n$ is a convergent power series and $(z_n)_{n\in\mathbb{N}}$ is a sequence of nonzero complex numbers converging to 0, such that $f(z_n)=0$ for all n. Then $a_k=0$ for every k.*

Recall that in a metric space X a cluster point (or sometimes limit point) of a set E is a point $a\in E$ such that $D(a,\epsilon)\setminus\{a\}$ contains points of E for all ϵ.

Theorem 2.3. ***(Identity theorem)*** *Let $D\subset\mathbb{C}$ be open and connected. If $f:D\longrightarrow\mathbb{C}$ and $g:D\longrightarrow\mathbb{C}$ are analytic functions that are equal on a set $\Omega\subset D$ and Ω has a cluster point in D then $f(z)=g(z)$ for all $z\in D$.*

Proof. Without loss of generality suppose Ω is the set of all points $z\in D$ such that $f(z)=g(z)$. Note that Ω must be closed as f and g are continuous.

Suppose Ω has a cluster point. Without loss of generality assume that 0 is this cluster point. Near we have the expansions

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \quad \text{and} \quad g(z) = \sum_{n=0}^{\infty} b_n z^n$$

which converge in some ball $D(0, r)$. Therefore the series

$$0 = f(z) - g(z) = f(z) = \sum_{n=0}^{\infty} (a_n - b_n) z^n$$

converges in $D(0, r)$ As 0 is a cluster point of Ω, there is a sequence of nonzero points $(z_n)_{n\in\mathbb{N}}$ such that $f(z_n) - g(z_n) = 0$ Hence, by the corollary above $a_n = b_n$ for all n. Therefore, $D(0, r) \subset \Omega$.

Thus the set of cluster points of Ω is open. The set of cluster points of Ω is also closed: A limit of cluster points of Ω is in Ω as it is closed, and it is clearly a cluster point of Ω. As D is connected, the set of cluster points of Ω is equal to D, or in other words $\Omega = D$. $\square$

2.6 EXERCISES

Exercise 2.1. Find the radius of convergence R and the disc of convergence D of the following power series:

$$\sum_{n=1}^{\infty} n^3 z^n, \ \sum_{n=1}^{\infty} \frac{2^n}{n!} z^n, \ \sum_{n=1}^{\infty} \frac{2^n}{n^2} z^n, \ \sum_{n=1}^{\infty} \frac{n^3}{3^n} z^n, \ \sum_{n=1}^{\infty} n^3 z^n, \ \sum_{n=1}^{\infty} \frac{z^n}{\ln n}.$$

Exercise 2.2. Find the radius of convergence R of the following power series:

$$\sum_{n=1}^{\infty} i^n z^n, \qquad \sum_{n=2}^{\infty} (1 + ni) z^n.$$

Exercise 2.3. Find the radius of convergence R of the following power series:

$$\sum_{n=0}^{\infty} n^{(-1)^n} z^n \qquad \sum_{n=0}^{\infty} \frac{sh\, n}{ch\, n} z^n \qquad \sum_{n=1}^{\infty} \left(\ln \frac{n^2 + a}{n^2} \right) z^n; \quad a > 0$$

$$\sum_{n=0}^{\infty} e^{-sh\, n} z^n \qquad \sum_{n=0}^{\infty} \frac{n!}{(a+1)(a+2)\dots(a+n)} z^n; \quad a > 0$$

Exercise 2.4. Write the power series expansion for the following functions:

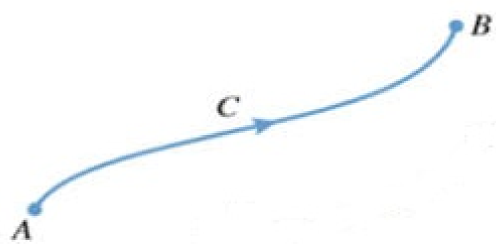
1) $f(z) = \dfrac{1}{2 - z}$ 2) $g(z) = \dfrac{1}{3 + 2z}$ 3) $h(z) = \dfrac{-1}{(1 + z)^2}$ 4) $k(z) = \arctan z$

Chapter 3

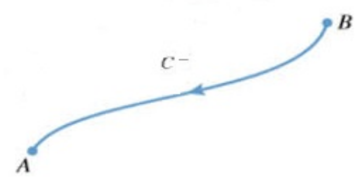
Cauchy's theory

3.1 Curvilinear Integral

Definition 3.1. A curve C connecting z_0 and z_1 is defined by a continuous and differentiable function $z(t) : [t_0; t_1] \longrightarrow \mathbb{C}$ such that $z(t_0) = z_0$ and $z(t_1) = z_1$.



Remark 3.1. The opposite curve C^- is the curve with the reverse orientation.

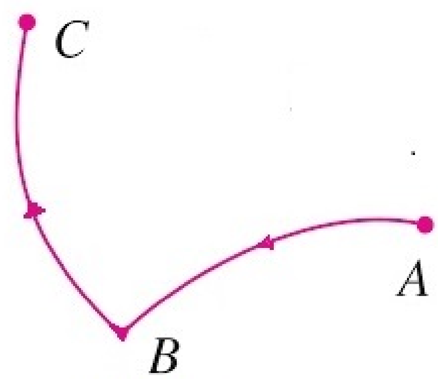


Example 3.1. The line segment from z_0 (the point A) to z_1 (the point B) is a curve C can be parametrised by

$$z(t) = (1-t)z_0 + tz_1 \quad \text{with} \quad t \in [0, 1]$$

Definition 3.2. A path γ is a union of curves: $\gamma = \bigcup\limits_{i=1}^{n} C_i$.

The path γ is said to be closed if its origin coincides with its end. $\gamma(z_0) = \gamma(z_1)$.



The path

Example 3.2. The circle $C(z_0, r)$ is a closed path can be parametrised by

$$z(\theta) = z_0 + re^{i\theta} \quad \text{with} \quad \theta \in [0, 2\pi]$$

Definition 3.3. Let $f : D \longrightarrow \mathbb{C}$ be a continuous function on D simply connected, and $\gamma \subset D$ a path defined by the equation

$z = z(t),\ a \leq t \leq b$. We defined the integral of f along γ by

$$\int\limits_{\gamma} f(z)dz = \int\limits_{a}^{b} f(z(t)).z^{'}(t)dt.$$

Example 3.3. Compute

1) $\int\limits_{\gamma} zdz$, where γ is the directed line segment from 0 to $1+i$.

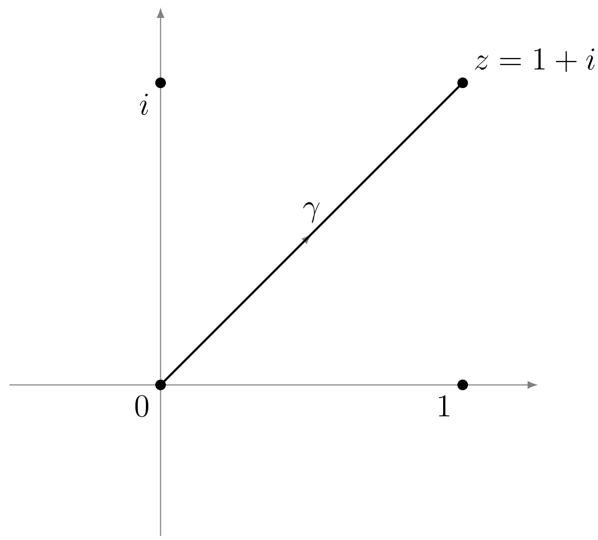
2) $\int\limits_{\gamma} \overline{z}dz$, where γ is the circle $|z-i| = 2$

Solution. We have

1) $\int\limits_{\gamma} zdz$

The directed line segment $[0, 1+i]$ is defined by the equation

$$z(t) = (1-t) \times 0 + t(1+i) = (1+i)\,t\,; \quad \text{with} \quad t \in [0, 1]$$



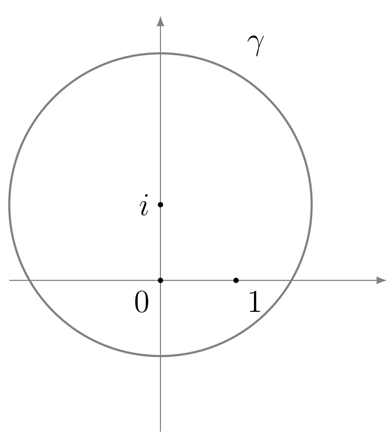
Then

$$\int_{\gamma} z dz = \int_0^1 z(t).z^{'}(t)dt = \int_0^1 (1+i)^2\, tdt = (1+i)^2 \left[\frac{t^2}{2}\right]_0^1 = \frac{(1+i)^2}{2} = i.$$

2) $\displaystyle\int_{\gamma} \overline{z} dz$

The path γ is the circle $C\,(i, 2)$ defined by

$$z\,(\theta) = z_0 + re^{i\theta} = i + 2e^{i\theta} \quad \text{with} \quad \theta \in [0, 2\pi]$$



So

$$\int_{\gamma} \overline{z} dz = \int_{0}^{2\pi} \overline{z(\theta)}.z^{'}(\theta) d\theta = \int_{0}^{2\pi} \left(-i + 2e^{-i\theta}\right)\left(2ie^{i\theta}\right) d\theta$$

$$= 2\int_{0}^{2\pi} \left(e^{i\theta} + 2i\right) d\theta$$

$$= 2\left[-ie^{i\theta} + 2i\theta\right]_{0}^{2\pi} = 8i\pi$$

Proposition 3.1. *The curvilinear Integral of continuous functions over path γ satisfies the following properties:*

- *It is linear, that is, if $\alpha, \beta \in \mathbb{C}$, then*

$$\int_{\gamma} (\alpha f(z) + \beta g(z))\, dz = \alpha \int_{\gamma} f(z) dz + \beta \int_{\gamma} f(z) dz.$$

- *If γ^{-} is γ with the reverse orientation, then*

$$\int_{\gamma^{-}} f(z) dz = -\int_{\gamma} f(z) dz.$$

- *If $\gamma = \gamma_1 + \gamma_2 + \cdots + \gamma_n$ (a subdivision of the path γ), then*

$$\int_{\gamma} f(z) dz = \int_{\gamma_1} f(z) dz + \int_{\gamma_2} f(z) dz + \cdots + \int_{\gamma_n} f(z) dz$$

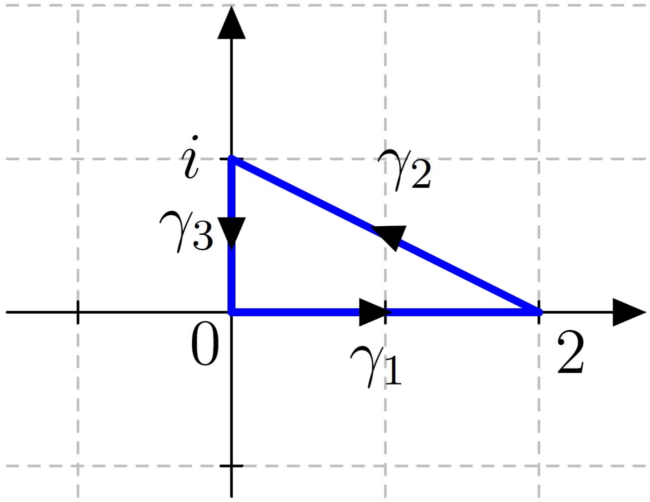
Example 3.4. Let γ be the path that is the triangle with vertices 0, 2, and i.

a) Compute $\displaystyle\int_{\gamma} |z|^2 \, dz$, $\displaystyle\int_{\gamma} (Re(z) + Im(z)) dz$,

b) Deduce the following integrals

$$\int_{\gamma} (|z|^2 - 2Re(z) - 2Im(z)) dz, \qquad \int_{\gamma^{-}} 3\,|z|^2 \, dz$$

Solution. a) We have $\gamma = \gamma_1 + \gamma_2 + \gamma_3$



1) γ_1 is the directed line segment $[0,2]$ is defined by the equation

$$z_1(t) = (1-t)\times 0 + t\times 2 = 2t\,; \quad \text{with} \quad t\in[0,1]$$

2) γ_2 is the directed line segment $[2,i]$ is defined by the equation

$$z_2(t) = (1-t)\times 2 + t\times i = 2-2t+it; \quad \text{with} \quad t\in[0,1]$$

3) γ_3 is the directed line segment $[i,0]$ is defined by the equation

$$z_3(t) = (1-t)\times i + t\times 0 = i(1-t)\,; \quad \text{with} \quad t\in[0,1]$$

So

$$\begin{aligned}
\int_{\gamma} |z|^2\,dz =& \int_{\gamma_1} |z|^2\,dz + \int_{\gamma_2} |z|^2\,dz + \int_{\gamma_3} |z|^2\,dz \\
=& \int_0^1 (2t)^2 2dt + \int_0^1 \left[(2-2t)^2+t^2\right](-2+i)dt + \int_0^1 (1-t)^2(-i)dt \\
=& 8\int_0^1 t^2dt + (-2+i)\int_0^1 \left(5t^2-8t+4\right)dt - i\int_0^1 (t^2-2t+1)dt \\
=& -\frac{2}{3}+2i
\end{aligned}$$

$$
\begin{aligned}
\int_{\gamma} (Re(z)+Im(z))\,dz =& \int_{\gamma_1} (Re(z)+Im(z))\,dz + \int_{\gamma_2} (Re(z)+Im(z))\,dz + \int_{\gamma_3} (Re(z)+Im(z))\,dz \\
=& \int_0^1 2t2dt + \int_0^1 (2-2t+t)\,(-2+i)dt + \int_0^1 (1-t)(-i)dt \\
=& 4\int_0^1 tdt + (-2+i)\int_0^1 (2-t)\,dt - i\int_0^1 (1-t)dt \\
=& -1+i
\end{aligned}
$$

b) By using the properties of the curvilinear Integral, we find

$$
\begin{aligned}
\int_{\gamma} (|z|^2 - 2Re(z) - 2Im(z))dz =& \int_{\gamma} |z|^2\,dz - 2\int_{\gamma} (Re(z)+Im(z))dz \\
=& -\frac{2}{3} + 2i - 2(-1+i) \\
=& \frac{4}{3}
\end{aligned}
$$

$$
\begin{aligned}
\int_{\gamma^-} 3\,|z|^2\,dz =& -\int_{\gamma} 3\,|z|^2\,dz \\
=& -3\int_{\gamma} |z|^2\,dz \\
=& -3(-\frac{2}{3}+2i) = 2-6i
\end{aligned}
$$

3.2 Cauchy's Theorem

Theorem 3.1. *Let $\gamma \subset D$ be a closed path, and $f : D \longrightarrow \mathbb{C}$a holomorphic (analytic) function in $Int(\gamma)$. Then*

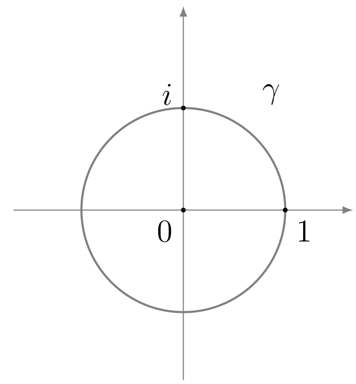
$$\int_{\gamma} f(z)dz = 0.$$

Example 3.5. Compute $\int_{\gamma} z^2 dz$, where γ is the circle $|z| = 1$

Solution. From Cauchy's Theorem we have

The path γ is the circle $C(0,1)$ so it is closed, and the function $f(z) = z^2$ is holomorphic (a polynomial) in $\mathbb{C}$. Then

$$\int_\gamma z^2 dz = 0$$



We verify this by direct computation.

The path γ is the circle $C(0,1)$ defined by

$$z(\theta) = z_0 + re^{i\theta} = e^{i\theta} \quad \text{with} \quad \theta \in [0, 2\pi]$$

And

$$\begin{aligned}\int_\gamma z^2 dz = \int_0^{2\pi} z^2(\theta)\, z'(\theta)\, d\theta &= \int_0^{2\pi} e^{2i\theta} i e^{i\theta} d\theta \\ &= \int_0^{2\pi} ie^{3i\theta} d\theta \\ &= \left[\frac{e^{3i\theta}}{3}\right]_0^{2\pi} = \frac{e^{i6\pi}}{3} - \frac{1}{3} = 0\end{aligned}$$

3.3 Cauchy's Integral Formula

Theorem 3.2. *Let γ be a closed path and let f be holomorphic (analytic) in an open domain containing γ. Then for every point a in $Int(\gamma)$*

$$\int_\gamma \frac{f(z)}{z-a} dz = 2\pi i f(a)$$

Example 3.6. Calculate integrals using Cauchy's integral formula

1) $\displaystyle\int_{|z|=2} \frac{e^z}{z-i} dz$ 2) $\displaystyle\int_{|z|=3} \frac{\cos \pi z}{z^2 - 3z + 2} dz$ 3) $\displaystyle\int_{|z-1|=2} \frac{z}{(z-i)(z+3i)} dz$

Solution. We have

1) $\displaystyle\int\limits_{|z|=2} \frac{e^{iz}}{z+i}dz$

We have the function $f(z) = e^{iz}$ is holomorphic in $\mathbb{C}$ $(f'(z) = ie^{iz})$, and the path γ is closed (a circle $C(0,2)$)

In addition

$$|a| = |-i| = 1 < 2 \Longrightarrow a \in Int(\gamma)$$

Then

$$\int\limits_{|z|=2} \frac{e^{iz}}{z+i}dz = 2\pi i f(-i) = 2\pi i e^{i(-i)} = 2\pi i e$$

2) $\displaystyle\int\limits_{|z|=3} \frac{\cos \pi z}{z^2 - 3z + 2}dz$

We have $z^2 - 3z + 2 = 0 \Longrightarrow z_1 = 1$ or $z_2 = 2$ and

$$\begin{cases} |z_1| = |1| = 1 < 3 & \Longrightarrow z_1 \in Int(\gamma) \\ |z_2| = |2| = 2 < 3 & \Longrightarrow z_2 \in Int(\gamma) \end{cases}$$

So we have to decompose $\dfrac{1}{z^2 - 3z + 2}$ into two simple elements

$$\frac{1}{z^2 - 3z + 2} = \frac{1}{(z-1)(z-2)} = \frac{1}{z-2} - \frac{1}{z-1}$$

Then

$$\begin{aligned} \int\limits_{|z|=3} \frac{\cos \pi z}{z^2 - 3z + 2}dz &= \int\limits_{|z|=3} \left(\frac{\cos \pi z}{z-2} - \frac{\cos \pi z}{z-1} \right) dz \\ &= \int\limits_{|z|=3} \frac{\cos \pi z}{z-2}dz - \int\limits_{|z|=3} \frac{\cos \pi z}{z-1}dz \end{aligned}$$

The function $f(z) = \cos \pi z$ is holomorphic in $\mathbb{C}$, so

$$\begin{aligned} \int\limits_{|z|=3} \frac{\cos \pi z}{z^2 - 3z + 2}dz &= 2\pi i \left[\cos 2\pi - \cos \pi\right] \\ &= 4\pi i \end{aligned}$$

3) $\displaystyle\int\limits_{|z-1|=2} \frac{z}{(z-i)(z+3i)}dz$

There are two singularities $a_0 = i$ and $a_1 = -3i$. It remains to verify whether:

$$a_0, \quad a_1 \in \text{Int}(\gamma)$$

$$\begin{cases} |i-1| = \sqrt{2} < 2 \implies i \in \text{Int}(\gamma). \\ |-3i-1| = \sqrt{10} > 2 \implies -3i \notin \text{Int}(\gamma). \end{cases}$$

Thus, the function $f(z) = \dfrac{z}{z+3i}$ is holomorphic in $\text{Int}(\gamma)$ because $-3i \notin \text{Int}(\gamma)$, and the path γ is closed since $\gamma = C(1,2)$.

$$\begin{aligned} \int_\gamma \frac{z}{(z-i)(z+3i)}\, dz = & \int_\gamma \frac{z/(z+3i)}{z-i}\, dz \\ =& 2\pi i f(i) \\ =& 2\pi i \left(\frac{i}{4i}\right) = \frac{\pi}{2} i. \end{aligned}$$

Theorem 3.3. *Let γ be a closed path and let f be holomorphic (analytic) in an open domain containing γ.*

Then for every point a in $Int(\gamma)$

$$\int\limits_\gamma \frac{f(z)}{(z-a)^{n+1}}dz = 2\pi i \frac{f^{(n)}(a)}{n!} \tag{3.1}$$

Example 3.7. Compute

$$\int\limits_{|z|=1} \frac{e^{\sin z}}{z^3}dz$$

Solution. By using the formula (3.1) we have

the function $f(z) = e^{\sin z}$ is holomorphic in $\text{int}(\gamma)$ because Because it is a composition of two functions, holomorphic , and the path γ is closed since $\gamma = C(0,1)$, with $0 \in Int(\gamma)$. Then

$$\int\limits_{|z|=1} \frac{e^{\sin z}}{z^3}dz = 2\pi i \frac{f''(0)}{2!}$$

where

$$f(z) = e^{\sin z} \Longrightarrow f'(z) = e^{\sin z}\cos z$$
$$\Longrightarrow f''(z) = e^{\sin z}\left(\cos^2 z - \sin z\right)$$

Thus $f''(0) = 1$, and so

$$\int\limits_{|z|=1} \frac{e^{\sin z}}{z^3} dz = \pi i.$$

3.4 EXERCISES

Exercise 3.1. Compute

1) $\int\limits_{\gamma} Re\,(z)\,dz,$ where γ is the directed line segment from $1+i$ to $3i$.

2) $\int\limits_{\gamma} Im\,(z)\,dz,$ where γ is the triangle with vertices 0, $1+i$, $4i$.

3) $\int\limits_{\gamma} z\overline{z}dz$, where γ is the circle $|z| = 2$

4) $\int\limits_{\gamma} z^2 dz$, where γ is the It is the semi-circle $|z+1| = 2$ from 1 to -3

5) $\int\limits_{\gamma} \frac{1}{1+z^2} dz$, where γ is the It is quarter circle $|z| = 2$ from 2 to $2i$

Exercise 3.2. Calculate integrals using Cauchy's integral formula

1) $\int\limits_{|z|=3} \frac{e^{\pi z}}{z-2i} dz,$ 2) $\int\limits_{|z|=3} \frac{\cos \pi z}{z^2+5z-6} dz,$ 3) $\int\limits_{|z|=2} \frac{e^{iz}}{z^2+1} dz.$

4) $\int\limits_{|z+1|=2} \frac{z}{z\,(z-i)} dz,$ 5) $\int\limits_{|z|=1} \frac{z^2}{2z-i} dz.$

Exercise 3.3. Calculate integrals

1) $\displaystyle\int\limits_{|z|=2} \frac{e^{iz}}{(z-i)^2}dz$, 2) $\displaystyle\int\limits_{|z|=1} \frac{\cos \pi z}{(z-1/2)^4}dz$, 3) $\displaystyle\int\limits_{|z+i|=2} \frac{e^{i\pi z}}{(z-1)^3}dz$.

Exercise 3.4. Let f be the following function

$$f(z) = \frac{e^z}{(z-1)(z+3)^2}$$

1- Determine the residues of f at each of its poles.

2- Deduce the following integrals

$$\int\limits_{|z|=\frac{1}{2}} \frac{e^z}{(z-1)(z+3)^2}dz \qquad \text{and} \qquad \int\limits_{|z|=4} \frac{e^z}{(z-1)(z+3)^2}dz$$

Chapter 4

Applications

4.1 Equivalence between holomorphy and analyticity.

We know that a real-valued function with a power series expansion around 0 is infinitely differentiable, so we will show that this result is still valid for functions of a complex variable

Theorem 4.1. *Let* $f(z) = \sum\limits_{n=0}^{\infty} a_n z^n$*, with radius of convergence* R*.*

- f is a holomorphic function inside its disc of convergence.
- The derivative of f is also a power series obtained by differentiating term by term the series for f, that is
$$f'(z) = \sum_{n=1}^{\infty} n a_n z^{n-1}$$
Moreover, f' has the same radius of convergence as f.

Example 4.1. We know that the function

$$\sum_{n=0}^{\infty} z^n = \frac{1}{1-z}; \quad \text{when } |z| < 1$$

Then the function $f(z) = \dfrac{1}{1-z}$ is holomorphic in $Int\,(D\,(0,1))$.

4.2 Maximum theorem.

Theorem 4.2. ***(Open Mapping Theorem).***
If f is holomorphic and non-constant in a region Ω, then it is open (i.e., f maps open sets to open sets).

Corollary 4.1. *Every holomorphic application is an open application*

Theorem 4.3. ***(The principle of maximum)***
Let Ω be a domain and $f \in \mathcal{H}(\Omega)$ be non-constant. Then $|f|$ does not attain its maximum value in Ω.

Proof. Suppose that $|f|$ reaches its maximum at $a \in \Omega$. i.e

$$|f(a)| = \max_{z\in\Omega} |f(z)| > 0$$

Since f is holomorphic and not constant, it is an open application, from which there exists $\delta > 0$ such that the disk $D\,(f(a); \delta) \subset f(\Omega)$.

Let

$$\omega = (1 + \frac{\delta}{2\,|f(a)|})f(a)$$

Then $\omega \in D\,(f(a); \delta)$ and consequently there exists $z \in \Omega$ such that $\omega = f(z)$ and on the other hand, $|f(z)| = |\omega| > |f(a)|$, which is absurd. $\square$

Example 4.2. Let

$$f(z) = z^2 + 1$$

be a function defined on the closed disk

$$\overline{D}(0, 2) = \{z \in \mathbb{C} \mid |z| \leq 2\}.$$

The function $f(z) = z^2 + 1$ is a polynomial, meaning it is holomorphic on $\mathbb{C}$, including the closed disk $\overline{D}(0, 2)$.

Evaluate $|f(z)|$ on the boundary

On the boundary $|z| = 2$, write $z = 2e^{i\theta}$. Then:

$$f(z) = (2e^{i\theta})^2 + 1 = 4e^{2i\theta} + 1$$

$$|f(z)| = |4e^{2i\theta} + 1|$$

Compute:

$$\begin{aligned}|f(z)|^2 &= |4e^{2i\theta} + 1|^2 \\ &= (4e^{2i\theta} + 1)(4e^{-2i\theta} + 1) \\ &= 16 + 4e^{2i\theta} + 4e^{-2i\theta} + 1 \\ &= 17 + 8\cos(2\theta)\end{aligned}$$

Therefore,

$$|f(z)| = \sqrt{17 + 8\cos(2\theta)}$$

The maximum of $\cos(2\theta)$ is 1, so:

$$\max |f(z)| = \sqrt{17 + 8 \cdot 1} = \sqrt{25} = 5$$

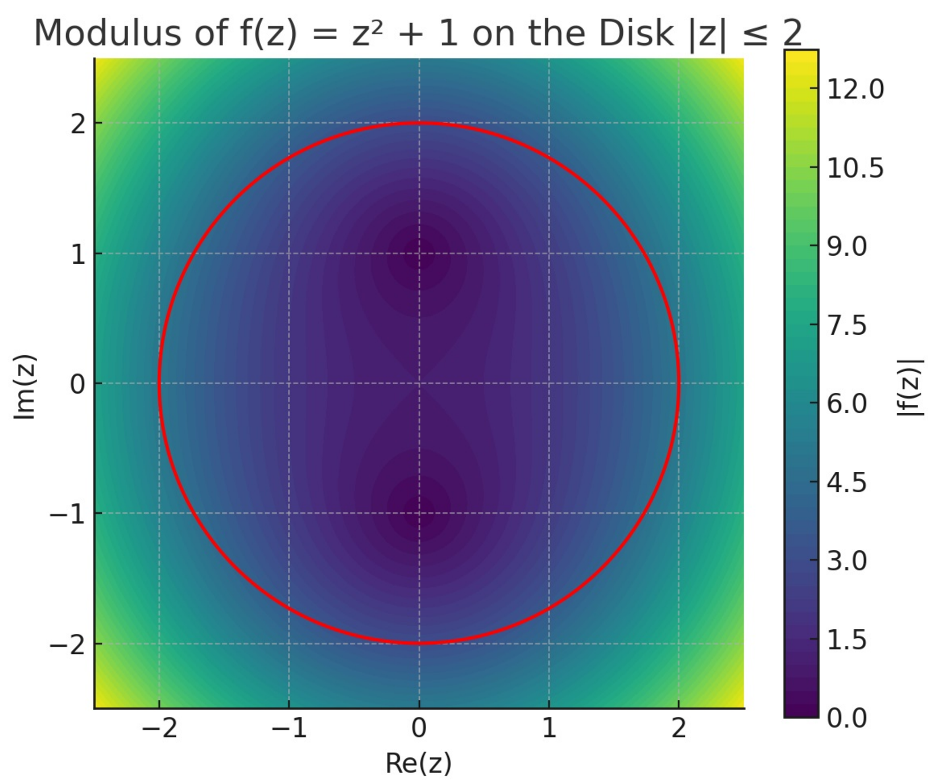
Interior of the disk

At the center $z = 0$, we have:

$$f(0) = 1 \quad \Rightarrow \quad |f(0)| = 1 < 5$$

Then

$|f(z)|$ cannot reach its maximum in $\overline{D}(0, 2)$, which confirms the **Maximum Theorem**.



4.3 Liouville's theorem.

Theorem 4.4. ***(Liouville's theorem)***
if f is holomorphic (analytic) on $\mathbb{C}$ and $|f| \leq M$ then f is constant.

Proof. Let f holomorphic (analytic) on $\mathbb{C}$, then f is a power series expansion

$$f(z) = \sum_{n=0}^{+\infty} a_n z^n$$

We have the Cauchy inequalities for the coefficients of the series:

$$|a_k| \leq \frac{1}{2\pi r^k} \sup\{|f(z)| : |a| = r\}$$

Since here $f(z) \leq M$ for all z, letting $r \longrightarrow +\infty$, we obtain

$$a_k = 0; \ \forall k \geq 1$$

Then $f(z) = a_0$, is constant. $\square$

4.4 Rouche's theorem.

Definition 4.1. A curve is a subset of $\mathbb{R}^2$ of the form $\gamma = \gamma(x) : x \in [0,1]$, where $\gamma : [0,1] \to \mathbb{R}^2$ is a continuous mapping from the closed interval $[0,1]$ to the plane. $\gamma(0)$ and $\gamma(1)$ are called the endpoints of curve γ. A curve is closed if its first and last points are the same. A curve is simple if it has no repeated points except possibly first = last. A closed simple curve is called a **Jordan-curve**.

Example 4.3. Line segments between $A, B \in \mathbb{R}^2$, circular arcs, Bezier-curves without self-intersection, etc...

Theorem 4.5. ***(Rouche's theorem)*** *Let $f, g : U \longrightarrow \mathbb{C}$ analytic on $U \subset \mathbb{C}$ open, $\gamma : I \to U$ a Jordan curve with $Int(\gamma) \subset U$. Assume f has no zero on $\gamma(I)$ and*

$$|f(z) - g(z)| \leq |g(z)|\,, \ \forall z \in \gamma(I).$$

Then f and g have the same number of zeros inside γ, counting multiplicities.

Example 4.4. Let $f(z) = 1 + 2z + 7z^2 + 3z^5$. Show that f has exactly two roots inside the unit disc.

Answer: Apply Rouché's theorem to $g(z) = 7z^2$ and $(f - g)(z) = 1 + 2z + 3z^5$.

4.5 Residue theorem.

4.5.1 The Poles

Definition 4.2. Let $f(z) = \dfrac{g(z)}{(z-a)^m}$ with $g(a) \neq 0$

The point a is called a pole of order m of f.

Remark 4.1. we have:

1. If $m = 1$ then a is a simple pole.
2. If $m = 1$ then a is a double pole.
3. If $m = 1$ then a is a triple pole.

Example 4.5. Determine the poles of the following functions

1) $f(z) = \dfrac{1}{(z+2)(z-3)}$ 2) $g(z) = \dfrac{\cos \pi z}{z^2 (z+1)}$ 3) $h(z) = \dfrac{e^z (z+2)}{(z^2-4)^3}$

Solution. We have

1) The function f has a simple pole at -2 and 3.

2) The function g has a simple pole at -1 and a double pole at 0.

3) $h(z) = \dfrac{e^z (z+2)}{(z^2-4)^3} = \dfrac{e^z}{(z+2)^2 (z-2)^3}$. Then the function h has a double pole at -2 and a triple pole at 2.

4.5.2 The residues

Definition 4.3. Let γ be a path. The residue of f in a is a complex number that verifies:

$$Res\,(f, a) = \frac{1}{2\pi i} \int_\gamma f(z) dz$$

with $a \in Int\,(\gamma)$.

If $f(z) = \dfrac{g(z)}{(z-a)^m}$ with $g(a) \neq 0$, then

$$Res\,(f, a) = \frac{1}{(m-1)!} \lim_{z \to a} \left[(z-a)^m f(z) \right]^{(m-1)}$$

If $m = 1$ then $Res\,(f, a) = \lim\limits_{z\to a} (z-a)\, f(z) = g(a)$

If $m = 2$ then $Res\,(f, a) = \lim\limits_{z\to a} \left((z-a)^2 f(z)\right)^{'} = g^{'}(a)$

If $m = 3$ then $Res\,(f, a) = \lim\limits_{z\to a} \frac{1}{2}\left((z-a)^3 f(z)\right)^{''} = \frac{g^{''}(a)}{2}$

Example 4.6. Calculate the residues of the following functions

1) $f(z) = \dfrac{z^3}{z-i}$ 2) $g(z) = \dfrac{z\cos\pi z}{(z+1)^2}$ 3) $h(z) = \dfrac{e^z}{(z-2)^3}$

Solution. We have

1) The function f has a simple pole at i, then

$$Res(f, i) = \lim_{z\to i} (z-i)\, f(z) = i^3 = -i$$

2) The function g has a double pole at -1, so

$$Res\,(f, -1) = \lim_{z\to -1} \left((z+1)^2 f(z)\right)^{'} = \lim_{z\to -1} (z\cos\pi z)^{'} = -1$$

3) The function h has a triple pole at 2, then

$$Res\,(f, 2) = \lim_{z\to 2} \frac{1}{2}\left((z-2)^3 f(z)\right)^{''} = \frac{e^2}{2}$$

Theorem 4.6. ***(Residue Theorem)*** *Let $f : \Omega \longrightarrow \mathbb{C}$, a holomorphic function in Ω except at a finite number of points $a_0, a_1, a_2, \ldots, a_n$, and $\gamma \subset \Omega$ a closed path. If $a_0, a_1, a_2, \ldots, a_n \in Int\,(\gamma)$ then*

$$\int_\gamma f(z)dz = 2\pi i \sum_{k=1}^{n} Res\,(f, a_k)$$

Example 4.7. Evaluate the integrals:

$$\int_\gamma \frac{z+1}{z\,(z-1)^2} dz; \quad \text{with } \gamma : |z| = 2$$

Solution 4.1. The function $f(z) = \dfrac{z+1}{z\,(z-1)^2}$ has a simple pole at 0 and a double pole at 1.

we have

$$\begin{cases} |0| = 0 < 2 & \Longrightarrow 0 \in Int\,(\gamma) \\ |1| = 1 < 2 & \Longrightarrow 1 \in Int\,(\gamma) \end{cases}$$

Then

$$\int\limits_{\gamma} \frac{z+1}{z\,(z-1)^2} dz = 2\pi i \left(Res\,(f,0) + Res\,(f,1) \right)$$

$$Res\,(f,0) = \lim_{z\to 0} (z-0)\, f(z) = \lim_{z\to 0} \frac{z+1}{(z-1)^2} = 1$$

$$Res\,(f,1) = \lim_{z\to 1} \left((z-1)^2 f(z) \right)' = \lim_{z\to 1} \frac{-1}{z^2} = -1$$

Then

$$\int\limits_{\gamma} \frac{z+1}{z\,(z-1)^2} dz = 2\pi i\,(1-1) = 0$$

4.6 Calculation of integrals using the residue method.

4.6.1 Integrals of the form $\int\limits_0^{2\pi} R\,(\cos\theta, \sin\theta)\, d\theta$

All integrals of the form

$$\int\limits_0^{2\pi} R\,(\cos\theta, \sin\theta)\, d\theta \tag{4.1}$$

where the integrated is a rational function of $\cos\theta$ and $\sin\theta$ can be easily evaluated by means of residues. It is very natural to make the substitution $z = e^{i\theta}$ which immediately transforms (4.1) into the line integral

$$\int\limits_{|z|=1} R\left[\frac{1}{2}\left(z + \frac{1}{z}\right), \frac{1}{2i}\left(z - \frac{1}{z}\right) \right] \frac{dz}{iz}$$

with $d\theta = \dfrac{dz}{iz}$, $\cos\theta = \dfrac{1}{2}\left(z + \dfrac{1}{z}\right)$ and $\sin\theta = \dfrac{1}{2i}\left(z - \dfrac{1}{z}\right)$.

It remains only to determine the residues which correspond to the poles of the integrated inside the unit circle.

Example 4.8. Evaluate the following integrals by the method of residues:

1) $\int\limits_0^{2\pi} \dfrac{d\theta}{5-4\cos\theta}$, 2) $\int\limits_0^{2\pi} \dfrac{\sin^2\theta}{5+4\cos\theta} d\theta$

Solution. we have

1) $\int\limits_0^{2\pi} \dfrac{d\theta}{5-4\cos\theta}$

$$\int\limits_0^{2\pi} \frac{d\theta}{5-4\cos\theta} = \int\limits_{|z.|=1} \frac{1}{5-4\left(z+z^{-1}\right)} \frac{dz}{iz} = \frac{2}{i} \int\limits_{|z|=1} \frac{1}{-4z^2+10z-4} dz = \frac{2}{i} \int\limits_{\gamma} f(z) dz$$

where $f(z) = \dfrac{1}{-4z^2+10z-4}$, and γ is the circle $|z| = 1$.

$$-4z^2+10z-4 = 0 \Longrightarrow \Delta = 36 \Longrightarrow \begin{cases} z_1 = \dfrac{-10+6}{-8} = \dfrac{1}{2} \\ z_2 = \dfrac{-10-6}{-8} = 2 \end{cases}$$

Then the function $f(z) = \dfrac{1}{-4\left(z-2\right)\left(z-1/2\right)}$ have 2 and $\dfrac{1}{2}$ a tow simple pole.

Only the pole $\dfrac{1}{2} \in Int\left(\gamma\right)$ ($\left|\dfrac{1}{2}\right| = \dfrac{1}{2} < 1$). So we need to calculate the residue of f in $\dfrac{1}{2}$

$$Res\left(f, \frac{1}{2}\right) = \lim_{z \to \frac{1}{2}} \left(z - \frac{1}{2}\right) f(z) = \lim_{z \to \frac{1}{2}} \frac{1}{-4\left(z-2\right)} = \frac{1}{6}.$$

Then

$$\begin{aligned} \int\limits_0^{2\pi} \frac{d\theta}{5-4\cos\theta} &= \frac{2}{i} \int\limits_{\gamma} f(z) dz \\ &= \frac{2}{i} \left(2\pi i Res\left(f, \frac{1}{2}\right)\right) \\ &= \frac{2\pi}{3} \end{aligned}$$

2) $\displaystyle\int_0^{2\pi} \frac{\sin^2\theta}{5+4\cos\theta}d\theta$

$$\int_0^{2\pi} \frac{\sin^2\theta}{5+4\cos\theta}d\theta = \int_{|z|=1} \frac{\left(1/2i\left(z-z^{-1}\right)\right)^2 dz}{5+4/2\left(z+z^{-1}\right)iz} = \frac{-1}{4i}\int_{|z|=1} \frac{\left(z^2-1\right)^2}{z^2\left(2z^2+5z+2\right)}dz = \frac{-1}{4i}\int_{|z|=1} f(z)dz$$

with $f(z) = \dfrac{\left(z^2-1\right)^2}{z^2\left(2z^2+5z+2\right)}$

we have $2z^2+5z+2=0 \Longrightarrow z=-\dfrac{1}{2}$ or $z=-2$

So $f(z) = \dfrac{\left(z^2-1\right)^2}{z^2\left(2z^2+5z+2\right)} = f(z) = \dfrac{\left(z^2-1\right)^2}{2z^2\left(z+\dfrac{1}{2}\right)(z+2)}$ has a double pole at $z=0$

and simple poles at $z=-\dfrac{1}{2}$ and $z=-2$.

Of these, only the poles at $z=0$ and $z=-\dfrac{1}{2}$ lie inside the unit disk. So we need to calculate the residue of f in $-\dfrac{1}{2}$ and 0.

For $z=0$

$$Res\,(f,0) = \lim_{z\to 0}\left(z^2 f(z)\right)' = \lim_{z\to 0}\left[\frac{\left(z^2-1\right)^2}{2\left(z+\dfrac{1}{2}\right)(z+2)}\right]' = -\frac{5}{4}$$

For $z=-\dfrac{1}{2}$

$$Res\left(f,-\frac{1}{2}\right) = \lim_{z\to -\frac{1}{2}}\left(z+\frac{1}{2}\right)f(z) = \lim_{z\to -\frac{1}{2}}\frac{\left(z^2-1\right)^2}{2z^2(z+2)} = \frac{3}{4}$$

Then

$$\begin{aligned}\int_0^{2\pi} \frac{\sin^2\theta}{5+4\cos\theta}d\theta &= \frac{-1}{4i}\int_\gamma f(z)dz \\ &= \frac{-1}{4i}\left[2\pi i\left(Res\,(f,0)+Res\left(f,-\frac{1}{2}\right)\right)\right] \\ &= \frac{\pi}{4}\end{aligned}$$

4.6.2 Integrals of the form $\int_{-\infty}^{+\infty} R\,(x)\,dx$

An integral of the form

$$\int_{-\infty}^{+\infty} R\,(x)\,dx$$

converges if and only if in the rational function $R(x) = \dfrac{P(x)}{Q(x)}$ the degree of the denominator is at least two units higher than the degree of the numerator, and if no pole lies on the real axis. Let us further assume that we have:

$$\deg Q \geq 2 + \deg P$$

we have

$$\int_{-\infty}^{+\infty} R\,(x)\,dx = 2\pi i\sum_{k=1}^{n} Res\,(f, a_k)$$

the a_k being the zeros of Q; with $Im\,(a_k) > 0$.

Example 4.9. Evaluate the following integrals by the method of residues:

1) $\displaystyle\int_{-\infty}^{+\infty} \frac{1}{x^2+4}dx$, 2) $\displaystyle\int_{-\infty}^{+\infty} \frac{1}{(x^2+1)^2}dx$

Solution. We have

1) $\displaystyle\int_{-\infty}^{+\infty} \frac{1}{x^2+4}dx$

Let $f(z) = \dfrac{1}{z^2+4} \Longrightarrow P(z) = 1, \quad Q(z) = z^2+4$

we have $\deg Q = 2 \geq \deg P + 2 = 2$, The roots of Q are $2i$ and $-2i$. Only pole $2i$ has a strictly positive imaginary part.

So we need to calculate the residue in $2i$

$$Res\,(f, 2i) = \lim_{z\to 2i} (z-2i)\, f(z) = \lim_{z\to 2i} \frac{1}{z+2i} = \frac{1}{4i}$$

Then

$$\int_{-\infty}^{+\infty} \frac{1}{x^2+4} dx = 2\pi i Res\,(f, 2i) = \frac{\pi}{2}$$

2) $\displaystyle\int_{-\infty}^{+\infty} \frac{1}{(x^2+1)^2} dx$

Let $f(z) = \dfrac{1}{(z^2+1)^2} \Longrightarrow P(z) = 1, \quad Q(z) = (z^2+1)^2$

we have $\deg Q = 4 \geq \deg P + 2 = 2$, The roots of Q are i and $-i$. Only pole i has a strictly positive imaginary part which is a double pole.

So we need to calculate the residue in $2i$

$$Res\,(f, i) = \lim_{z\to i} \left((z-i)^2 f(z)\right)' = \lim_{z\to i} \left(\frac{1}{(z+i)^2}\right)' = \frac{1}{4i}$$

Then

$$\int_{-\infty}^{+\infty} \frac{1}{(x^2+1)^2} dx = 2\pi i Res\,(f, i) = \frac{\pi}{2}$$

4.6.3 Integrals of the form $\displaystyle\int_{-\infty}^{+\infty} R\,(x)\, e^{i\alpha x} dx$

Let $R(x) = \dfrac{p(x)}{Q(x)}$ whose deminator $Q(x)$ does not have real roots and

$$\deg(Q(x)) \geq \deg(P(x)) + 1$$

. The calculation of the simple integral

$$\int_{-\infty}^{+\infty} R(x)e^{i\alpha x}dx$$

will be done according to the sign of the parameter α:

1) If $\alpha > 0$, then

$$\int_{-\infty}^{+\infty} R(x)e^{i\alpha x}dx = 2\pi i \sum_{k=1}^{n} Res(R(z)e^{i\alpha z}, a_k)$$

where a_k being the zeros of Q; with $Im\,(a_k) > 0$.

2) If $\alpha < 0$, then

$$\int_{-\infty}^{+\infty} R(x)e^{i\alpha x}dx = -2\pi i \sum_{k=1}^{n} Res(R(z)e^{i\alpha z}, b_k)$$

where b_k being the zeros of Q; with $Im\,(b_k) < 0$.

Example 4.10. Evaluate the following integral

$$\int_{-\infty}^{+\infty} \frac{\cos x}{x^2+1}dx$$

Solution. Let the integral

$$\int_{-\infty}^{+\infty} \frac{e^{ix}}{x^2+1}dx$$

We have $\displaystyle\int_{-\infty}^{+\infty} \frac{e^{ix}}{x^2+1}dx = \int_{-\infty}^{+\infty} \frac{\cos x + i\sin x}{x^2+1}dx = \int_{-\infty}^{+\infty} \frac{\cos x}{x^2+1}dx + i\int_{-\infty}^{+\infty} \frac{\sin x}{x^2+1}dx$

From this we notice

$$\int_{-\infty}^{+\infty} \frac{e^{ix}}{x^2+1}dx = Re\left(\int_{-\infty}^{+\infty} \frac{e^{ix}}{x^2+1}dx\right)$$

Let $R(z) = \dfrac{1}{z^2+1}$, The roots of $R(z)$ are i and $-i$. Only pole i has a strictly positive imaginary part which is a simple pole. Then

$$\int_{-\infty}^{+\infty} \frac{e^{ix}}{x^2+1}dx = \int_{-\infty}^{+\infty} R(x)e^{ix}dx = 2\pi i Res\left(R(z)e^{iz}, i\right)$$

We have

$$Res\left(R(z)e^{iz}, i\right) = \lim_{z\to i}(z-i)\,\frac{e^{iz}}{z^2+1} = \lim_{z\to i}\frac{e^{iz}}{z+i} = \frac{1}{2ie}$$

Then

$$\int_{-\infty}^{+\infty} \frac{e^{ix}}{x^2+1}dx = 2\pi i\frac{1}{2ie} = \frac{\pi}{e}$$

Finally $\displaystyle\int_{-\infty}^{+\infty} \frac{e^{ix}}{x^2+1}dx = \frac{\pi}{e}.$

4.7 EXERCISES

Exercise 4.1. Find the poles and residues of the following functions:

1) $f(z) = \dfrac{1}{z^2+5z+6}$ 2) $g(z) = \dfrac{1}{(z^2-1)^2}$ 3) $h(z) = \dfrac{(z-i)\,e^z}{z^2\,(z^2+1)^3}$

Exercise 4.2. Evaluate the following integrals by the method of residues:

1) $\displaystyle\int_0^{2\pi} \frac{d\theta}{5+3\sin\theta}$ 2) $\displaystyle\int_0^{2\pi} \frac{d\theta}{(2+\cos\theta)^2}$ 3) $\displaystyle\int_0^{2\pi} \frac{d\theta}{13+5\sin\theta}$ 4) $\displaystyle\int_0^{2\pi} \frac{\sin\theta}{5+\sin\theta}d\theta$

Exercise 4.3. Evaluate the following integrals by the method of residues:

1) $\displaystyle\int_{-\infty}^{+\infty} \frac{1}{x^6+1}dx$ 2) $\displaystyle\int_{-\infty}^{+\infty} \frac{x^2}{(x^2+a^2)\,(x^2+b^2)}dx$ 3) $\displaystyle\int_{-\infty}^{+\infty} \frac{1}{(x^2+1)^3}dx$

Exercise 4.4. 1) Evaluate the following integral

$$\int_{-\infty}^{+\infty} \frac{xe^{i\pi x}}{x^2+2x+5}dx$$

2) Deduce the integrals

$$\int_{-\infty}^{+\infty} \frac{x\cos(\pi x)}{x^2+2x+5}dx \qquad \int_{-\infty}^{+\infty} \frac{x\sin(\pi x)}{x^2+2x+5}dx$$

Chapter 5

Harmonic Functions

5.1 Harmonic Functions

Definition 5.1. Let U be an open subset of $\mathbb{R}^2$. A function $\psi : U \longrightarrow \mathbb{R}$ is called harmonic if

- ψ has continuous second order partial derivatives in U;
- ψ satisfies Laplace's equation:

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0$$

Example 5.1. Let $\psi : \mathbb{R}^2 \longrightarrow \mathbb{R}$ be a function defined by

$$\psi(x, y) = x^3 - 3xy^2 - 2xy + 2$$

We easily see that

$$\begin{cases} \dfrac{\partial \psi}{\partial x} = 3x^2 - 3y^2 - 2 \\ \dfrac{\partial \psi}{\partial y} = -6xy - 2x \end{cases} \Longrightarrow \begin{cases} \dfrac{\partial^2 \psi}{\partial x^2} = 6x \\ \dfrac{\partial^2 \psi}{\partial y^2} = -6x \end{cases} \Longrightarrow \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 6x - 6x = 0$$

Then ψ is harmonic in $\mathbb{R}^2$

Theorem 5.1. *Let f be holomorphic in an open set Ω, with real and imaginary parts P and Q. Then both P and Q are harmonic in Ω.*

Proof. We are supposing that $P, Q : \mathbb{R}^2 \longrightarrow \mathbb{R}$ are such that

$$f(z) = P(x, y) + iQ(x, y)$$

Since f is holomorphic, and by the Cauchy-Riemann equations we have

$$\frac{\partial^2 P}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial P}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{\partial Q}{\partial y}\right) = \frac{\partial}{\partial y}\left(\frac{\partial Q}{\partial x}\right) = \frac{\partial}{\partial y}\left(-\frac{\partial P}{\partial y}\right) = -\frac{\partial^2 P}{\partial y^2}$$

Then $\dfrac{\partial^2 P}{\partial x^2} + \dfrac{\partial^2 P}{\partial y^2} = 0.$

and similarly

$$\frac{\partial^2 Q}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial Q}{\partial x}\right) = \frac{\partial}{\partial x}\left(-\frac{\partial P}{\partial y}\right) = -\frac{\partial}{\partial y}\left(\frac{\partial P}{\partial x}\right) = -\frac{\partial}{\partial y}\left(\frac{\partial Q}{\partial y}\right) = -\frac{\partial^2 Q}{\partial y^2}$$

So $\dfrac{\partial^2 Q}{\partial x^2} + \dfrac{\partial^2 Q}{\partial y^2} = 0.$

Thus both P and Q are harmonic functions. □

Theorem 5.2. *Let Ω be an open disc, and suppose that $P : \Omega \longrightarrow \mathbb{R}$ is harmonic. Then there exists a complex function f, holomorphic in Ω, such that $P = Ref$.*

Remark 5.1. The function $Q = Imf$, which is also harmonic, is called a **harmonic conugate** for P.

Example 5.2. Verify that P is harmonic, and determine a function Q such that $f = P + iQ$ is holomorphic with $f(0,0) = 2 + 3i$.

1) $P(x, y) = x^3 - 3xy^2 - 2y + 2$

2) $P(x, y) = y^3 - 3x^2y + x^2 - y^2 + 2$

Solution. 1) $P(x, y) = x^3 - 3xy^2 - 2y + 2$

We have

$$\begin{cases} \dfrac{\partial P}{\partial x} = 3x^2 - 3y^2 \\ \dfrac{\partial P}{\partial y} = -6xy - 2 \end{cases} \Longrightarrow \begin{cases} \dfrac{\partial^2 P}{\partial x^2} = 6x \\ \dfrac{\partial^2 P}{\partial y^2} = -6x \end{cases}$$

Then $\dfrac{\partial^2 P}{\partial x^2} + \dfrac{\partial^2 P}{\partial y^2} = 6x - 6x = 0$ and P is harmonic function.

We are supposing that $f = P + iQ$ is holomorphic then

$$\begin{cases} \dfrac{\partial P}{\partial x} = \dfrac{\partial Q}{\partial y} & \ldots\ldots(1) \\ \dfrac{\partial P}{\partial y} = -\dfrac{\partial Q}{\partial x} & \ldots\ldots(2) \end{cases}$$

By integration in (1) we deduce that:

$$Q(x, y) = \int \left(3x^2 - 3y^2\right) dy = 3x^2y - y^3 + \varphi(x)$$

By using (2) we have

$$-6xy - 2 = -\left(6xy + \varphi^{'}(x)\right) \Longrightarrow \varphi^{'}(x) = 2 \Longrightarrow \varphi(x) = 2x + k$$

Then

$$Q(x, y) = 3x^2y - y^3 + 2x + k$$

From $f(0, 0) = 2 + 3i$ by solving, we find $k = 3$

Finally

$$Q(x, y) = 3x^2y - y^3 + 2x + 3$$

Observe that

$$f(z) = x^3 - 3xy^2 - 2y + 2 + i\left(3x^2y - y^3 + 2x + 3\right)$$

To express f as a function of z, we define: $x = z$ and $y = 0$. Then we get

$$f(z) = z^3 + 2iz + 2 + 3i$$

2) $P(x, y) = y^3 - 3x^2y + x^2 - y^2 + 2$

We have

$$\begin{cases} \dfrac{\partial P}{\partial x} = -6xy + 2x \\ \dfrac{\partial P}{\partial y} = 3y^2 - 3x^2 - 2y \end{cases} \Longrightarrow \begin{cases} \dfrac{\partial^2 P}{\partial x^2} = -6y + 2 \\ \dfrac{\partial^2 P}{\partial y^2} = 6y - 2 \end{cases}$$

Then $\dfrac{\partial^2 P}{\partial x^2} + \dfrac{\partial^2 P}{\partial y^2} = -6y + 2 - (6y - 2) = 0$ and P is harmonic function.

We are supposing that $f = P + iQ$ is holomorphic then

$$\begin{cases} \dfrac{\partial P}{\partial x} = \dfrac{\partial Q}{\partial y} & \ldots\ldots(1) \\ \dfrac{\partial P}{\partial y} = -\dfrac{\partial Q}{\partial x} & \ldots\ldots(2) \end{cases}$$

By integration in (1) we deduce that:

$$Q\left(x,y\right)=\int\left(-6xy+2x\right)dy=-3xy^{2}+2xy+\varphi\left(x\right)$$

By using (2) we have

$$3y^{2}-3x^{2}-2y=-\left(-3y^{2}+2y+\varphi^{'}\left(x\right)\right)\Longrightarrow\varphi^{'}\left(x\right)=3x^{2}\Longrightarrow\varphi\left(x\right)=x^{3}+k$$

Then

$$Q\left(x,y\right)=-3xy^{2}+2xy+x^{3}+k$$

From $f\left(0,0\right)=2+3i$ by solving, we find $k=3$

Finally

$$Q\left(x,y\right)=x^{3}-3xy^{2}+2xy+3$$

Observe that

$$f(z)=y^{3}-3x^{2}y+x^{2}-y^{2}+2+i\left(x^{3}-3xy^{2}+2xy+3\right)$$

To express f as a function of z, we define: $x=z$ and $y=0$. Then we get

$$f(z)=iz^{3}+z^{2}+2+3i$$

5.2 EXERCISES

Exercise 5.1. Verify that the following functions P are harmonic

1) $P\left(x,y\right)=5x^{2}–5y^{2}-3y+1;\quad x,y\in\mathbb{R}$

2) $P\left(x,y\right)=-y^{3}+3x^{2}y+7x+1;\quad x,y\in\mathbb{R}$

3) $P\left(x,y\right)=e^{x^{2}-y^{2}}cos(2xy)$

Exercise 5.2. Prove that v is a harmonic conjugate of u if and only $-u$ is a harmonic conjugate of v.

Exercise 5.3. Verify that the following functions u are harmonic, and determine a function v such that $u+iv$ is a holomorphc function.

1. $u(x,y)=x(1+2y)$
2. $u(x,y)=e^{x}cosy$
3. $u(x,y)=x-\dfrac{y}{x^{2}+y^{2}}.$

Chapter 6

Exams

6.1 Exam 2022

Ibn Khaldoun University of Tiaret - Faculty of Applied Sciences
Department of Science and Technology.

Exercise 01

Let $z = x + iy$ where $x, y \in \mathbb{R}$ and consider the function:

$$f(z) = e^{-y} \cos x + ie^{-y} \sin x.$$

1. Show that f is holomorphic using the Cauchy-Riemann conditions.
2. Compute the modulus and argument of $f(z)$.
3. Express $f(z)$ as a function of z.

Exercise 02

Solve in $\mathbb{C}$ the following equation:

$$2 \cosh z - 3e^{-z} = -1$$

Exercise 03

Let $P(x,y) = x^4 + y^4 - 6x^2y^2 - 5y + 1$.

1. Show that P is harmonic on $\mathbb{R}^2$.
2. Find the function Q such that f is a holomorphic function on $\mathbb{C}$ given in algebraic form by
$$f(z) = f(x+iy) = P(x,y) + iQ(x,y),$$
where $z = x+iy$, $P = \text{Re}(f)$ and $Q = \text{Im}(f)$.
3. Express $f(z)$ as a function of z such that $f(0,0) = 1-2i$.
4. Compute $f'(z)$ using two methods.

Exercise 04

Compute the following integrals using Cauchy's integral formula:

1. $\displaystyle\int_C \frac{\cos \pi z}{z+\frac{1}{4}}\,dz$ where C is the circle $|z| = 1$.
2. $\displaystyle\int_C \frac{e^{i\pi z}}{z^2+5z+6}\,dz$ where C is the circle $|z-i| = 3$.

Solution

Exercise 01

Let $P(x,y) = e^{-y}\cos x$ and $Q(x,y) = e^{-y}\sin x$. Thus, $P = \Re(f)$, $Q = \Im(f)$.

1. Show that f is holomorphic on $\mathbb{C}$:
$$\frac{\partial P}{\partial x} = -e^{-y}\sin x, \quad \frac{\partial Q}{\partial y} = -e^{-y}\sin x \Rightarrow \frac{\partial P}{\partial x} = \frac{\partial Q}{\partial y}$$
$$\frac{\partial P}{\partial y} = -e^{-y}\cos x, \quad \frac{\partial Q}{\partial x} = e^{-y}\cos x \Rightarrow \frac{\partial P}{\partial y} = -\frac{\partial Q}{\partial x}$$
So f satisfies the Cauchy-Riemann equations.

2. (a) The modulus of f:

$$|f(z)| = \sqrt{P^2 + Q^2} = \sqrt{e^{-2y}(\cos^2 x + \sin^2 x)} = e^{-y}$$

(b) The argument of f:

$$\cos\theta = \frac{e^{-y}\cos x}{e^{-y}} = \cos x, \quad \sin\theta = \frac{e^{-y}\sin x}{e^{-y}} = \sin x \Rightarrow \arg(f) = x + 2k\pi, \quad k \in \mathbb{Z}$$

3. The expression of $f(z)$:

$$f(z) = e^{-y}\cos x + ie^{-y}\sin x = \cos z + i\sin z = e^{iz}$$

Exercise 02

Solve the equation $\quad 2\cosh z - 3e^{-z} = -1$:

$$2\cosh z - 3e^{-z} = -1 \Rightarrow 2\left(\frac{e^z + e^{-z}}{2}\right) - 3e^{-z} = -1 \Rightarrow e^z - 2e^{-z} = -1$$

Multiply both sides by e^z:

$$e^{2z} + e^z - 2 = 0$$

Let $M = e^z$, then:

$$M^2 + M - 2 = 0 \Rightarrow \Delta = 1 + 8 = 9$$

So

$$M = 1 \Rightarrow z = \ln 1 + i2k\pi = i2k\pi$$

$$M = -2 \Rightarrow z = \ln|2| + i(\pi + 2k\pi) = \ln 2 + i(2k+1)\pi$$

Or:

$$z_k = i2k\pi \quad \text{or} \quad z_k = \ln 2 + i(2k+1)\pi, \quad k \in \mathbb{Z}$$

Exercise 03

1. Let $P(x,y) = x^4 + y^4 - 6x^2y^2 - 5y + 1$:

$$\frac{\partial P}{\partial x} = 4x^3 - 12xy^2, \quad \frac{\partial^2 P}{\partial x^2} = 12x^2 - 12y^2$$

$$\frac{\partial P}{\partial y} = 4y^3 - 12x^2y - 5, \quad \frac{\partial^2 P}{\partial y^2} = 12y^2 - 12x^2$$

$$\Rightarrow \frac{\partial^2 P}{\partial x^2} + \frac{\partial^2 P}{\partial y^2} = 0 \Rightarrow P \text{ is harmonic on } \mathbb{R}^2$$

2. Since f is holomorphic on $\mathbb{C}$, the pair (P,Q) satisfies the Cauchy-Riemann conditions:
$$\frac{\partial P}{\partial x} = \frac{\partial Q}{\partial y}, \qquad (1)$$
$$\frac{\partial P}{\partial y} = -\frac{\partial Q}{\partial x}, \qquad (2)$$
From (1): $\dfrac{\partial Q}{\partial y} = 4x^3 - 12xy^2 \Rightarrow Q(x,y) = \displaystyle\int (4x^3 - 12xy^2)\,dy = 4x^3y - 4xy^3 + C(x)$

From (2): $\dfrac{\partial P}{\partial y} = -\dfrac{\partial Q}{\partial x} \Rightarrow 4y^3 - 12x^2y - 5 = -(12x^2y - 4y^3 + C'(x)) \Rightarrow C'(x) = 5 \Rightarrow C(x) = 5x + c$

Final expression: $Q(x,y) = 4x^3y - 4xy^3 + 5x + c$

3. Since $f(0,0) = 1 - 2i \Rightarrow P(0,0) + iQ(0,0) = 1 + ic = 1 - 2i \Rightarrow c = -2$

   Therefore:
$$f(z) = x^4 + y^4 - 6x^2y^2 - 5y + 1 + i(4x^3y - 4xy^3 + 5x - 2) = z^4 + 5iz + 1 - 2i$$

4. Derivative of f:

   Method 1:
$$f'(z) = 4z^3 + 5i$$
   Method 2:
$$f'(z) = \frac{\partial P}{\partial x} + i\frac{\partial Q}{\partial x} = 4x^3 - 12xy^2 + i(12x^2y - 4y^3 + 5) = 4z^3 + 5i$$

Exercise 0.4

1. Evaluate $\displaystyle\int_C \frac{\cos(\pi z)}{z + \frac{1}{4}}\,dz$

   The function $f(z) = \cos(\pi z)$ is holomorphic inside C, which is the circle $C(0,1)$. Since $|-\frac{1}{4}| < 1$, the point is inside C.
$$\int_C \frac{\cos(\pi z)}{z + \frac{1}{4}}\,dz = 2\pi i \cos\left(-\frac{\pi}{4}\right) = \pi i\sqrt{2}$$

2. Evaluate $\displaystyle\int_C \frac{e^{i\pi z}}{z^2 + 5z + 6}\,dz$

   Factor the denominator: $z^2 + 5z + 6 = (z+2)(z+3)$

   Singularities at $z = -2, -3$

Check if: $\quad -2,\ -3 \overset{?}{\in} C(i,3)$

$$|-2-i| = \sqrt{5} < 3, \qquad |-3-i| = \sqrt{10} > 3$$

Only $z = -2 \in \text{int}(C)$

Method 1:

$$\int_C \frac{e^{i\pi z}}{(z+2)(z+3)}\,dz = \int_C \frac{e^{i\pi z}/(z+3)}{z+2}\,dz = 2\pi i \cdot \frac{e^{-2i\pi}}{1} = 2\pi i$$

Method 2: Partial Fractions:

$$\frac{1}{(z+2)(z+3)} = \frac{1}{z+2} - \frac{1}{z+3}$$

Thus:

$$\int_C \frac{e^{i\pi z}}{z^2+5z+6}\,dz = \int_C \frac{e^{i\pi z}}{z+2}\,dz - \int_C \frac{e^{i\pi z}}{z+3}\,dz = 2\pi i e^{-2\pi i} - 0 = 2\pi i$$

6.2 Exam 2023

Ibn Khaldoun University of Tiaret - Faculty of Applied Sciences Department of Science and Technology.

Exercise 01

Let $z = x + iy$, where $x, y \in \mathbb{R}$, and consider the function:

$$f(z) = e^{-iz}$$

1. Write $f(z)$ in algebraic form: $P(x,y) + iQ(x,y)$
2. Show that f is holomorphic on $\mathbb{C}$ using two methods.
3. Compute the modulus and argument of $f(z)$.
4. Solve in $\mathbb{C}$ the following equations:

$$e^z = i \qquad \cos z = i \sin z$$

Exercise 02

Let
$$P(x,y) = x^3 - 3xy^2 - 7y + 2$$

1. Show that this function P is harmonic on $\mathbb{R}^2$.
2. Find the function $Q(x,y)$ such that $f(z) = P(x,y) + iQ(x,y)$ is holomorphic on $\mathbb{C}$, where $z = x + iy$, $P = \Re(f)$, and $Q = \Im(f)$.
3. Express $f(z)$ in terms of z, given that $f(0,0) = 2 + 5i$.
4. Compute $f'(z)$ using two different methods.

Exercise 03

Let:
$$f(z) = \frac{1}{(2+i)z^2 + 6iz - 2 + i}$$

1. Determine the poles of f.
2. Compute the following integral using the residue theorem:
$$I = \int_0^{2\pi} \frac{1}{3 + \cos\theta + 2\sin\theta}\, d\theta$$

Solution

Exercise 01

Let $z = x + iy$, where $x, y \in \mathbb{R}$.

1. $$f(z) = e^{-i(x+iy)} = e^y e^{-ix} = e^y[\cos(x) - i\sin(x)] = e^y\cos(x) - ie^y\sin(x)$$
   Hence:
   $$\Re(f) = P(x,y) = e^y\cos(x), \quad \Im(f) = Q(x,y) = -e^y\sin(x)$$

2. Prove that f is holomorphic on $\mathbb{C}$:

   Method 1:

$$f'(z) = -ie^{-iz} \quad \forall z \in \mathbb{C}$$

   Method 2: Cauchy-Riemann Conditions

$$\frac{\partial P}{\partial x} = -e^y \sin(x), \quad \frac{\partial Q}{\partial y} = -e^y \sin(x) \Rightarrow \frac{\partial P}{\partial x} = \frac{\partial Q}{\partial y}$$

$$\frac{\partial P}{\partial y} = e^y \cos(x), \quad \frac{\partial Q}{\partial x} = -e^y \cos(x) \Rightarrow \frac{\partial P}{\partial y} = -\frac{\partial Q}{\partial x}$$

3. **Modulus:**

$$|f(z)| = \sqrt{P^2 + Q^2} = \sqrt{e^{2y}[\cos^2(x) + \sin^2(x)]} = e^y$$

   Argument:

$$\cos\theta = \frac{e^y \cos x}{e^y} = \cos x, \quad \sin\theta = \frac{-e^y \sin x}{e^y} = -\sin x \Rightarrow \arg(f) = -x+2k\pi, \quad k \in \mathbb{Z}$$

4. Solve in $\mathbb{C}$:

   (a) $e^z = i \Rightarrow z = \ln(i) = \ln 1 + i(\frac{\pi}{2} + 2k\pi) = i(\frac{\pi}{2} + 2k\pi), \quad k \in \mathbb{Z}$

   (b) $\cos z = i \sin z \Rightarrow$

$$\frac{e^{iz} + e^{-iz}}{2} = i \cdot \frac{e^{iz} - e^{-iz}}{2i} \Rightarrow 2e^{-iz} = 0$$

   Contradiction: $e^{-iz} \neq 0 \quad \forall z \in \mathbb{C} \Rightarrow$ No solution.

Exercise 02

Let $P(x, y) = x^3 - 3xy^2 - 7y + 2$

1. Show that P is harmonic on $\mathbb{R}^2$:

$$\frac{\partial^2 P}{\partial x^2} = 6x, \quad \frac{\partial^2 P}{\partial y^2} = -6x \Rightarrow \Delta P = 0 \Rightarrow P \text{ is harmonic}$$

2. Since f is holomorphic on $\mathbb{C}$, the pair (P, Q) satisfies the Cauchy-Riemann conditions:

$$\frac{\partial P}{\partial x} = \frac{\partial Q}{\partial y}, \quad \frac{\partial P}{\partial y} = -\frac{\partial Q}{\partial x}$$

From the first:

$$\frac{\partial Q}{\partial y} = 3x^2 - 3y^2 \Rightarrow Q(x, y) = \int (3x^2 - 3y^2)dy = 3x^2y - y^3 + C(x)$$

From the second:

$$\frac{\partial P}{\partial y} = -6xy - 7 = -\left(6xy + C'(x)\right) \Rightarrow C'(x) = 7 \Rightarrow C(x) = 7x + c$$

So:

$$Q(x, y) = -y^3 + 3x^2y + 7x + c$$

3. Given: $f(0, 0) = 2 + 5i \Rightarrow c = 5$

Final expression:

$$f(z) = x^3 - 3xy^2 - 7y + 2 + i(-y^3 + 3x^2y + 7x + 5) = z^3 + 7iz + 2 + 5i$$

4. Derivative of f:

Method 1:

$$f'(z) = 3z^2 + 7i$$

Method 2:

$$f'(z) = \frac{\partial P}{\partial x} + i\frac{\partial Q}{\partial x} = 3x^2 - 3y^2 + i(6xy + 7) = 3z^2 + 7i$$

Exercise 03

1. Find the poles of $f(z) = \dfrac{1}{(2+i)z^2 + 6iz - 2 + i}$

Solve the quadratic:

$$(2+i)z^2+6iz-2+i = 0 \Rightarrow \Delta = (-6i)^2-4(2+i)(-2+i) = -36+20 = -16 = (4i)^2$$

Roots:

$$z_0 = \frac{-6i + 4i}{2(2 + i)} = \frac{-i}{2 + i}, \quad z_1 = \frac{-6i - 4i}{2(2 + i)} = \frac{-5i}{2 + i}$$

2. Let $z = e^{i\theta}$, then:

$$\cos\theta = \frac{z+z^{-1}}{2}, \quad \sin\theta = \frac{z-z^{-1}}{2i}, \quad d\theta = \frac{dz}{iz}$$

$$I = \int_0^{2\pi} \frac{1}{3+\cos\theta+2\sin\theta} d\theta = \int_C \frac{1}{3+\frac{z+z^{-1}}{2}+2\cdot\frac{z-z^{-1}}{2i}} \cdot \frac{dz}{iz}$$

Simplifies to:

$$I = 2\int_C \frac{1}{(2+i)z^2+6iz-2+i} dz = 2\int_C f(z)dz$$

Only the pole $z_0 = \frac{-i}{2+i}$ is inside the unit circle because:

$$\left|\frac{-i}{2+i}\right| = \frac{1}{\sqrt{5}} < 1$$

Compute the residue at z_0:

$$\text{Res}(f, z_0) = \lim_{z\to z_0}(z-z_0)f(z) = \frac{1}{(2+i)(z_0-z_1)} = \frac{1}{4i}$$

Final result:

$$I = 2\cdot 2\pi i \cdot \frac{1}{4i} = \pi$$

6.3 Make-up Exam 2023

Ibn Khaldoun University of Tiaret - Faculty of Applied Sciences Department of Science and Technology.

Exercise 01: Let $z = x+iy$ where $x, y \in \mathbb{R}$, and consider the function:

$$f(z) = \cos(iz)$$

1) Express $f(z)$ in algebraic form: $P(x,y)+iQ(x,y)$.

2) Show that f is holomorphic on $\mathbb{C}$ using the Cauchy-Riemann conditions.

3) Find all values of z such that $f(z)$ is real.

4) Solve the following equation in $\mathbb{C}$:

$$f(z) = i$$

Exercise 02: Let

$$P(x, y) = x^2 - y^2 - 2xy - 2x + 3y + 5$$

1) Prove that this function P is harmonic on $\mathbb{R}^2$.

2) Find the function Q such that f is a holomorphic function on $\mathbb{C}$ given in algebraic form:

$$f(z) = f(x + iy) = P(x, y) + iQ(x, y)$$

where $z = x + iy$, $P = \Re(f)$, and $Q = \Im(f)$.

3) Express $f(z)$ as a function of z such that $f(0, 0) = 5 + 2i$.

4) Compute $f'(z)$ using two different methods.

Exercise 03: Let

$$f(z) = \frac{z^2}{z^4 + 3z^2 + 2}$$

1) Determine the poles of f.

2) Calculate the following integral using the residue theorem:

$$I = \int_{-\infty}^{+\infty} \frac{x^2}{x^4 + 3x^2 + 2}\, dx$$

Solution

Exercise 01

Let $z = x + iy$ where $x, y \in \mathbb{R}$.

1. Algebraic form of $f(z)$:

$$f(z) = \cos(iz) = \cos(ix - y) = \cosh x \cos y + i \sinh x \sin y$$

$$\Rightarrow Re(f) = P(x, y) = \cosh x \cos y \quad \text{and} \quad Im(f) = Q(x, y) = \sinh x \sin y$$

2. Show f is holomorphic on $\mathbb{C}$:

$$\begin{cases} \dfrac{\partial P}{\partial x} = \sinh x \cos y = \dfrac{\partial Q}{\partial y} \\ \dfrac{\partial P}{\partial y} = -\cosh x \sin y = -\dfrac{\partial Q}{\partial x} \end{cases}$$

The Cauchy-Riemann conditions are satisfied, so f is holomorphic.

3. Find z such that $f(z)$ is real:

$$Im(f) = 0 \Rightarrow \sinh x \sin y = 0 \Rightarrow x = 0 \text{ or } y = k\pi, k \in \mathbb{Z}$$

Solutions:

$$z = iy \text{ or } z = x + ik\pi, \quad x, y \in \mathbb{R}, k \in \mathbb{Z}$$

4. Solve $f(z) = i$:

$$f(z) = i \Longrightarrow \cos(iz) = i \Longrightarrow \frac{e^{-z} + e^{z}}{2} = i$$

$$\Longrightarrow e^{2z} - 2ie^{z} + 1 = 0 \overset{M=e^z}{\Longrightarrow} M^2 - 2iM + 1 = 0$$

$\Delta = -4 - 4 = -8 = (i2\sqrt{2})^2$

$$\begin{cases} M_1 = (1+\sqrt{2})i & \Longrightarrow e^z = (1+\sqrt{2})i \Longrightarrow z = \log((1+\sqrt{2})i) \\ M_2 = (1-\sqrt{2})i & \Longrightarrow e^z = (1-\sqrt{2})i \Longrightarrow z = \log((1-\sqrt{2})i) \end{cases}$$

Then $z_k = \ln(1+\sqrt{2}) + i(\frac{\pi}{2} + 2k\pi)$ and $z_k = \ln(\sqrt{2}-1) + i(-\frac{\pi}{2} + 2k\pi)$; $k \in \mathbb{Z}$

Exercise 02

Let $P(x, y) = x^2 - y^2 - 2xy - 2x + 3y + 5$.

1. Show P is harmonic on $\mathbb{R}^2$:

$$\begin{cases} \dfrac{\partial P}{\partial x} = 2x - 2y - 2 \\ \dfrac{\partial^2 P}{\partial x^2} = 2 \end{cases} \quad \text{et} \quad \begin{cases} \dfrac{\partial P}{\partial y} = -2y - 2x + 3 \\ \dfrac{\partial^2 P}{\partial y^2} = -2 \end{cases}$$

$$\Longrightarrow \frac{\partial^2 P}{\partial x^2} + \frac{\partial^2 P}{\partial y^2} = 0 \Longrightarrow P \text{ is harmonic in } \mathbb{R}^2$$

2. Find Q such that $f = P + iQ$ is holomorphic:

   we have

$$\begin{cases} \dfrac{\partial P}{\partial x} = \dfrac{\partial Q}{\partial y} \cdots\cdots (01) \\ \dfrac{\partial P}{\partial y} = -\dfrac{\partial Q}{\partial x} \cdots\cdots (02) \end{cases}$$

   From (1) we have $\dfrac{\partial Q}{\partial y} = 2x - 2y - 2$, then:

$$Q(x, y) = \int (2x - 2y - 2)\, dy = 2xy - y^2 - 2y + C(x).$$

   From (2):

   (2) $\Longrightarrow -2y - 2x + 3 = -(2y + C'(x)) \Longleftrightarrow C'(x) = 2x - 3 \Longrightarrow C(x) = x^2 - 3x + c.$

   Finally: $Q(x, y) = x^2 - y^2 + 2xy - 3x - 2y + c \;\; ; \;\; c \in \mathbb{R}..$

3. Express $f(z)$ in terms of z:

   we have

$$f(0, 0) = P(0, 0) + iQ(0, 0) \Longrightarrow 5 + ic = 5 + 2i \Longrightarrow c = 2$$

   Then

$$f(z) = (1 + i)z^2 - (2 + 3i)z + 5 + 2i$$

4. Compute $f'(z)$ by two methods:

   - Direct differentiation: $f'(z) = 2(1 + i)z - 2 - 3i$
   - Using Cauchy-Riemann:

$$\begin{aligned} f'(z) =& \frac{\partial P}{\partial x} + i\frac{\partial Q}{\partial x} \\ =& 2x - 2y - 2 + i(2x - 3) \\ =& 2z - 2 + i(2z - 3) = 2(1 + i)z - 2 - 3i \end{aligned}$$

Exercise 03

$$f(z) = \frac{z^2}{z^4 + 3z^2 + 2}$$

1. Poles of f:

   En posant $z^2 = t$ on obtient :

   $z4 + 3z2 + 2 = 0 \Longrightarrow t^2 + 3t + 2 = 0 \Longrightarrow \Delta = 9 - 8 = 1$

   $\begin{cases} t_1 = -1 \\ t_2 = -2 \end{cases} \Longrightarrow \begin{cases} z^2 = -1 \\ z^2 = -2 \end{cases} \Longrightarrow \begin{cases} z_0 = i & \text{ou } \ z_1 = -i \\ z_2 = i\sqrt{2} & \text{ou } \ z_3 = -i\sqrt{2} \end{cases}$

2. Calculate the integral using residue theorem:

   The function $f(z)$ has 4 simple poles i, $-i$, $i\sqrt{2}$, et $-i\sqrt{2}$

   we have $P(z) = z^2$ and $Q(z) = z^4 + 3z^2 + 2$ with $\deg Q - \deg P = 4 - 2 \geq 2$

   $$f(z) = \frac{z^2}{(z+i)(z-i)(z+i\sqrt{2})(z-i\sqrt{2})}$$

   Only poles i and $i\sqrt{2}$ have a strictly positive imaginary part.

   Then we need to calculate the residue at i and $i\sqrt{2}$

   $$\begin{aligned} Res(f,i) =& \lim_{z\to i}(z-i)f(z) \\ =& \lim_{z\to i}\frac{z^2}{(z+i)(z+i\sqrt{2})(z-i\sqrt{2})} = \frac{-1}{2i} \end{aligned}$$

   $$\begin{aligned} Res(f,i\sqrt{2}) =& \lim_{z\to i\sqrt{2}}(z-i\sqrt{2})f(z) \\ =& \lim_{z\to i\sqrt{2}}\frac{z^2}{(z+i)(z-i)(z+i\sqrt{2})} = \frac{1}{i\sqrt{2}} \end{aligned}$$

   Then :

   $$\begin{aligned} I = \int_{-\infty}^{+\infty} \frac{x^2}{x^4+3x^2+2}\, dx =& 2\pi i \left[Res(f,i) + Res(f,i\sqrt{2})\right] \\ =& 2\pi i \left(\frac{-1}{2i} + \frac{1}{i\sqrt{2}}\right) \\ =& \frac{2-\sqrt{2}}{\sqrt{2}}\pi = (\sqrt{2}-1)\pi \end{aligned}$$

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