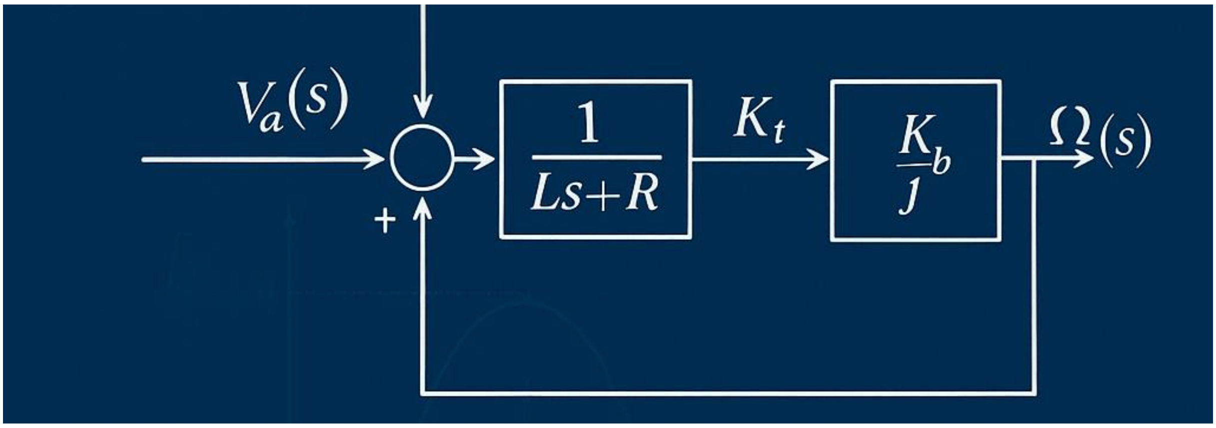
The People's Democratic Republic of Algeria
The Ministry of Higher Education and Scientific Research

Ibn Khaldoun University, Tiaret, Algeria
Applied Science Faculty
Electrical Engineering Departement

LINEAR CONTROL SYSTEMS :
A Comprehensive Course with Solved Exercises

For third-year undergraduate electrical engineering students



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Preface

This course, *Control of Linear Systems,* is designed for third-year undergraduate students in Electrical Engineering. It follows the official program established by the Ministry of Higher Education, which is divided into two main parts:

- The first part covers the modeling and representation of linear systems, providing the fundamental tools needed to describe dynamic behavior, such as transfer functions, Laplace transforms, and state-space representations.
- The second part focuses on the analysis of linear systems, including stability, frequency response, time-domain analysis, and feedback control techniques.

The course offers an introduction to linear control theory, aiming to give students a strong foundation for more advanced studies and practical applications in control engineering.
To support learning, the course includes numerous exercises, each accompanied by detailed solutions.

I hope this course helps students develop a solid understanding of linear control principles and prepares them for future academic and professional challenges.

Chapter 1: Introduction to linear control system

1.1 Introduction

This section provides context for the course, introduces the preliminary definitions for the study, highlights the key points that will be addressed later, and helps build your intuition about control theory.

1.2 Definitions:

Control Engineering: The branch of science and engineering concerned with the design and application of systems that operate autonomously without human intervention.

We can break down the key elements of this definition as follows:

1. **Scientific Principles:** This indicates that control engineering involves theoretical work to:

    - Develop a mathematical model of a system.
    - Analyze its behavior based on the model.
    - Design a control strategy using the model.

2. **System Design and Implementation:** This refers to the practical aspects, which may involve fields like electronics, computer engineering, and related disciplines.
3. **Autonomous Operation:** This emphasizes the concept of automated systems, which:

    - Improve performance and user convenience (e.g., climate control, power steering).
    - Enhance safety (e.g., autopilot, ABS braking).

1.3 Why we need a control engineering:

The demand for control engineers has skyrocketed with the rise of automation and digital industrial technologies. This has created many career opportunities across a diverse array of industries. Controls engineers are indispensable across a range of verticals, including but not limited to:

- Manufacturing: Robotic systems, industrial machinery, and process control technologies.

- Renewable Energy: Solar and wind energy systems that require sophisticated controls for efficiency and power grid integration.
- Defense and Aerospace: Guidance systems for aircraft and spacecraft, and control systems for unmanned vehicles.
- Biomedical Engineering: Medical devices, prosthetics, and imaging equipment that demand precise control for safe and effective operation.
- Automotive: Antilock Braking Systems (ABS), cruise control, and other high-tech vehicle systems

The Bureau of Labor Statistics projects a 6% job growth rate for control engineers through 2030, much faster than the average for all occupations and salaries typically reach six figures early in the career.

1.4 System concept

In control systems engineering, the concept of a system is fundamental. Its definition closely aligns with the classical one in physics. Generally, a system is an entity that interacts with its environment, producing various dynamic behaviors. Certain external physical quantities influence the system; these are referred to as input**s**. Conversely, other quantities are generated by the system and affect its surroundings; these are known as output**s**. Input signals are typically denoted by u, while output signals are represented by **y**. A system's inputs can generally be manipulated. However, some inputs are beyond direct control and cannot be altered. These are called disturbances and are commonly denoted by d**.**

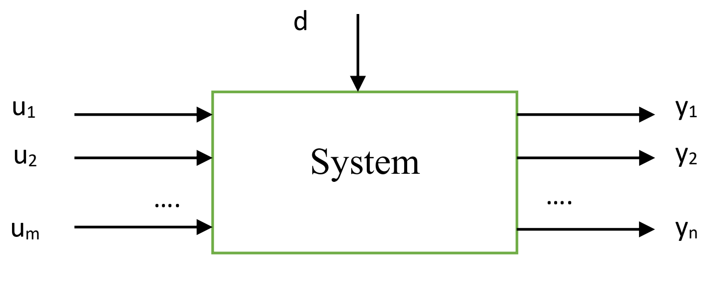

Figure 1: system with inputs and outputs

In practice, a system can represent a mechanical, electronic, or chemical device, and it is usually straightforward to distinguish it from its environment as well as to define its inputs and outputs.

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For example:

- In mechanical systems, inputs can be a force or torque, while outputs might be velocity or torque.
- In electrical systems, inputs could be voltage or current, with corresponding outputs being voltage or current.
- In chemical systems, an input might be the concentration of a reactant, while an output could be the concentration of the final product.

Typical disturbances include factors such as aerodynamic drag in mechanical systems, electrical noise in electronic circuits, or unaccounted impurities in chemical processes.

4.1.1 Linear system :

A linear system in control theory is a system that satisfies the principles of superposition and homogeneity. This means that the system's response to a combination of inputs is the sum of the individual responses to each input, scaled accordingly.

Formally, a system is linear if it obeys the following two properties:

- **Superposition (Additivity):** If an input $u_1(t)$ produces an output $y_1(t)$, and an input $u_2(t)$ produces an output $y_2(t)$, then the response to the combined input $u_1(t)+u_2(t)$ is:

$$y(t)=y_1(t)+y_2(t) \qquad (1.1)$$

- **Homogeneity (Scaling):** If an input $u(t)$ produces an output $y(t)$, then for any scalar α\alphaα, the response to the scaled input $\alpha\, u(t)$ is:

$$y(t)=\alpha y(t) \qquad (1.2)$$

4.1.2 Dynamic system

A dynamic system is a system in which the relationship between inputs and outputs is described by differential equations. This means that the system's behavior evolves over time based on its internal state and external inputs.

In control theory, dynamic systems can be classified into:

- Continuous-time systems, where the evolution is governed by ordinary or partial differential equations.

- Discrete-time systems, where the system dynamics are described by difference equations

4.1.3 Linear time-invariant system

A Linear Time-Invariant (LTI) system is a system that satisfies two key properties:

1. **Linearity**: The system adheres to the principles of superposition and scaling, meaning that the output for a weighted sum of inputs is the weighted sum of the outputs for each individual input.
2. **Time-Invariance**: The system's behavior and characteristics do not change over time. In other words, if the input is shifted in time, the output will also shift by the same amount without any change in its form or structure.

Figure 2: Time invariant system

Remark: An LTI system is a system whose mathematical model is governed by differential equations with constant coefficients.

1.5 Feedback concept :

Feedback is a process in which a portion of a system's output is returned to its input to regulate its behavior. In control systems, feedback is used to improve stability, accuracy, and performance by continuously correcting errors between the desired output and the actual output.

Example: Necessity of a Closed-Loop System

Consider the problem of maintaining a car's speed:

- **Open-Loop Control (Without Feedback):** Suppose a driver wants to maintain a speed of 60 km/h and sets the throttle to a fixed position. However, external factors such as road incline, wind resistance, and vehicle load can cause speed variations, making it difficult to maintain exactly 60 km/h.

- **Closed-Loop Control (With Feedback):** In cruise control, a speed sensor continuously measures the car's actual speed and compares it to the desired speed (setpoint). If there is a deviation, the system automatically adjusts the throttle to correct the speed, ensuring that the vehicle stays at 60 km/h despite disturbances.

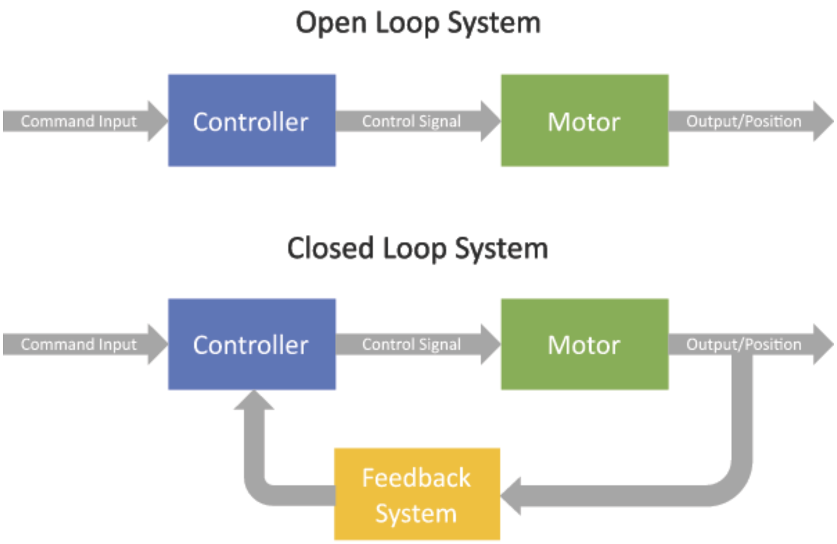


Figure 3: open-loop *Vs* closed loop

From this example, we can understand the importance of the feedback loop. In control systems, feedback enables two essential functions:

1. **Tracking:** Ensuring that the system's outputs follow given reference signals as closely as possible.
2. **Regulation:** Minimizing the effect of disturbances on the system's outputs to maintain stability and performance.

In both cases, certain performance criteria are often required, such as:

- **Stability** – Making sure the system remains controlled and doesn't behave unpredictably.
- **Response Time** – How fast the system reacts to changes.
- **Smooth Output** – Avoiding unnecessary oscillations or fluctuations.
- **Accuracy** – Keeping the output as close as possible to the desired value.

These aspects will be discussed in more detail in the following chapters.

1.6 Structure of Closed-Loop System (feedback system)

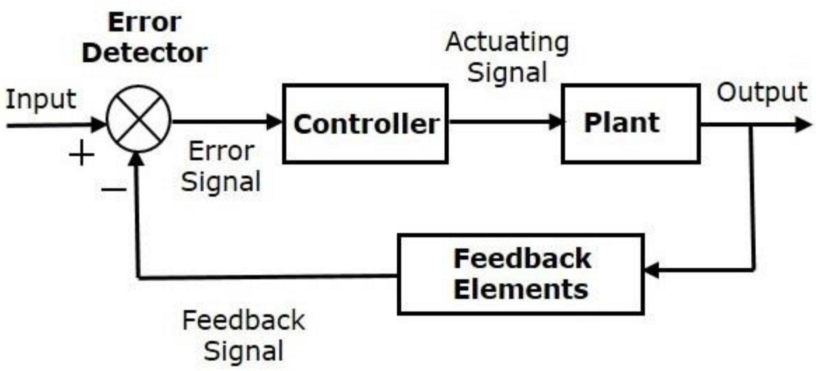


Figure 4: basic structure of closed-loop control system

A typical closed-loop control system consists of the following key components:

1. **Reference Input** The desired value or setpoint that the system should follow.
2. **Controller** – Processes the error signal and generates a control action to minimize the deviation from the reference. This could be a PID controller, state-space controller, etc.
3. **Plant** – The system being controlled, such as a motor, robotic arm, or any dynamic system.
4. **Sensor (Measurement System)** – Measures the actual output of the system.
5. **Feedback Loop** – Compares the actual output with the reference and adjusts the control action accordingly.

Working Principle

1. The reference input (r(t)) is compared with the actual output (y(t)) to compute the error signal.
2. The controller processes the error and generates a control signal.
3. The actuator (or plant) responds to the control signal and produces an output.
4. The sensor measures the actual output and sends feedback to the controller.
5. The process continues until the output closely matches the reference, ensuring stability and precision.

This closed-loop structure allows the system to track setpoints accurately and reject disturbances, making it essential in automation, robotics, and industrial control applications.

This structure is very intuitive because it mirrors human behavior in everyday actions. For example, when you reach for an object, your brain (controller) sends signals to your muscles (actuator) to move your hand. Your eyes (sensor) continuously monitor the movement and provide feedback, allowing your brain to adjust your motion in real time. If the object is farther than expected, your brain automatically corrects the movement until you successfully grasp it.

1.7 Methodology in Control Systems

In control engineering, the methodology generally follows several key steps:

1. **Specification (Requirement Analysis):**
   The control engineer must fully understand the problem and its specifications. This includes clearly defining the system, identifying its inputs and outputs, and setting performance objectives (e.g., stability, speed, accuracy).
2. **Modeling:**
   The next step is to describe the system's behavior (open-loop system) using physical laws. This results in algebraic and differential equations, which are then reformulated into a standard control system model, such as a transfer function or state-space representation.
3. **Analysis:**
   Using control techniques, the system's performance is evaluated based on stability, response time, oscillations, and accuracy. This step helps determine whether the system meets the desired specifications.
4. **Controller Design (Synthesis):**
   The final step is to design a control law—an intelligent feedback mechanism that ensures the closed-loop system achieves the desired performance, if possible.

Remark: In this course, we will focus only on the modeling and analysis phases

Part1: Modeling and Linear Control System Representation

- **Chapter 2:** Modeling – From Differential Equations to Transfer Functions
- **Chapter 3:** Block Diagram Representation
- **Chapter 4:** State Space Representation
-

2.1 Introduction :

Modeling is a fundamental step in control system analysis. It involves describing the system's behavior using mathematical equations and transforming them into standard representations used in control theory.

2.2 Modeling or Equation Formulation

To model a system, we must establish the relationships between input and output variables using the fundamental laws of the relevant domains (mechanical, electrical, thermal, etc.).

$$\text{Mechanics: Newton's Laws as: } \sum F = m\frac{\partial^2 x}{\partial t^2}$$

$$\text{Electricity: Kirchhoff's Laws as; } \sum V_i = 0 \text{ and } \sum I_n = \sum I_{out}$$

Note: All physical models are generally governed by ordinary differential equations (ODEs) or partial differential equations (PDEs).

Example 2.1: RC Circuit

Consider the **RC circuit** shown in Figure 2.1, where the input is denoted by Ve(t) (the applied voltage), and the output is Vs(t) (the voltage across the capacitor).

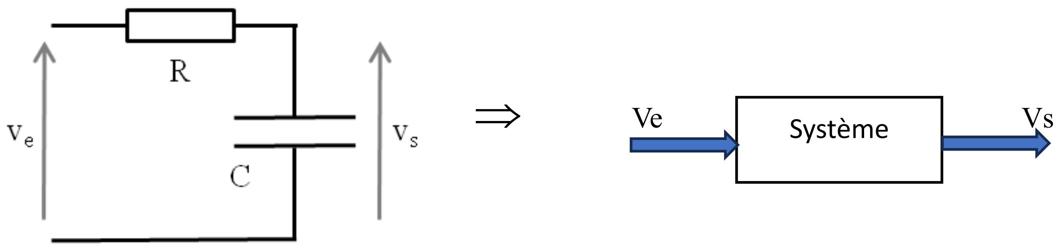


Figure 2.1: RC circuit

We always aim to determine the differential equation that directly relates the **output** to the **input**. To do this, we apply Kirchhoff's laws—specifically, Kirchhoff's Voltage Law (KVL)—to the circuit.

$$\sum V_i = 0 \Rightarrow V_e(t) - Ri(t) - V_s(t) = 0. \quad i = C\frac{dV_s(t)}{dt}$$

$$V_e = RC\frac{dV_s(t)}{dt} + V_s(t) \Rightarrow \frac{dV_s(t)}{dt} + \frac{1}{RC}V_s(t) = \frac{1}{RC}V_e(t) \tag{2.1}$$

This first-order ordinary differential equation describes the model of the system, whose variables are the input Ve and the output Vs.

Example 2.2: Consider a separately excited DC motor, where the input is the armature voltage and the output is the rotational speed.

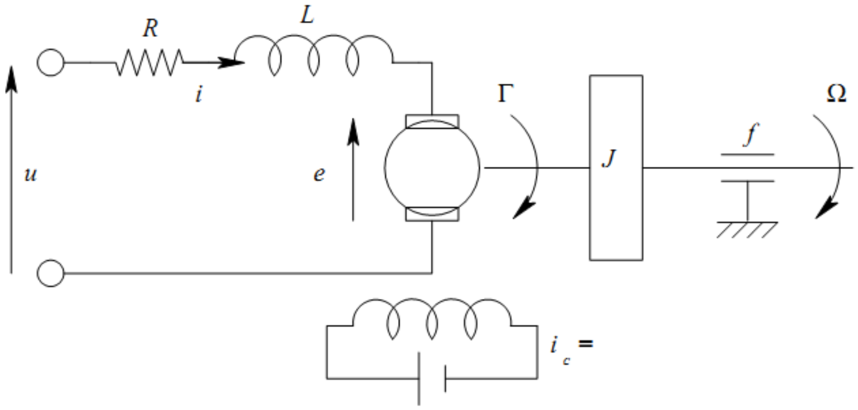


Figure 2.2: separately excited DC motor

u : power supply voltage , i : stator current , R : stator resistor , L : inductance of the stator , e : electromotive force , J : moment of inertia, f : viscous friction coefficient, et Ω : angular velocity (rotational speed), Tem ; electromechanical torque.

We aim to derive the direct differential equation that links the input to the output.

The model of the separately excited DC motor (DCM**)** is described by the following equations

Electrical Equation (Armature Circuit):

$$L\frac{di}{dt}+Ri+e=u \tag{2.2}$$

Mechanical Equation (Rotational Dynamics):

$$J\frac{d\Omega}{dt}=T_{em}-f\Omega \tag{2.3}$$

Electromechanical Equations

$$e=K\Omega, \text{ and } \quad T_{em}=Ki \tag{2.4}$$

Where k is constant

Let's now derive the differential equation that directly relates the input to the output**.**

Form equation (2.2) :

$$L\frac{di}{dt}+Ri+e=u \Rightarrow \frac{di}{dt}=\frac{1}{L}(u-Ri-e) \tag{2.5}$$

By differentiating the equation (2.3), we obtain :

$$
\begin{aligned}
J\frac{d^2\Omega}{dt^2} = \frac{dT_{em}}{dt} - f\frac{d\Omega}{dt} \Rightarrow J\frac{d^2\Omega}{dt^2} + f\frac{d\Omega}{dt} &= K\frac{di}{dt} \\
\Rightarrow J\frac{d^2\Omega}{dt^2} + f\frac{d\Omega}{dt} &= \frac{K}{L}(u - Ri - e) \\
\Rightarrow J\frac{d^2\Omega}{dt^2} + f\frac{d\Omega}{dt} &= \frac{K}{L}u - \frac{R}{L}T_{em} - \frac{K^2}{L}\Omega \qquad (2.6) \\
\Rightarrow J\frac{d^2\Omega}{dt^2} + f\frac{d\Omega}{dt} &= \frac{K}{L}u - \frac{R}{L}(J\frac{d\Omega}{dt} + f\Omega) - \frac{K^2}{L}\Omega
\end{aligned}
$$

By rearranging the terms, we obtain

$$JL\frac{d^2\Omega}{dt^2} + (fL + RJ)\frac{d\Omega}{dt} + (K^2 + Rf)\Omega = Ku \qquad (2.7)$$

This differential equation represents the model of the DC machine, governed by a second-order ordinary differential equation that directly relates the input to the output. In control theory, the solution to this type of differential equation involves a simple and effective tool, namely the Laplace transform.

2.3 Laplace Transform:

Each physical quantity is described by a time-domain signal (as a function of time). Note that in this course, only causal functions are considered.

A causal function is defined as: $f(t) = \begin{cases} f(t) & t >= 0 \\ 0 & t < 0 \end{cases}$ (2.8)

Any causal function can be subjected to a transformation called the Laplace transform, denoted as £, and is defined as follows:

$$f(t) \to F(s) = \int_0^{+\infty} f(t)e^{-st}dt\,. \qquad (2.9)$$

Where *F(s)* is called the Laplace transform of the function *f(t)* where *s* is the complex Laplace operator

2.3.1 Properties of the Laplace Transform

1. Linearity :

$$f_1(t)+kf_2(t)\to F_1(s)+kF_2(s) \qquad (2.10)$$

2. Derivation theorem

$$\begin{aligned}&\dot{f}(t)\to sF(s)-f(0)\\&\ddot{f}(t)\to s^2F(s)-sf(0)-\dot{f}(0)\\&f^{(n)}(t)\to s^nF(s)-s^{n-1}f(0)-s^{n-2}\dot{f}(0)+........+f^{(n-1)}(t)\end{aligned} \qquad (2.11)$$

3. Integration theorem

$$\int_0^{t>0} f(t)dt\to\frac{F(s)}{s} \qquad (2.12)$$

4. Theorem of Time Delay

$$f(t-\theta)\to F(s)e^{-\theta s} \qquad (2.13)$$

5. Initial Value Theorem

$$\lim_{t\to 0} f(t)=\lim_{t\to+\infty} sF(s) \qquad (2.14)$$

6. Final Value Theorem

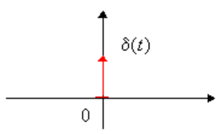
$$\lim_{t\to+\infty} f(t)=\lim_{t\to 0} sF(s) \qquad (2.15)$$

7. Frequency Shift Theorem

$$f(t)e^{\omega t}\to F(s+\omega) \qquad (2.16)$$

2.3.2 Laplace Transform of Common Signals

Let the table summarize the Laplace transform of important signals (typically test signals) in control systems.

Signal : f(t), t>=0	**Laplace Transform**
Dirac Impulsion $f(t)=\begin{cases}1 & \text{si } t=0\\ 0 & \text{si } t>0\end{cases}$	1 
Step signal $f(t)=\begin{cases}1 & \text{si } t\geq 0\\ 0 & \text{si } t<0\end{cases}$	$\frac{1}{s}$

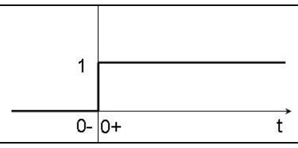
	
Ramp $f(t)=t$	$\frac{1}{s^2}$
e^{-at}	$\frac{1}{s+a}$
e^{at}	$\frac{1}{s-a}$
$te^{\pm at}$	$\frac{1}{(s\mp a)^2}$
$t^q e^{\pm at}$	$\frac{q!}{(s\mp a)^{q+1}}$
$\sin(\omega t)$	$\frac{\omega}{s^2+\omega^2}$
$\cos(\omega t)$	$\frac{s}{s^2+\omega^2}$
$e^{-at}\sin(\omega t)$	$\frac{\omega}{(s+a)^2+\omega^2}$
$e^{-at}\cos(\omega t)$	$\frac{s+a}{(s+a)^2+\omega^2}$

Table 2.1: Laplace transform table

2.4 Concept of the Transfer Function

The transfer function is the Laplace transform of the system’s differential equation that relates the input to the output. In other words, it is the ratio of the output to the input after applying the Laplace transform to the system's differential equation**.**

Example 2.3: R-C Circuit

The system's differential equation is: $\frac{dV_s(t)}{dt}+\frac{1}{RC}V_s(t)=\frac{1}{RC}V_e(t)$ (2.17)

By applying the Laplace transform to this model, we obtain:

$$sV_c(s)+\frac{1}{RC}V_c(s)=\frac{1}{RC}V_e(s)\Rightarrow V_c(s)(s+\frac{1}{RC})=\frac{1}{RC}V_e(s) \qquad (2.18)$$

Thus, the transfer function of the R-C circuit is:

$$G(s)=\frac{V_c(s)}{V_e(s)}=\frac{1/RC}{s+1/RC} \qquad (2.19)$$

Example 2.4: The Direct Current (DC) Machine

Based on the model of the DC machine given in equation (2.7), the transfer function of the system can be expressed as:

$$G(s)=\frac{\Omega(s)}{U(s)}=\frac{K}{JLs^2+(fL+RJ)s+K^2+fR} \qquad (2.20)$$

In general, consider a linear time-invariant (LTI) system described by the following differential equation:

$$a_n\frac{d^n y(t)}{dt^n}+a_{n-1}\frac{d^{n-1}y(t)}{dt^{n-1}}+...........+a_1\frac{dy(t)}{dt}+a_0y(t)=b_m\frac{d^m e(t)}{dt^m}+b_{m-1}\frac{d^{m-1}e(t)}{dt^{m-1}}+............+b_1\frac{de(t)}{dt}+b_0e(t)$$

Where the coefficients are constant. The Laplace transform of this differential equation is:

$$a_ns^nY(s)+a_{n-1}s^{n-1}Y(s)+...........+a_1sY(s)+a_0Y(s)=b_ms^mE(s)+b_{m-1}s^{m-1}E(s)+............+b_1sE(s)+b_0E(s)$$
(2.22)

Therefore, the transfer function of the system is:

$$G(s)=\frac{Y(s)}{E(s)}=\frac{b_ms^m+b_{m-1}s^{m-1}+........+b_1s+b_0}{a_ns^n+a_{n-1}s^{n-1}+.........+a_1s+a_0} \qquad (2.23)$$

Remark: *n* is the order of the system.
The transfer function can be expressed in a factored form as follows:

$$G(s)=\frac{Y(s)}{E(s)}=\frac{(s-z_1)(s-z_2)...........(s-z_{m-1})(s-z_m)}{(s-p_1)(s-p_2)............(s-p_{n-1})(s-p_n)} \qquad (2.24)$$

Where the z_i are the zeros of the system, and the p_i are the poles of the system.

2.5 Concept of Causality:

A linear time-invariant system whose transfer function is of the form of equation (2.23) is said to be causal if $n \geq m$. The transfer function is considered proper $n > m$, the transfer function is strictly proper. Note that most physical systems are strictly proper.

2.6 Inverse Laplace Transform

Starting with a transfer function *F(s)* it is possible to determine *f(t)* by evaluating the integral in the complex plane as follows:

$$F(s) \rightarrow f(t) = \frac{1}{2\pi j}\int_{c-j\infty}^{c+j\infty} F(s)e^{st}ds \qquad (2.25)$$

With *c* being a real constant. Note that, in practice, this expression is not commonly used. Typically, in control systems, *f(t)* is calculated from *F(s)* using partial fraction expansion.

2.7 Partial Fraction Expansion :

Let the transfer function *F(s)* of order *n* be of the following general form:

$$G(s) = \frac{Y(s)}{E(s)} = \frac{(s-z_1)(s-z_2)............(s-z_{m-1})(s-z_m)}{(s-p_1)(s-p_2).............(s-p_{n-1})(s-p_n)}, \text{ where } n>m$$

This function can be decomposed into partial fractions of the following form: $\frac{A_i}{(s-p_i)}$

Depending on the nature of the poles, three cases are distinguished:

2.7.1 Case of Distinct Real Poles

In this case, the function F(s) can be written as :

$$F(s) = \frac{A_1}{(s-p_1)} + \frac{A_2}{(s-p_2)} + ... + \frac{A_n}{(s-p_n)} \qquad (2.26)$$

The constants A_i are called residues . Where:

$$A_i = (s-p_i).F(s)\big|_{s=p_i} \qquad (2.27)$$

The inverse transform of the equation (2.26) is :

$$f(t) = A_1e^{p_1t} + A_2e^{p_2t} + ... + A_ne^{p_nt} \qquad (2.28)$$

Example 2.5: Let the transfer function be:

$G(s) = \frac{2}{(s+2)(s+1)} = \frac{A_1}{s+1} + \frac{A_2}{s+2}$ The poles are: $p_1 = -1, p_2 = -2$ .Therefore :

$$A_1 = (s+1)G(s)\big|_{s=-1} = 1$$
$$A_2 = (s+2)G(s)\big|_{s=-2} = -1$$

So $G(s) = \frac{1}{s+1} - \frac{1}{s+2}$

The inverse Laplace transform is written as:

$g(t) = e^{-t} + e^{-2t}$

2.7.2 Case of Complex Poles

Let the transfer function be: $F(s) = \frac{A_1}{(s-p_1)} + \frac{A_2}{(s-p_2)} + \dots\dots + \frac{A_n}{(s-p_n)}$ where the poles are complex. The residues are calculated in the same way as in the case of distinct real poles, with the difference that the residues are also complex, taking the form: $A_i = \alpha_i + j\beta_i$

Example 2.6: $G(s) = \frac{2}{(s^2+4)} = \frac{A_1}{s-2j} + \frac{A_2}{s+2j}$. The poles are $p_1 = -2j, p_2 = 2j$

$A_1 = (s-2j)G(s)\big|_{s=2j} = -\frac{1}{2}j$

$A_2 = (s+2j)G(s)\big|_{s=-2j} = \frac{1}{2}j$

so : $G(s) = -\frac{1}{2}\frac{j}{s-2j} + \frac{1}{2}\frac{j}{s+2j}$

The inverse Laplace transform is written as:

$$g(t) = -\frac{1}{2}je^{2jt} + \frac{1}{2}je^{-2jt}$$

2.7.3 Case of Double Poles:

Let the transfer function be:

$$F(s) = \frac{N(s)}{(s-p_1)^r(s-p_2)(s-p_3)\dots\dots\dots(s-p_n)} \tag{2.29}$$

where the pole p_1 is a double pole, and the other poles are either real or complex.

The partial expansion of the equation 2.29 is

$$F(s) = \frac{A_{1r}}{(s-p_1)} + \frac{A_{2r}}{(s-p_1)^2} + \dots\dots \frac{A_r}{(s-p_1)^r} + \sum_{i=2}^{n}\frac{A_i}{(s-p_i)} \tag{2.30}$$

The residues are calculated as follows:

$A_i = (s-p_i).F(s)\big|_{s=p_i}$

(2.31)

$$A_r = (s-p_1)^r.F(s)\big|_{s=p_1} \tag{2.32}$$

$$A_{r-1} = \frac{\partial}{\partial s}\left[(s-p_1)^r .F(s)\Big|_{s=p_1} \right] \tag{2.33}$$

$$A_{r-2} = \frac{1}{2!}\frac{\partial^2}{\partial s^2}\left[(s-p_1)^r .F(s)\Big|_{s=p_1} \right] \tag{2.34}$$

In general,

$$A_{r-i} = \frac{1}{i!}\frac{\partial^i}{\partial s^i}\left[(s-p_1)^r .F(s)\Big|_{s=p_1} \right]$$
(2.35)

The inverse Laplace transform is written as follows:

$$f(t) = A_{1r}e^{p_1 t} + A_{2r}te^{p_1 t} + ...+ A_r \frac{t^{r-1}}{(r-1)!}e^{p_1 t} + \sum_{i=2}^{n} A_i e^{p_i t} \tag{2.36}$$

Example 2.7:
Let the transfer function be:

$G(s) = \dfrac{s}{(s+1)^2} = \dfrac{A_1}{s+1} + \dfrac{A_2}{(s+1)^2}$ **,** Therefore :

$$A_2 = (s+1)^2 G(s)\Big|_{s=-1} = -1 \tag{2.37}$$

$$A_1 = \frac{\partial}{\partial s}\left[(s+1)^2 G(s)\Big|_{s=-1} \right] = 1$$

So , *g(t)* is :

$$g(t) = e^{-t} - te^{-t} \tag{2.38}$$

2.8 Solving a Problem Using the Transfer Function

Consider a system whose model is a linear time-invariant differential equation. The determination of the output should follow these steps:

1. Calculate the transfer function from the differential equation. $G(s) = \dfrac{Y(s)}{E(s)}$
2. Determine *E(s)* from the Laplace transform table.
3. Decompose $Y(s) = G(s)E(S)$ into partial fractions.
4. Deduce y(t) from the Laplace transform table.

2.9 Exercises

Exercise 2.1: Direct Calculation of the Laplace Transform

1. What is the condition for functions (or signals) $f(t)$ to be causal?

2. Calculate the Laplace transform of the following functions:

$$f(t)=Au(t),\ f(t)=bt,\ f(t)=\sin(\omega t),\ f(t)=\cos(\omega t),\ f(t)=e^{-at},\ f(t)=e^{-at}\sin(\omega t)$$

Exercise 2.2: Calculation of the Laplace Transform of a Real Pulse

We consider a pulse of width T and height A (see Figure 2.3).

- s(t) = A for $0 \leq t \leq T$ and
- s(t) = 0 otherwise.

1. Is this signal s(t) causal? Justify your answer.

2. Calculate the Laplace transform of this signal.

Figure 2.3: Real Pulse.

Exercise 2.3: Calculation of a Simple Transfer Function

We consider a system governed by the following differential equation:

$$\frac{d^3y}{dt^3}+3\frac{d^2y}{dt^2}+3\frac{dy}{dt}+y(t)=2\frac{de}{dt}+e(t)$$

1. Is this system linear? Justify your answer.

2. What are the inputs and outputs of this system?

3. What is the order of this system?

4. Represent this system as a simple input-output block diagram.

5. Determine the initial conditions necessary to calculate the transfer function of the system.

6. Calculate the transfer function of this system and determine its poles and zeros.

Exercise 2.4: Equation of an Electrical System

We consider the electrical circuit shown in Figure 2. An input signal u(t) corresponding to a step voltage from 0 to 5 V is applied to the system.

1. Is the system causal?

2. Determine the differential equation linking the input u(t) to the output voltage v(t).

3. Derive the system's transfer function and its poles and zeros based on the system's parameters.

4. Do the system's parameters affect the poles and zeros? (Can the poles and zeros of the system be modified by changing the circuit parameters?)

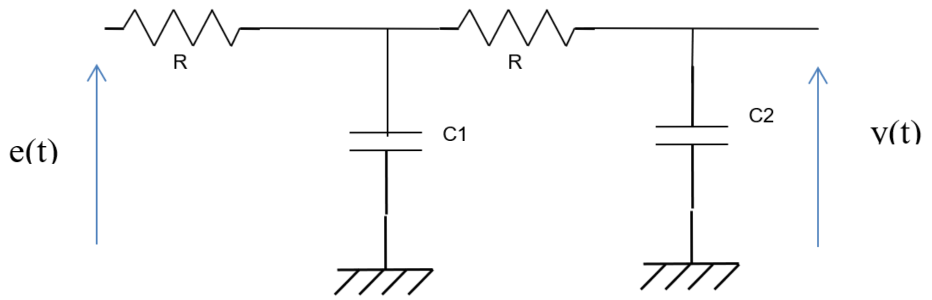


Figure 2.4: The Electrical Circuit.

Exercise 2.5: Direct Calculation of the Inverse Laplace Transform

- What condition must the transfer function satisfy for partial expansion to be possible?
- Compute the inverse Laplace transform of the following functions.

$$G(s)=\frac{s^2}{s^2+4}\quad , \; G(s)=\frac{5}{(s^2+2)(s^2-4s+3)}, \; G(s)=\frac{s+1}{(s-3)^3(s^2+4s+3)(s+2)}$$

Exercise 2.6

Consider the electrical circuit shown in Figure 2.5, with R=1KΩ, L= 10mH, C=6 μ F. An input signal *v(t)*. The input signal *v(t)* is a step voltage ranging from 0 to 5V

- Derive the differential equation that relates the input voltage v(t) to the output voltage vc(t).
- Determine the system's transfer function
- Compute the output response vc(t).

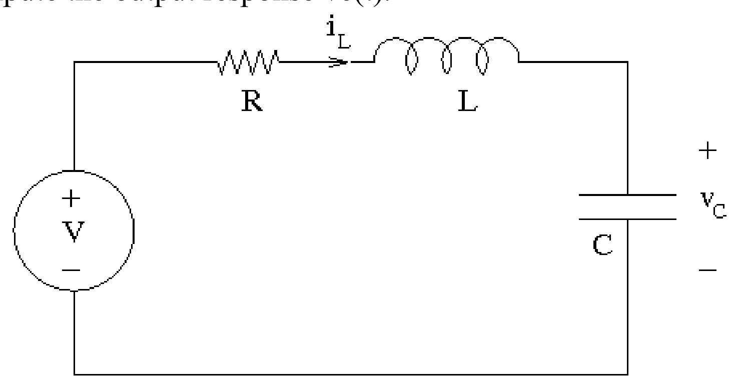


Figure 2.5: R-L-C circuit

Exercise 2.8 :

Consider the electrical circuit shown in Figure 2, where R=10 Ω , L= 1mH.

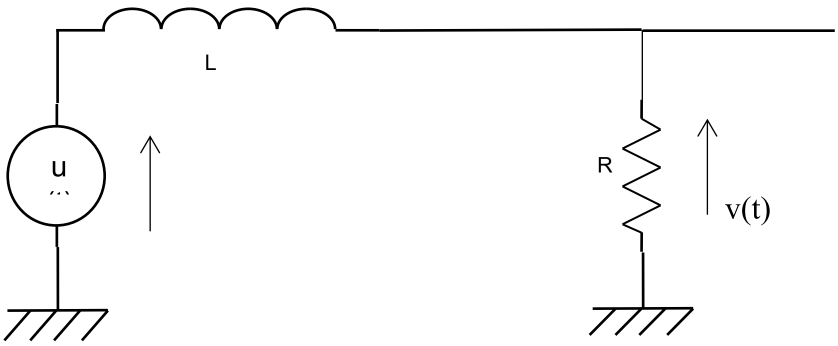


Figure.2.6: L-R Circuit

- Determine the system's transfer function.
- Calculate the output voltage v(t) when the input signal is *u(t)=sin(10t)*

2.10 Exercises solutions

Exercise 2.1: Direct Calculation of the Laplace Transform

1. The condition for functions (or signals) $f(t)$ to be causal is $f(t)=\begin{cases} f(t) & t\geq 0 \\ 0 & t<0 \end{cases}$

2. The Laplace transform of the following functions:

a. Step function $f(t)=Au(t)$

$$F(s)=\int_0^{+\infty} f(t)e^{-st}dt \Rightarrow F(s)=\int_0^{+\infty} Au(t)e^{-st}dt$$

$$\Rightarrow F(s)=A\int_0^{+\infty} 1.e^{-st}dt \tag{2.39}$$

$$\Rightarrow F(s)=A\left[-\frac{1}{s}e^{-st}\right]_0^{+\infty}=\frac{A}{s}$$

b. The ramp function : $f(t)=bt$

$$F(s)=\int_0^{+\infty} f(t)e^{-st}dt \Rightarrow F(s)=\int_0^{+\infty} bte^{-st}dt \tag{2.40}$$

Applying integration by part :

$u = bt \qquad u' = b$

$v = -\frac{1}{s}e^{-st} \qquad v' = e^{-st}$

$$F(s) = \int_0^{+\infty} bte^{-st}dt \Rightarrow F(s) = -b\frac{t}{s}e^{-st} - \int_0^{+\infty} -\frac{b}{s}e^{-st}dt$$

$$\Rightarrow F(s) = -b\frac{t}{s}e^{-st}\Big|_0^{+\infty} + \left[\frac{b}{s^2}e^{-st}\right]_0^{+\infty} \qquad (2.41)$$

$$\Rightarrow F(s) = 0 + \frac{1}{s^2} = \frac{1}{s^2}$$

c. The exponential function $f(t) = e^{-at}$

$$F(s) = \int_0^{+\infty} f(t)e^{-st}dt \Rightarrow F(s) = \int_0^{+\infty} e^{-at}e^{-st}dt$$

$$\Rightarrow F(s) = \int_0^{+\infty} e^{-(a+s)t}dt \qquad (2.42)$$

$$\Rightarrow F(s) = \left[-\frac{1}{(s+a)}e^{-(a+s)t}\right]_0^{+\infty} = \frac{1}{s+a}$$

d. The sine and the cosine functions $f(t) = \sin(\omega t),\ f(t) = \cos(\omega t)$

In this case, we will use the Euler formula for sine and cosine such as :

$$\sin(\omega t) = \frac{e^{j\omega t} - e^{-j\omega t}}{2j}$$

$$\cos(\omega t) = \frac{e^{j\omega t} + e^{-j\omega t}}{2} \qquad (2.43)$$

Therefore

$$F(s) = \int_0^{+\infty} f(t)e^{-st}dt \Rightarrow F(s) = \int_0^{+\infty} \sin(\omega t)e^{-st}dt \Rightarrow F(s) = \int_0^{+\infty}\left(\frac{e^{j\omega t} - e^{-j\omega t}}{2j}\right)e^{-st}dt$$

$$\Rightarrow F(s) = \frac{1}{2j}\int_0^{+\infty}(e^{-(s-j\omega)t} - e^{-(s+j\omega)t})dt$$

$$\Rightarrow F(s) = \frac{1}{2j}\left[-(\frac{1}{(s-j\omega)}e^{-(s-j\omega)t}) - (-\frac{1}{(s+j\omega)}e^{-(s+j\omega)t})\right]_0^{+\infty} \qquad (2.44)$$

$$\Rightarrow F(s) = \frac{1}{2j}\left[(\frac{1}{(s-j\omega)}) - (\frac{1}{(s+j\omega)})\right] = \frac{\omega}{s^2+\omega^2}$$

$$F(s)=\int_0^{+\infty} f(t)e^{-st}dt \Rightarrow F(s)=\int_0^{+\infty}\cos(\omega t)e^{-st}dt \Rightarrow F(s)=\int_0^{+\infty}\left(\frac{e^{j\omega t}+e^{-j\omega t}}{2}\right)e^{-st}dt$$

$$\Rightarrow\ F(s)=\frac{1}{2}\int_0^{+\infty}(e^{-(s-j\omega)t}+e^{-(s+j\omega)t})dt$$

$$\Rightarrow\ F(s)=\frac{1}{2}\left[-(\frac{1}{(s-j\omega)}e^{-(s-j\omega)t})+(-\frac{1}{(s+j\omega)}e^{-(s+j\omega)t})\right]_0^{+\infty} \qquad (2.45)$$

$$\Rightarrow F(s)=\frac{1}{2}\left[(\frac{1}{(s-j\omega)})+(\frac{1}{(s+j\omega)})\right]=\frac{s}{s^2+\omega^2}$$

e. The function $f(t)=e^{-at}\sin(\omega t)$

$$F(s)=\int_0^{+\infty} f(t)e^{-st}dt \Rightarrow F(s)=\int_0^{+\infty}\sin(\omega t)e^{-a}e^{-st}dt$$

$$\Rightarrow\ F(s)=\int_0^{+\infty}\left(\frac{e^{j\omega t}-e^{-j\omega t}}{2j}\right)e^{-(s+a)t}dt \qquad (2.46)$$

$$\Rightarrow\ F(s)=\frac{1}{2j}\int_0^{+\infty}(e^{-(s+a-j\omega)t}-e^{-(s+a+j\omega)t})dt$$

$$\Rightarrow\ F(s)=\frac{1}{2j}\left[-(\frac{1}{(s-j\omega)}e^{-(s+a-j\omega)t})-(-\frac{1}{(s+j\omega)}e^{-(s+a+j\omega)t})\right]_0^{+\infty}$$

$$\Rightarrow F(s)=\frac{1}{2j}\left[(\frac{1}{(s+a-j\omega)})-(\frac{1}{(s+a+j\omega)})\right]=\frac{\omega}{(s+a)^2+\omega^2}$$

Exercise 2.2

1. The signal s(t) is causal. It can verify the causality properties such as :

$$s(t)=\begin{cases} s(t)>0 & \text{when } t\geq 0 \\ 0 & \text{when } t<0 \end{cases}$$

2. The Laplace transform of this signal. The signal of Figure 2.3 can be decomposed as follows:

$s(t)=s_1(t)-s_2(t)$ (See the figure 2.7)

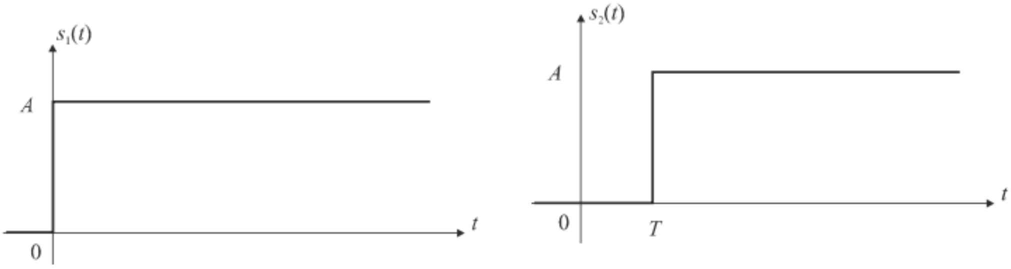


Figure 2.7: decomposition of *s(t)*

Using the time delay theorem $f(t-\theta) \to F(s)e^{-\theta s}$, we can obtain **:**

$$S(s) = S_1(s) - S_2(s) = \frac{A}{s} - \frac{A}{s}e^{-Ts} = \frac{A}{s}(1-e^{-Ts}) \tag{2.47}$$

Exercise 2.3: Calculation of a Simple Transfer Function

We consider a system governed by the following differential equation:

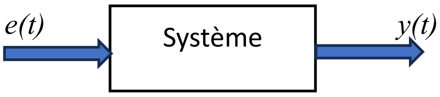
$$\frac{d^3y}{dt^3} + 3\frac{d^2y}{dt^2} + 3\frac{dy}{dt} + y(t) = 2\frac{de}{dt} + e(t) \tag{2.48}$$

1. The system is linear because the coefficients are constants and the linearity properties can be verified (superposition and homogeneity)

2. The input is *e(t)* and the output is *y(t)*

3. The order of this system is : 3

4. Representation of the system as a simple input-output block diagram.



5. The initial conditions necessary to calculate the transfer function of the system are :
   $y(0) = 0, y^{'}(0) = 0, y^{"}(0) = 0$
   $e(0) = 0$
6. The transfer function of the system:
   $$G(s) = \frac{2s+1}{s^3+3s^2+3s+1} \tag{2.49}$$
   The poles are :
   $p_1 = p_2 = p_3 = -1$
   The zero ;
   $z_1 = -\frac{1}{2}$

Exercise 2.4: Equation of an Electrical System

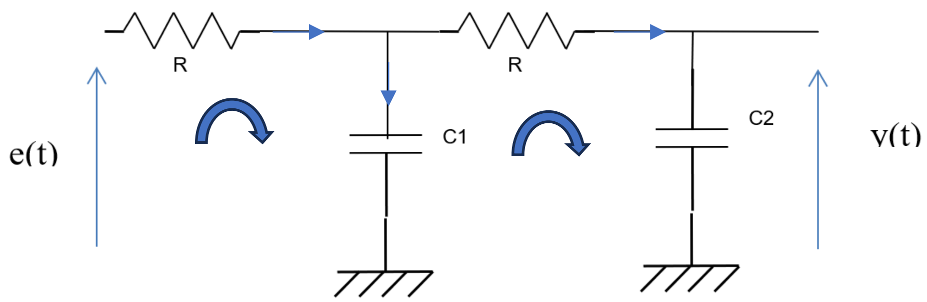


Figure 2.8**:** The Electrical Circuit

1. The system is causal: all real systems are causal

2. The differential equation linking the input u(t) to the output voltage v(t): applying the Kiricchoff's law , we obtain :

$$\begin{cases} i = i_1 + i_2 \\ e(t) = Ri + v_{c_1} \\ v_{c_1} = Ri_2 + v_{c_2} \\ i_1 = c_1 \dfrac{dv_{c_1}}{dt}, i_2 = c_2 \dfrac{dv_{c_2}}{dt} \end{cases} \tag{2.50}$$

Manipulating these equations, it can be written :

$$\begin{cases} i = c_1 \dfrac{dv_{c_1}}{dt} + c_2 \dfrac{dv_{c_2}}{dt} \\ e(t) = Ri + Ri_2 + v_{c_2} = R(c_1 \dfrac{dv_{c_1}}{dt} + c_2 \dfrac{dv_{c_2}}{dt}) + Rc_2 \dfrac{dv_{c_2}}{dt} + v_{c_2} \\ e(t) = 2Rc_2 \dfrac{dv_{c_2}}{dt} + v_{c_2} + Rc_1 \dfrac{dv_{c_1}}{dt} \end{cases} \tag{2.51}$$

Therefore the differential equation is :

$$R^2 c_1 c_2 \frac{d^2 v_{c_2}}{dt^2} + R(c_1 + 2c_2) \frac{dv_{c_2}}{dt} + v_{c_2} = e(t) \tag{2.52}$$

3. The transfer function :

   $$G(s) = \frac{Vc_2(s)}{E(s)} = \frac{1}{R^2 c_1 c_2 s^2 + R(c_1 + 2c_2)s + 1}$$

   - The poles are :

   $$p_1 = \frac{-(c_1 + c_2) - \sqrt{c_1^{\,2} + 4c_2^{\,2}}}{2Rc_1c_2} \text{ and } p_2 = \frac{-(c_1 + c_2) + \sqrt{c_1^{\,2} + 4c_2^{\,2}}}{2Rc_1c_2}$$

4. Yes, the system's parameters affect the poles and zeros

Exercise 2.5:

1. Let the transfer function be $G(s) = \dfrac{Y(s)}{E(s)} = \dfrac{b_m s^m + b_{m-1}s^{m-1} + \dots\dots + b_1 s + b_0}{a_n s^n + a_{n-1}s^{n-1} + \dots\dots + a_1 s + a_0}$, so the condition that must satisfy G(s) to be decomposed is $n \geq m$:

If $n > m$, $G(s) = \sum_{i=1}^{n} \dfrac{A_i}{(s - p_i)}$, where p_i are poles.

If $n = m$, $G(s) = \dfrac{b_m}{a_n} + \sum_{i=1}^{n} \dfrac{A_i}{(s - p_i)}$

2. The inverse Laplace transform of :

a. $$G(s)=\frac{s^2}{s^2+4} \quad (2.53)$$

L'equation (2.53) can be written as :

$$G(s)=1-\frac{4}{s^2+4}=1-2\frac{2}{s^2+(2)^2} \quad (2.54)$$

From Laplace table transform , the inverse laplace transform of *G(s):*

$$g(t)=\delta(t)-2\sin(2t) \quad (2.55)$$

b. $$G(s)=\frac{5}{(s^2+2)(s^2-4s+3)} \quad (2.56)$$

The poles of the transfer function of equation (2.56):

$$p_1=j\sqrt{2},\ p_2=-j\sqrt{2},\ p_3=1,\ p_4=3$$

L'equation (2.56) can be written as :

$$G(s)=\frac{A_1}{(s-j\sqrt{2})}+\frac{A_2}{(s+j\sqrt{2})}+\frac{A_3}{(s-1)}+\frac{A_4}{(s-3)} \quad (2.57)$$

Where

$$A_1=(s-j\sqrt{2})G(s)\Big|_{s=j\sqrt{2}}=0.3030 - 0.0536j$$

$$A_2=(s+j\sqrt{2})G(s)\Big|_{s=-j\sqrt{2}}=0.3030 + 0.0536j$$

$$A_3=(s-1)G(s)\Big|_{s=1}=\frac{-5}{6}$$

$$A_4=(s-3)G(s)\Big|_{s=3}=\frac{5}{22}$$

(2.58)

The inverse laplace transform of *G(s):*

$$g(t)=A_1e^{j\sqrt{2}t}+A_2e^{-j\sqrt{2}t}+A_3e^{t}+A_4e^{3t} \quad (2.59)$$

c. $$G(s)=\frac{s+1}{(s-3)^3(s^2+4s+3)(s+2)} \quad (2.60)$$

The poles of the transfer function of equation (2.60):

$$p_1=p_2=p_3=3,\ p_4=-1,\ p_5=-3,\ p_6=-2$$

So *G(s)* can be written as :

$$G(s)=\frac{1}{(s-3)^3(s+3)(s+2)} \quad (2.61)$$

L'equation (2.61) can be written as :

$$G(s)=\frac{A_1}{(s-3)}+\frac{A_2}{(s-3)^2}+\frac{A_3}{(s-3)^3}+\frac{A_4}{(s+3)}+\frac{A_5}{(s+2)} \tag{2.62}$$

Where

$$A_5=(s+2)G(s)\big|_{s=-2}=-\frac{1}{125}$$
$$A_4=(s+3)G(s)\big|_{s=-3}=\frac{1}{216}$$
$$A_3=(s-3)^3G(s)\big|_{s=3}=\frac{1}{30}$$
$$A_2=\frac{d}{ds}\left[(s-3)^3G(s)\right]\Big|_{s=3}=0.0122$$
$$A_1=\frac{1}{2}\frac{d^2}{ds^2}\left[(s-3)^3G(s)\right]\Big|_{s=3}=3,7.10^{-4}$$

(2.63)

The inverse Laplace transform of *G(s):*

$$g(t)=A_1e^{3t}+A_2te^{3t}+A_3\frac{t^2}{2}e^{3t}+A_4e^{-2t}+A_5e^{-3t} \tag{2.64}$$

Exercise 2.6

a. the differential equation that relates the input voltage v(t) to the output voltage vc(t):

Applying the Kiricchoff's law, we obtain :

$$\begin{cases} v(t)=Ri_L(t)+L\dfrac{di_L(t)}{dt}+v_c(t) \\ i_L=C\dfrac{dv_c(t)}{dt} \end{cases} \tag{2.65}$$

Therefore, we can drive the differential equation:

$$LC\frac{d^2v_c(t)}{dt}+RC\frac{dv_c(t)}{dt}+v_c(t)=v(t) \tag{2.66}$$

b. The transfer function of the system :

$$G(s)=\frac{1}{LCs^2+RCs+1}=\frac{1/LC}{s^2+(R/L)s+1/LC} \tag{2.67}$$

c. The output *vc(t)*

$$G(s)=\frac{V_c(s)}{V(s)}\Rightarrow V_c(s)=G(s)V(s) \tag{2.68}$$

Where $V(s)=\dfrac{5}{s}$

Thus :

$$V_c(s)=\frac{5}{s}\frac{1/LC}{s^2+(R/L)s+1/LC} \tag{2.69}$$

Also, the output can be written as :

$$V_c(s)=\frac{5}{s}\frac{1/LC}{(s-p_1)(s-p_2)} \tag{2.70}$$

Where $p_1=\text{-}9.9833.10^4,\ p_2=\text{-}0.0167.10^4$

The partial fraction of the output is

$$V_c(s)=\frac{A_1}{s}+\frac{A_2}{s-p_1}+\frac{A_3}{s-p_2} \tag{2.71}$$

Where

$$\begin{aligned}
A_1 &= sV_c(s)\big|_{s=0}=5\\
A_2 &= (s-p_1)G(s)\big|_{s=p_1}=0.0084\\
A_3 &= (s-p_2)G(s)\big|_{s=p_2}=\text{-}5.0067
\end{aligned} \tag{2.72}$$

The inverse Laplace transform of the output is

$$v_c(t)=5+A_1e^{p_1t}+A_2e^{p_2t} \tag{2.73}$$

Exercise 2.8 :

1. The transfer function: from the circuit of Figure 2.6, we can write :

$$\begin{aligned}
u(t) &= L\frac{di(t)}{dt}+v(t)\\
i(t) &= \frac{v(t)}{R}
\end{aligned} \tag{2.73}$$

The differential equation of the system is :

$$u(t)=\frac{L}{R}\frac{dv(t)}{dt}+v(t) \tag{2.73}$$

So, the transfer function of the system will be :

$$G(s)=\frac{V(s)}{U(s)}=\frac{1}{\frac{L}{R}s+1}=\frac{1}{10^{-3}s+1}=\frac{10^3}{s+10^3} \tag{2.74}$$

2. The output voltage v(t) when the input signal is *u(t)=sin(10t);*

We know that: $V(s) = U(s)G(s)$, where in this case $U(s) = \dfrac{10}{s^2+100}$,

Thus :

$$V(s) = \frac{10}{s^2+100}\frac{10^3}{s+10^3} = \frac{10^4}{(s^2+100)(s+10^3)} \tag{2.75}$$

The poles of the transfer function of equation (2.75):

$p_1 = -10^3,\ p_2 = -j10,\ p_3 = j10$

The voltage output can be written as :

$$V(s) = \frac{A_1}{(s+10^3)} + \frac{A_2}{(s+j10)} + \frac{A_3}{(s-j10)} \tag{2.76}$$

Where :

$$\begin{aligned} A_1 &= (s+10^3)V(s)\big|_{s=-10^3} = 0.01 \\ A_2 &= (s+j10)V(s)\big|_{s=-j10} = 0.005+0.5\text{j} \\ A_3 &= (s-j10)V(s)\big|_{s=j10} = 0.005-0.5\text{j} \end{aligned} \tag{2.77}$$

The inverse Laplace transform of the output voltage :

$$v(t) = 0.01e^{-10^3 t} + (0.005+j0.5)e^{-j10t} + (0.005-j0.5)e^{j10t} \tag{2.78}$$

The equation (2.78) can transformed as follows:

$$v(t) = 0.01e^{-10^3 t} + 2\text{x}0.005\left(\frac{e^{j10t}+e^{-j10t}}{2}\right) - 2j\text{x}j0.5\left(\frac{e^{j10t}+e^{-j10t}}{2j}\right) \tag{2.79}$$

Using the Euler's formula (equation 2.43), the output voltage can be :

$$v(t) = 0.01e^{-10^3 t} + 0.01\cos(10t) + \sin(10t) \tag{2.80}$$

Remark: For a linear system, when the input is a sinusoidal signal, the steady-state output is also sinusoidal with the same frequency, but possibly with a different amplitude and phase.

- **Chapter 3:** Block Diagram Representation

3.1 Introduction:

In control systems engineering, block diagram representation is a fundamental tool used to model and visualize the functional relationships between different components of a system. It provides a graphical means of representing the flow of signals and the interconnection of dynamic elements, where each block denotes a mathematical operation or a system element (e.g., transfer function, gain, summation, or integrator).

3.2 Block Diagram Representation of a Transfer Function

A transfer function describes the input-output relationship of a linear time-invariant (LTI) system in the Laplace domain. To represent this relationship graphically, we use a block diagram:

$$G(s) = \frac{Y(s)}{E(s)} \Rightarrow Y(s) = G(s).E(s) \Rightarrow$$

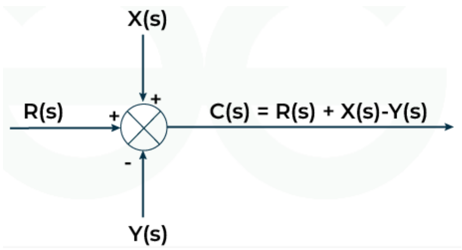
3.2.1 Formalism

a. branch: can be represented by *E(s)* $\longrightarrow$ *:* The branches represent the variables

b. bloc: 

a block represents a transfer function

c. Summing point:



A summing point **is** a key element used to add or subtract multiple signals. It is typically represented by a small circle with one or more input arrows and a single output arrow.

3.2.2 Block diagram reduction rules:

To form a control system, the blocks are interconnected according to the system's structure and signal flow. The process of simplifying such a system to obtain an overall transfer function is known as the Block Diagram Reduction Method. The block diagram reduction method is a powerful technique used to determine the transfer function of a complex control system. It simplifies intricate interconnections into an equivalent, simpler representation, allowing for easier analysis of system stability and performance characteristics.

a. Bloc in cascade:

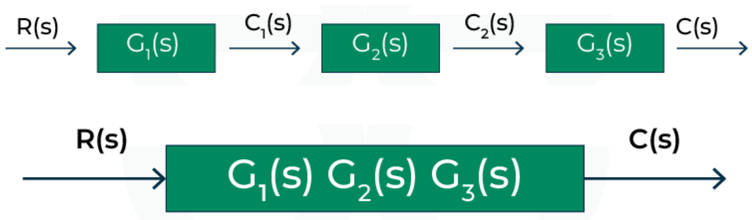


Figure 3.1: blocs in series

b. Blocks in parallel:

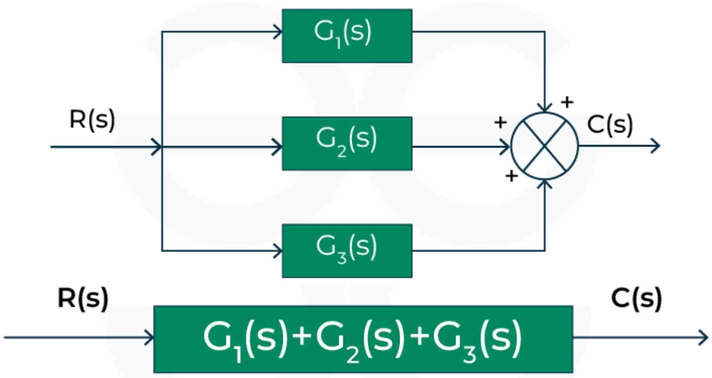


Figure 3.2: blocs in paralle

c. Shifting of take-off point

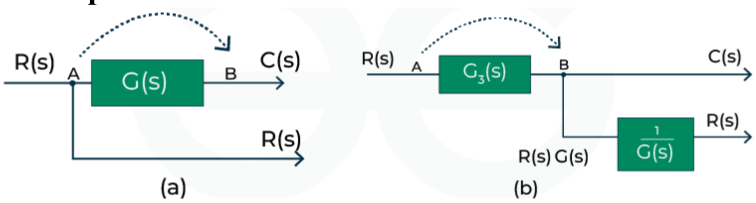


Figure 3.3: shifting of take-off from A to B

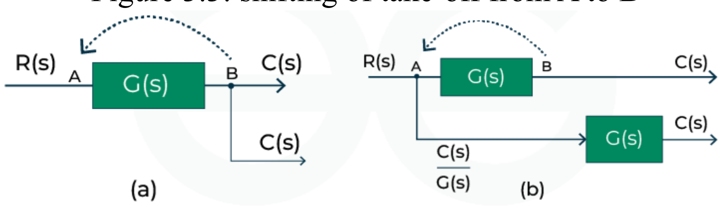


Figure 3.4: shifting of take-off from B to A

d. Shifting of summing point

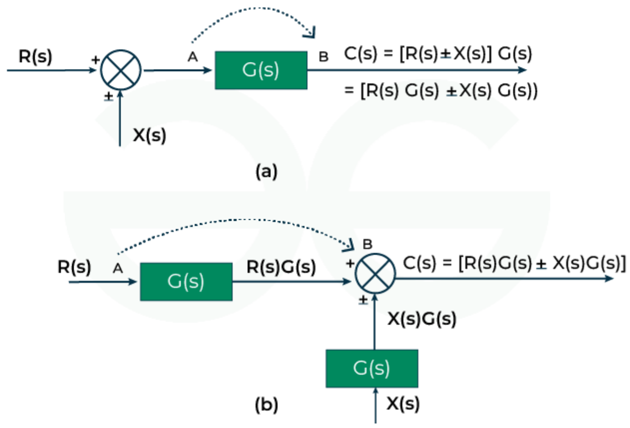


Figure 3.5: shifting of summing point from A to B

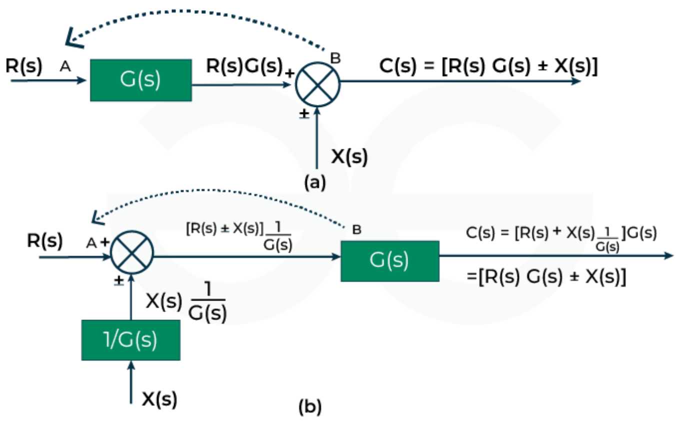


Figure 3.6: shifting of summing point from B to A

e. Canonical Form of a Closed-Loop System

The canonical (standard) form of such a structure is a fundamental configuration used for analysis and design as shown in Figure 3.7

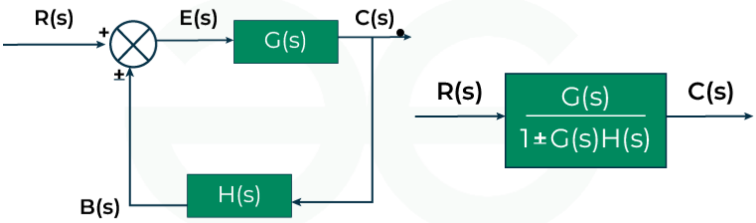


Figure 3.7: the canonical form of a closed-loop system

G(s) is the transfer function of the forward path (also called the direct path),

H(s) is the transfer function of the feedback path (also called the return path or feedback loop).

According to the canonical block diagram, we can write:

$$
\begin{aligned}
&Y(s)=G(s).E(s) &&\Rightarrow \quad Y(s)=G(s)(R(s)-B(s))\\
&B(s)=H(s).Y(s) &&\Rightarrow \quad Y(s)=G(s)(R(s)-H(s)Y(s))\\
&E(s)=R(s)-B(s) &&\Rightarrow \quad Y(s)=G(s)R(s)-G(s)H(s)Y(s)\\
&(1+G(s)H(s))Y(s)=G(s)R(s) \Rightarrow \frac{Y(s)}{R(s)}=\frac{G(s)}{1+G(s)H(s)}
\end{aligned}
\tag{3.1}
$$

Example 3.1: Reduce the following block diagram using the block diagram reduction rule

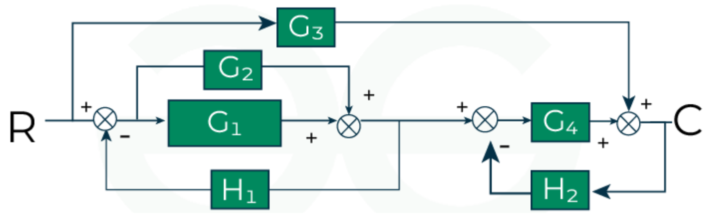


Figure 3.8: block diagram to be simplified

Step 1: G1 and G2 are connected in parallel, find its equivalent block G1 + G2

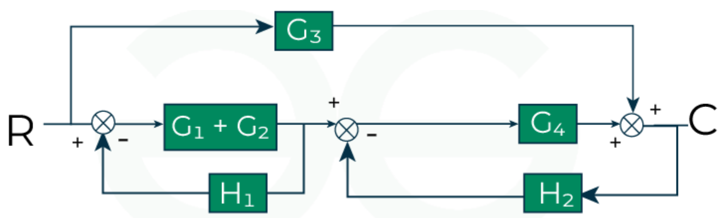


Figure 3.9: Step1

Step 2: Elimination of summing point before G1 + G2 having negative feedback H1 in the closed loop system.

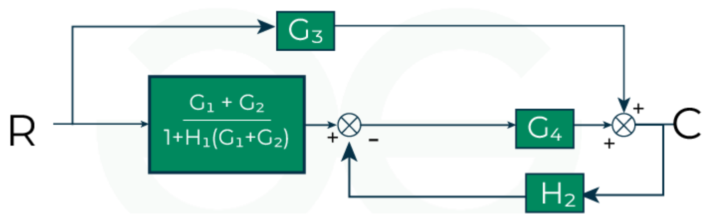


Figure 3.10: Step 2

Step 3: Move the summing point after G4 to before G4

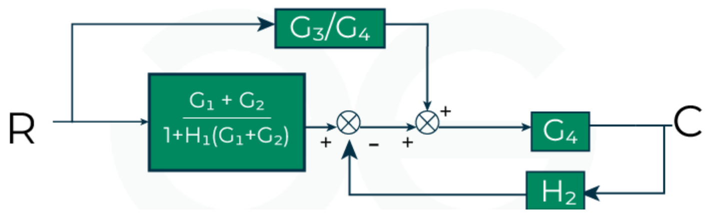


Figure 3.11: Step 3

Step 4: Exchange the position of both the summing point before G4

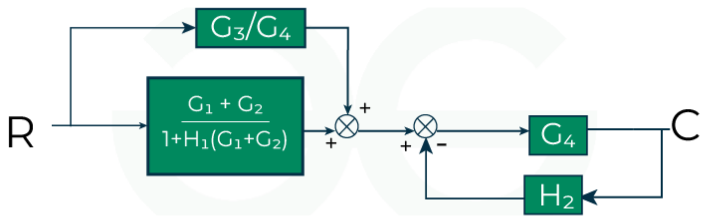


Figure 3.12: Step 4

Step 5 : G3 /G4 and (G1 +G2) / (1+G1H1+G2H2) both blocks are connected in parallel

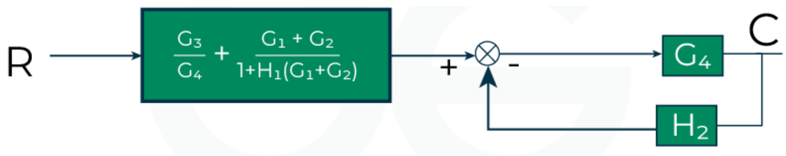


Figure 3.13: Step 5

Step 6: Eliminate the summing point before G4 has negative feedback in the closed-loop system

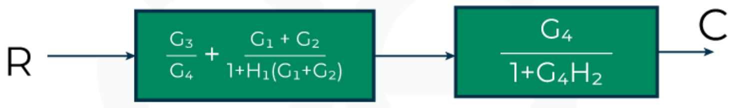


Figure 3.14: Step 6

Step 7: Single block diagram representation as shown below.

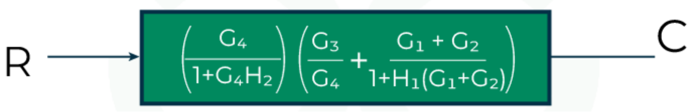


Figure 3.15: Step 7

3.3 Case of Multiple Input Systems

In the case of systems with multiple inputs and one output, block diagrams can become more complex, involving several inputs. However, the block diagram reduction method can still be applied by handling each input-output relationship individually and systematically reducing the interconnections by applying the superposition rule.

Example 3.2: Reduce the following block diagram using the block diagram reduction rule

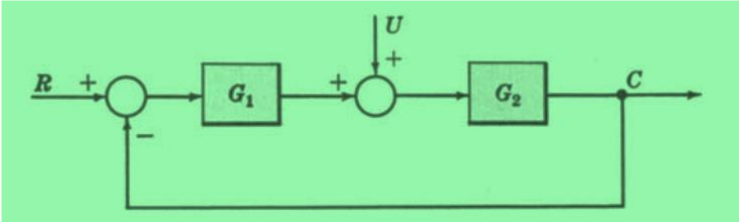


Figure 3.16: block diagram with multiple inputs

By applying the superposition principle, we should follow the following steps :

a. Step1 : put $U=0$
b. Step 2: The system reduces to

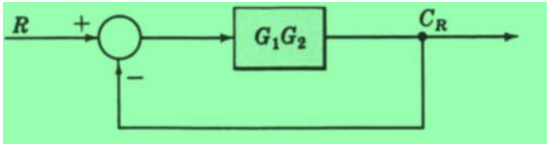


Figure 3.17: step2

c. Step 3: By Equation (3.1), the output C_R, due to input R is

$$C_R = \frac{G_1G_2}{1+G_1G_2}R \tag{3.2}$$

d. Step 4: put R=0, Put -1into a block, representing the negative feedback effect

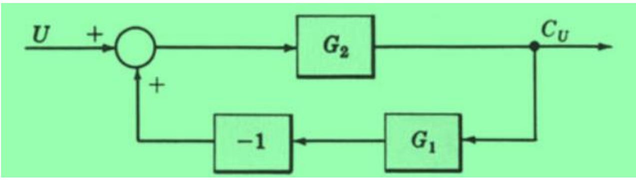


Figure 3.18: step3

e. Step 5: By Equation (3.1), the output C_v, due to input U is **:**

$$C_v = \frac{G_2}{1-G_1G_2}U \tag{3.3}$$

f. The final output is

$$C = C_R + C_v = \frac{G_1G_2}{1+G_1G_2}R + \frac{G_2}{1-G_1G_2}U \tag{3.4}$$

3.4 Exercises

Exercise 3.1: Block Diagram Representation of a DC Motor

Consider the separately excited DC motor shown in Figure 3.19, where the input is the supply voltage u and the output is the angular velocity Ω .w

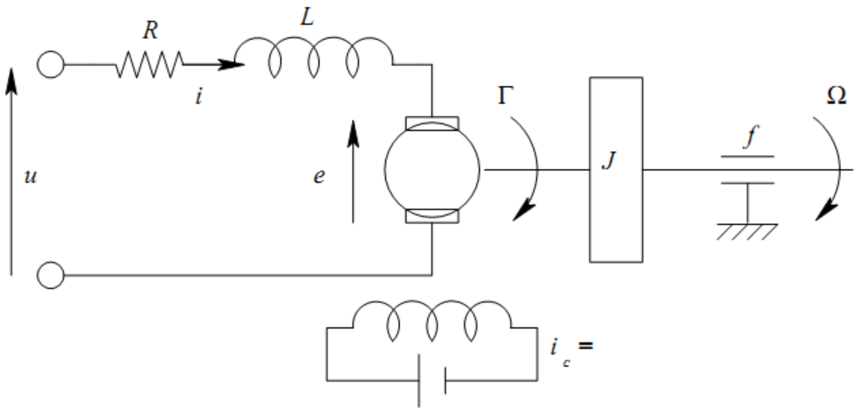


Figure 3.19: DC machine

- Represent this system using a block diagram.
- Determine the transfer function of the system.

Exercise 3.2: Block Diagram Simplification

Consider the systems represented by the block diagrams in Figure 3.20

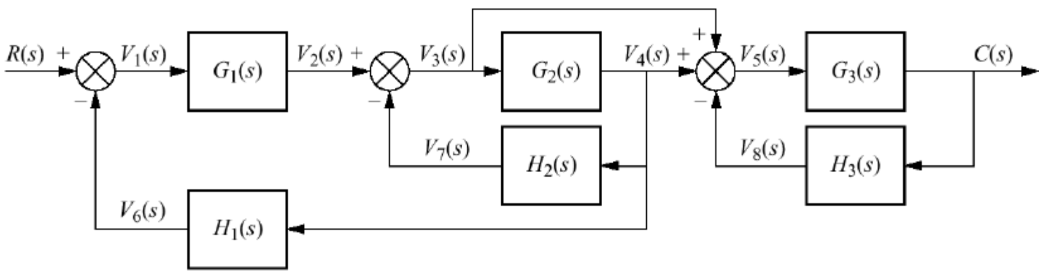


Figure 3.20: Block diagram

- Using block diagram transformation rules, compute the transfer function of each system.

Exercise 3.3: Multi-Input Systems

Consider the system represented by the block diagram in Figure 3.21

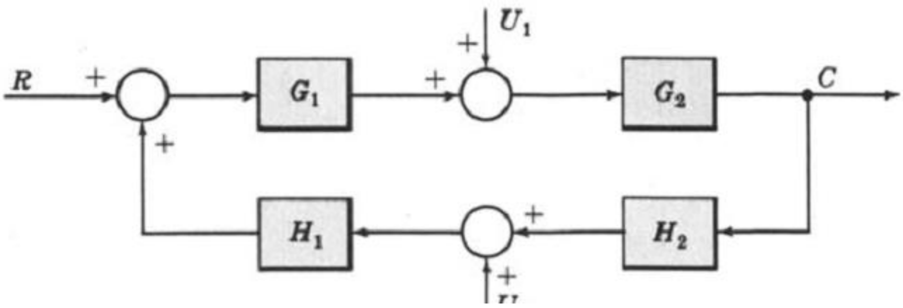


Figure 3.21: Block diagram of multiple inputs

1) Determine the output C as a function of the inputs *R, U1, U2.*

3.5 Exercises solution

Exercise 3.1: Block Diagram Representation of a DC Motor

Let us consider the DC motor model developed in Chapter 2. For each governing equation, we will construct the corresponding block diagram:

Electrical Equation

$$L\frac{di}{dt} + Ri + e = u \tag{3.5}$$

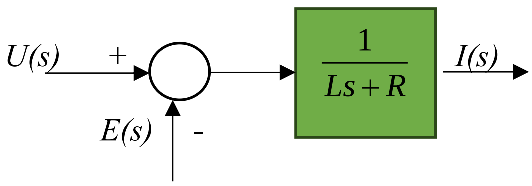
The Laplace transform of the equation (3.5) is:

$$LsI(s) + RI(s) + E(s) = U(s) \tag{3.6}$$

Thus:

$$I(s) = \frac{1}{(Ls+R)}(U(s) - E(s)) \tag{3.7}$$

The block diagram of the equation (3.7):



Mechanical Equation

$$J\frac{d\Omega}{dt} = T_{em} - f\Omega \tag{3.8}$$

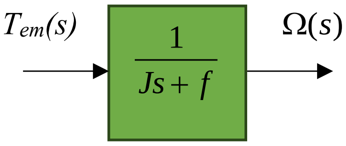
The Laplace transform of the equation (3.8) is:

$$Js\Omega(s) + f\Omega(s) = T_{em}(s) \tag{3.9}$$

Thus:

$$\Omega(s) = \frac{1}{Js+f}T_{em}(s) \tag{3.10}$$

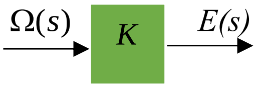
The block diagram of the equation (3.10):

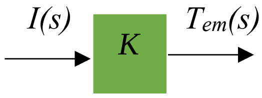


Electromechanical Equations

$$\begin{aligned} e = K\Omega &\Rightarrow E(s) = K\Omega(s) \\ T_{em} = Ki &\Rightarrow T_{em}(s) = KI(s) \end{aligned} \tag{3.11}$$

The blocks diagram of the equation (3.13):





By assembling all the individual components, the overall block diagram of the DC motor system is obtained as follows:

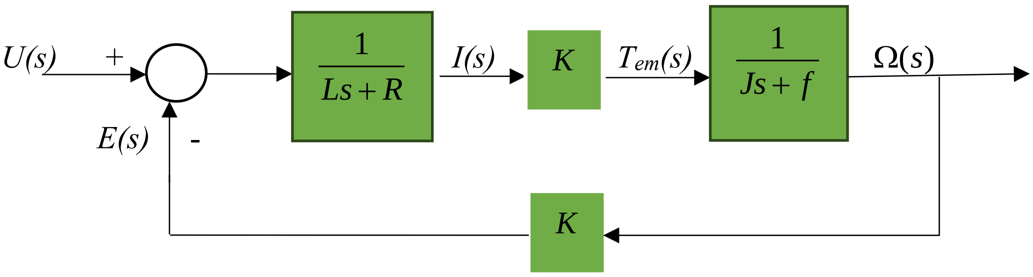


Figure 3.22: Block diagram of DC motor

Applying the block transformation rules, the transfer function of the system is :

$$G(s) = \frac{\Omega(s)}{U(s)} = \frac{\frac{1}{(Ls+R)} K \frac{1}{Js+f}}{1 + K \frac{1}{(Ls+R)} K \frac{1}{Js+f}} \tag{3.12}$$

Therefore :

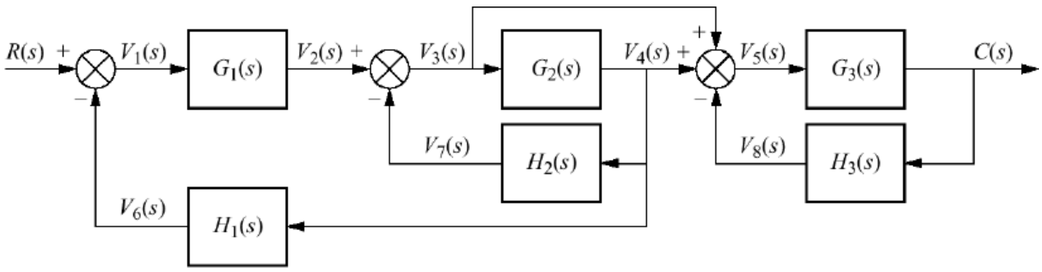
$$G(s) = \frac{\Omega(s)}{U(s)} = \frac{K}{(Ls+R)(Js+f)+K^2} = \frac{K}{JLs^2 + (Lf+RJ)s + K^2 + Rf} \tag{3.13}$$

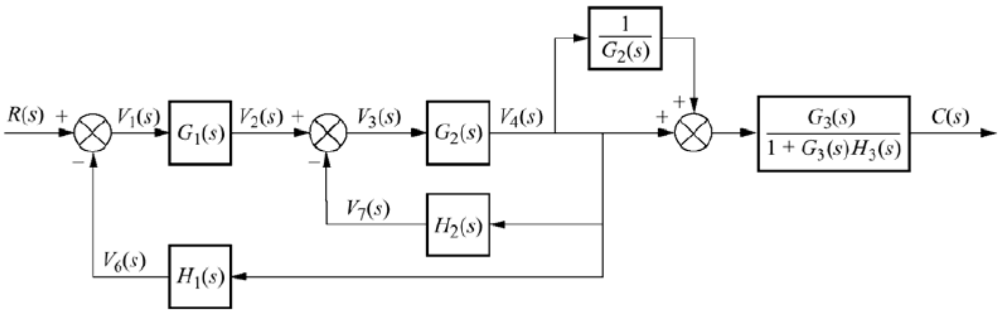
Notice that all parameters of the function transfer are defined in Chapter 2.

Exercise 3.2: Block Diagram Simplification

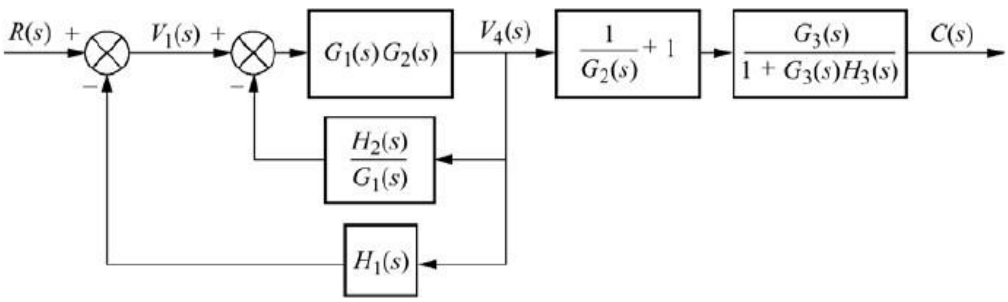
Applying the block transformation rules to the diagram of the figure 3.20:

Step 1 :

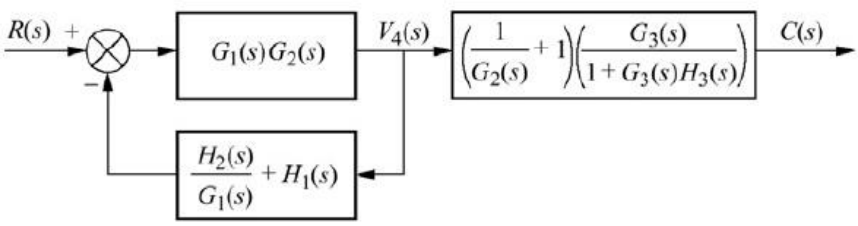




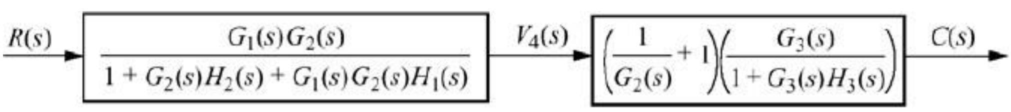
Step 2:



Step 3:



Step 4:



Step 5:

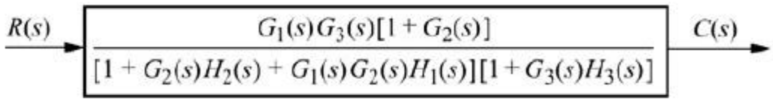


Figure 3.23: Block diagram transformation

Exercise 3.3: Multi-Input Systems

The output C as a function of the inputs *R, U1, U2:*

Since the block diagram in Figure 3.21 includes multiple outputs, we will employ the principle of superposition to analyze each output individually, as detailed below:

a. *U1= U2=0*

$$C1 = \frac{G_1 G_2}{1 - G_1 G_2 H_1 H_2} R \tag{3.14}$$

b. *R= U2=0*

$$C_2 = \frac{G_2}{1 - G_1 G_2 H_1 H_2} U_1 \tag{3.15}$$

c. *R= U1=0*

$$C_3 = \frac{H_1}{1 - G_1 G_2 H_1 H_2} U_2 \tag{3.16}$$

The output C can be written as:

$$C = C_1 + C_2 + C_3 \tag{3.17}$$

$$C = \frac{G_1 G_2}{1 - G_1 G_2 H_1 H_2} R + \frac{G_2}{1 - G_1 G_2 H_1 H_2} U_1 + \frac{H_1}{1 - G_1 G_2 H_1 H_2} U_2 \tag{3.18}$$

- **Chapter 4:** State Space Representation

4.1 Introduction

- The transfer function is used to represent a system by describing the relationship between its input and output, typically in a Single Input, Single Output (SISO) framework.
- One limitation of the transfer function approach is that it provides no insight into the internal behavior of the system—how internal variables evolve over time.
- The state-space representation addresses this shortcoming by modeling the dynamics of the system's internal variables, known as state variables, using a set of first-order linear differential equations.

4.2 Definition of a State-Space System

A **state-space system** refers to the minimal set of state variables required to completely describe the dynamic behavior of a system.

The standard form of a state-space model consists of a system of first-order ordinary differential equations given by the general form:

$$\begin{aligned}
\dot{x}_1(t) &= f_1(t,x,u) \\
\dot{x}_2(t) &= f_2(t,x,u) \\
&\ldots\ldots\ldots \\
\dot{x}_n(t) &= f_n(t,x,u)
\end{aligned} \tag{4.1}$$

Where, x_i are the state variables*, and u are* the inputs

In the case of Linear Time-Invariant (LTI) systems, the state-space representation takes the following standard form:

$$\begin{aligned}
\dot{x} &= Ax + Bu \\
y &= Cx + Du
\end{aligned} \tag{4.2}$$

Such as :

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \ldots \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & \ldots. & a_{1n} \\ a_{21} & a_{22} & \ldots. & a_{2n} \\ \ldots & \ldots & \ldots. & \ldots \\ a_{n1} & a_{n2} & \ldots. & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \ldots \\ x_n \end{bmatrix} + \begin{bmatrix} b_{11} & b_{12} & \ldots. & b_{1r} \\ b_{21} & b_{22} & \ldots. & b_{2r} \\ \ldots & \ldots & \ldots. & \ldots \\ b_{r1} & b_{r2} & \ldots. & b_{rr} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ \ldots \\ u_r \end{bmatrix} \tag{4.3}$$

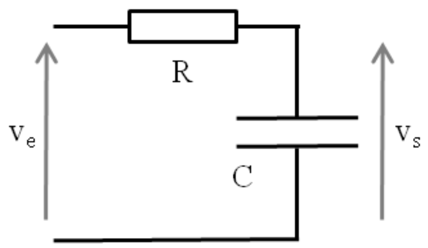
$$\begin{bmatrix} y_1 \\ y_2 \\ \cdots \\ y_m \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1m} \\ c_{21} & c_{22} & \cdots & c_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ c_{m1} & c_{n2} & \cdots & c_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \cdots \\ x_m \end{bmatrix} + \begin{bmatrix} d_{11} & d_{12} & \cdots & d_{1r} \\ d_{21} & d_{22} & \cdots & d_{2r} \\ \cdots & \cdots & \cdots & \cdots \\ d_{r1} & d_{r2} & \cdots & d_{rr} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ \cdots \\ u_r \end{bmatrix} \tag{4.4}$$

Where: x(t) is the state vector, u(t) is the input vector, y(t) is the output vector, A is the system matrix, B is the input matrix, C is the output matrix, D is the feedthrough (or direct transmission) matrix.

Remark 4.1: The number of state variables *n* in a physical system is equal to the number of independent energy storage elements (such as capacitors, inductors, or masses) present in the system.

Remark 4.2: In most physical systems, the feedthrough matrix *D* is equal to zero, indicating that the input does not directly affect the output instantaneously.

Example4.1: RC Circuit – State-Space Representation



$$V_e = Ri + L\frac{di}{dt} + V_s \Rightarrow \begin{cases} \dfrac{di}{dt} = \dfrac{1}{L}(Ri - V_s + V_e) \\ \dfrac{dV_s}{dt} = \dfrac{1}{C}i \end{cases} \tag{4.5}$$

$$i = C\frac{dV_s}{dt}$$

By choosing the state variables as $x_1 = i, x_2 = V_s, u = V_e$ et $y = V_s$, the state space representation:

$$\begin{cases} \dot{x}_1 = -\dfrac{R}{L}x_1 - \dfrac{1}{L}x_2 + \dfrac{1}{L}u \\ \dot{x}_2 = \dfrac{1}{C}x_1 \end{cases} \tag{4.6}$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -\dfrac{R}{L} & -\dfrac{1}{L} \\ \dfrac{1}{C} & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} \dfrac{1}{L} \\ 0 \end{bmatrix} u \tag{4.7}$$

$$y = [0 \quad 1]\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \tag{4.8}$$

Hence, we can deduce the state-space matrices as follows:

$$A=\begin{bmatrix}-\frac{R}{L} & -\frac{1}{L}\\ \frac{1}{C} & 0\end{bmatrix}, B=\begin{bmatrix}\frac{1}{L}\\ 0\end{bmatrix}u,\quad C=[0\quad 1], \text{et } D=0$$

Remark 4.3: The state-space representation is not unique; a given system can have multiple valid state-space representations that describe the same input-output behavior.

4.3 Block Diagram of an LTI System Described by State-Space Representation:

The block diagram of the state-space representation of an LTI system is illustrated in the following figure:

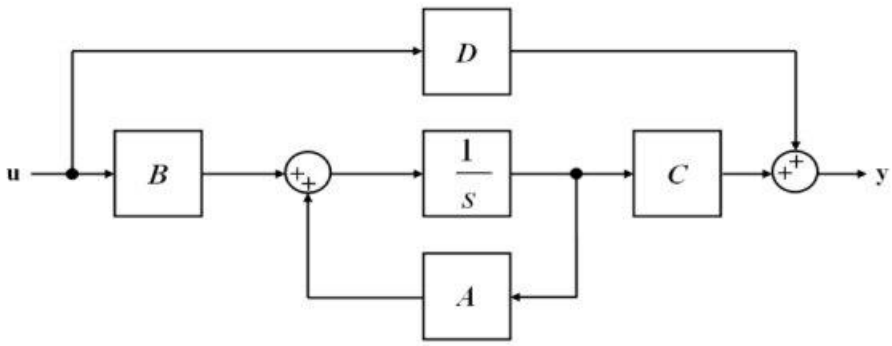


Figure 4.1**:** Block diagram of a state-space system.

4.4 Conversion from State-Space Representation to Transfer Function

Given the state-space equations for a system:

$$\begin{aligned}\dot{x} &= Ax+Bu\\ y &= Cx+Du\end{aligned}\qquad (4.9)$$

To derive the transfer function G(s) (the system's input-output relationship in the Laplace domain), follow these steps:

- First, apply the Laplace transform to the state-space equations, assuming zero initial conditions

$$\begin{aligned}\dot{x} = Ax + Bu \Rightarrow sX(s) &= AX(s) + BU(s) \\ (sI - A)X(s) &= BU(s) \\ X(s) &= (sI - A)^{-1}BU(s)\end{aligned} \tag{4.10}$$

Multiplying both terms of the equation (4.10) by the matrix C:

$$CX(s) = C(sI - A)^{-1}BU(s) \Rightarrow Y(s) = C(sI - A)^{-1}BU(s) \tag{4.11}$$

Thus, the transfer function relating the output to the input is given by:

$$G(s) = \frac{Y(s)}{U(s)} = C(sI - A)^{-1}B \tag{4.12}$$

4.5 Conversion from Transfer Function to State-Space Representation

Given the transfer function G(s) of a system, the goal is to derive the corresponding state-space representation of the system. The transfer function is generally expressed as:

$$G(s) = \frac{Y(s)}{U(s)} = \frac{k}{a_n s^n + a_{n-1}s^{n-1} + ... + a_1 s + a_0} \tag{4.13}$$

From the transfer function G(s), we can write:

$$Y(s)(a_n s^n + a_{n-1}s^{n-1} + ... + a_1 s + a) = kU(s) \tag{4.14}$$

Where k is constant. The inverse Laplace transform of the equation (4.14) is :

$$a_n y^{(n)}(t) + a_{n-1}y^{(n-1)}(t) + ... + a_1 y^{'}(t) + a_0 y(t) = ku(t) \tag{4.15}$$

By choosing the states variables as : $x_1 = y, x_2 = y^{'}, x_3 = y^{''}, ..., x_n = y^{(n-1)}$

Thus, the state-space representation can be written as:

$$
\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= x_3 \\
&\dots\dots\dots \\
\dot{x}_{n-1} &= x_n \\
\dot{x}_n &= \frac{1}{a_n}(ku(t) - a_0x_1 - a_1x_2 - a_2x_3 - \dots\dots - a_{n-1}x_n)
\end{aligned}
\tag{4.16}
$$

The matrix form of the state-space representation:

$$
\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dots \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0\dots. & 0 \\ 0 & 0 & 1\dots. & 0 \\ \dots & \dots & \dots. & \dots \\ -a_0/a_n & -a_1/a_n & \dots. & -a_{n-1}/a_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \dots \\ x_n \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \dots \\ k/a_n \end{bmatrix} u(t) \tag{4.17}
$$

$$
y = [1 \quad 0 \quad \dots. \quad 0] \begin{bmatrix} x_1 \\ x_2 \\ \dots \\ x_n \end{bmatrix} \tag{4.18}
$$

Example 4.2 : Given the transfer function G(s) such as

$$
G(s) = \frac{1}{s^2 + 2s + 1} \tag{4.19}
$$

The **state-space representation** of this transfer function is determined as follows:

$$
\frac{Y(s)}{U(s)} = \frac{1}{s^2 + 2s + 1} \Rightarrow (s^2 + 2s + 1)Y(s) = U(s) \tag{4.20}
$$

The inverse Laplace transform of the equation (4.20) is:

$$
y''(t) + 2y'(t) + y(t) = u(t) \tag{4.21}
$$

By choosing the state variables $x_1 = y, x_2 = y'$, we obtain:

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -2x_2 - x_1 + u(t) \end{aligned} \qquad (4.22)$$

The matrix form of the state-space representation:

$$\begin{aligned} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} &= \begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \\ y &= \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \end{aligned} \qquad (4.23)$$

4.6 Characteristic Polynomial and Eigenvalues :

The characteristic polynomial of a linear system is related to the state matrix A of the system's state-space representation. It is defined as the determinant of the matrix A minus a variable s multiplied by the identity matrix:

$$\det(sI - A) \qquad (4.24)$$

This polynomial called the characteristic polynomial, is of degree n (where n is the order of the system). The eigenvalues are the roots of the characteristic polynomial, i.e., the values of s that satisfy the equation:

$$\det(sI - A) = 0 \qquad (4.25)$$

The eigenvalues of matrix A are linked to the stability and dynamic behavior of the system. If all the eigenvalues have negative real parts, the system is stable.

Remark 4.4: The eigenvalues of matrix A are the poles of the system.

4.7 Exercises :

Exercise 4.1: Given the linear system described by the following equations:

$$\begin{pmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{pmatrix} = \begin{pmatrix} -1 & -0.5 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} + \begin{pmatrix} 0.5 \\ 0 \end{pmatrix} u(t) \quad ; \quad y = \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}$$

- Determine the eigenvalues of this system.
- Compute the transfer function relating the input to the output, as well as the poles of the system.

Exercise 4.2: Consider the linear time-invariant system described by the following differential equation:

$$\frac{d^3C(t)}{dt^3} + 3\frac{d^2C(t)}{dt^2} - 3\frac{dC(t)}{dt} - C(t) = 2\frac{de(t)}{dt} + e(t)$$

- Determine the transfer function of this system.
- Write the state-space representation of the system and provide the matrices A, B, C, and D.
- Compute the characteristic polynomial of A and its eigenvalues (poles).

Exercise 4.3: Consider the separately excited DC motor shown in Figure 1, where the input is the supply voltage u and the output is the position θ

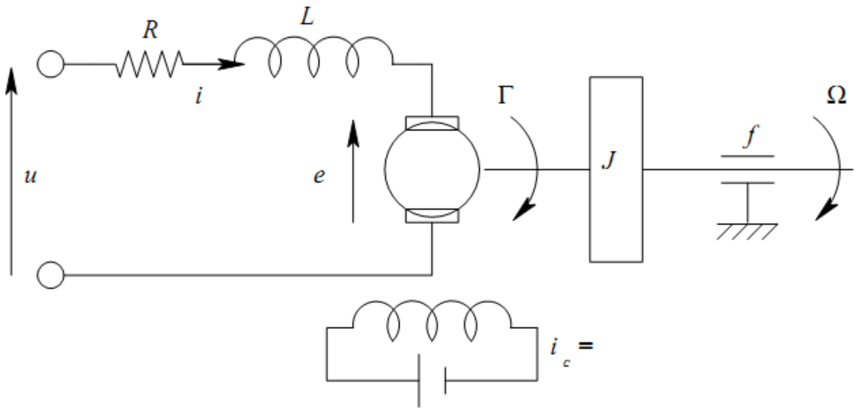


Figure 4.2: DC machine

We assume that the state variables of the system are:

1) i(t): armature current,
2) Ω(t): angular velocity,

3) θ(t): position.

- Drive the state space model of the system (DC motor)
- Compute the characteristic polynomial of A and its eigenvalues (poles)
- R=6.67 Ω , L=0,198 H, J=0,0398, *f*=0, K=1,281.

4.8 Exercises solution

Given the linear system described by the following equations:

$$\begin{pmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{pmatrix} = \begin{pmatrix} -1 & -0.5 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} + \begin{pmatrix} 0.5 \\ 0 \end{pmatrix} u(t) \quad ; \quad \text{y}=\begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} \tag{4.26}$$

a. The eigenvalues of the system are :

$$\det(\lambda I - A) = 0 \Rightarrow \det\begin{pmatrix} \lambda+1 & 0.5 \\ -1 & \lambda \end{pmatrix} = \lambda(\lambda+1) + 0.5 = 0 \tag{4.27}$$

The roots (eigenvalues) of the equation (4.27) are :

$$\begin{cases} \lambda_1 = \text{-0.5 + j0.5} \\ \lambda_2 = \text{-0.5 - j0.5} \end{cases} \tag{4.28}$$

b. The transfer function relating the input to the output:

Applying the Laplace transform to the state-space equations (2.26):

$$\begin{aligned} \dot{x} = Ax + Bu \Rightarrow sX(s) &= AX(s) + BU(s) \\ (sI - A)X(s) &= BU(s) \\ X(s) &= (sI - A)^{-1} BU(s) \end{aligned} \tag{4.29}$$

Multiplying both terms of the equation (4.29) by the matrix C:

$$CX(s) = C(sI - A)^{-1} BU(s) \Rightarrow Y(s) = C(sI - A)^{-1} BU(s) \tag{4.30}$$

Thus, the transfer function relating the output to the input is given by:

$$G(s) = \frac{Y(s)}{U(s)} = C(sI - A)^{-1}B \tag{4.31}$$

Thus :

$$(sI - A)^{-1} = \frac{1}{\det((sI - A))} C^{t}_{(sI-A)} \tag{4.32}$$

The cofactors of the matrix $(sI - A)$ are :

$$C_{(sI-A)} = \begin{pmatrix} s & -1 \\ 0.5 & s+1 \end{pmatrix} \Rightarrow C^{T}{}_{(sI-A)} = \begin{pmatrix} s & 0.5 \\ -1 & s+1 \end{pmatrix} \tag{4.33}$$

So :

$$(sI - A)^{-1} = \frac{1}{s^2 + s + 0.5} \begin{pmatrix} s & 0.5 \\ -1 & s+1 \end{pmatrix} \tag{4.34}$$

The transfer function is :

$$G(s) = C(sI - A)^{-1}B = \frac{0.5}{s^2 + s + 0.5} \tag{4.35}$$

The poles of the transfer function of equation (4.35) are :

$$\begin{cases} p_1 = \text{-0.5 + j0.5} \\ p_2 = \text{-0.5 - j0.5} \end{cases} \tag{4.36}$$

Exercise 4.2: Consider the linear time-invariant system described by the following differential equation:

$$\frac{d^3C(t)}{dt^3} + 3\frac{d^2C(t)}{dt^2} - 3\frac{dC(t)}{dt} - C(t) = 2\frac{de(t)}{dt} + e(t) \tag{4.37}$$

a. The transfer function of equation (4.37):

$$G(s) = \frac{2s+1}{s^3 + 3s^2 - 3s - 1} \tag{4.38}$$

b. The state-space representation of the system:

Let $G(s)$ written as $G(s)=\frac{C(s)}{V(s)}\frac{V(s)}{E(s)}$, where ;

$$\begin{cases}\frac{V(s)}{E(s)}=\frac{1}{s^3+3s^2-3s-1}\\ \frac{C(s)}{V(s)}=2s+1\end{cases} \tag{4.39}$$

From the equation (4.39), we can write:

$$V(s)(s^3+3s^2-3s-1)=U(s) \tag{4.40}$$

The inverse Laplace transform of equation (4.40):

$$\frac{d^3v(t)}{dt^3}+3\frac{d^2v(t)}{dt^2}-3\frac{dv(t)}{dt}-v(t)=e(t) \tag{4.41}$$

Let's choose the state variable as :

$$x_1=v(t), x_2=\frac{dv(t)}{dt}, x_3=\frac{d^2v(t)}{dt^2}$$

The state equation can be written as :

$$\begin{cases}\dot{x}_1=x_2\\ \dot{x}_2=x_3\\ \dot{x}_3=x_1+3x_2-3x_3+e(t)\end{cases} \tag{4.42}$$

The matrix form of the state-space representation:

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\end{bmatrix}=\begin{bmatrix}0&1&0\\0&0&1\\1&3&-3\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}+\begin{bmatrix}0\\0\\1\end{bmatrix}e(t) \tag{4.43}$$

Where $A=\begin{bmatrix}0&1&0\\0&0&1\\1&3&-3\end{bmatrix}$ and $B=\begin{bmatrix}0\\0\\1\end{bmatrix}$

The output can be found as:

$$\frac{C(s)}{V(s)}=2s+1 \Rightarrow C(s)=(2s+1)V(s) \tag{4.44}$$

The inverse transform of equation (4.44) can be written as :

$$C(t) = 2\frac{dv(t)}{dt} + v(t) = 2x_2 + x_1 \tag{4.45}$$

Therefore, the output is:

$$C(t) = \begin{bmatrix} 1 & 2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \quad , \text{D=0} \tag{4.46}$$

c. The characteristic polynomial of A and its eigenvalues (poles):

$$\det(\lambda I - A) \Rightarrow \det \begin{pmatrix} \lambda & -1 & 0 \\ 0 & \lambda & -1 \\ -1 & -3 & \lambda + 3 \end{pmatrix} = \lambda^3 + 3\lambda^2 - 3\lambda - 1 \tag{4.47}$$

The eigenvalues are:

$$\lambda_1 = 1, \lambda_2 = -3.7321, \lambda_3 = -0.2679$$

Exercise 4.3

a. The state space model of the system (DC motor): From Chapter 2 the equations of the DC motor are :

$$\begin{aligned} &L\frac{di}{dt} + Ri + e = u \\ &J\frac{d\Omega}{dt} = T_{em} - f\Omega \\ &e = K\Omega, \text{ and } \quad T_{em} = Ki \end{aligned} \tag{4.48}$$

Where all parameters are defined in the chapter 2.

Let's consider the state variable :

$$x_1 = i(t), x_2 = \Omega(t), x_3 = \theta(t)$$

From the equations (4.48), the state equation can be written as :

$$\begin{cases} \dot{x}_1 = \dfrac{1}{L}(-Rx_1 - kx_2 + u) \\ \dot{x}_2 = \dfrac{1}{J}(Kx_1 - fx_2) \\ \dot{x}_3 = x_2 \end{cases} \tag{4.49}$$

The matrix form of the state-space representation

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} -R/L & -K/L & 0 \\ K/J & -f/J & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 1/L \\ 0 \\ 1 \end{bmatrix} u(t) \tag{4.50}$$

Where $A = \begin{bmatrix} -R/L & -K/L & 0 \\ K/J & -f/J & 0 \\ 0 & 1 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 1/L \\ 0 \\ 1 \end{bmatrix}$, $C = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}$

b. The characteristic polynomial of A and its eigenvalues (poles):

$$\det(\lambda I - A) \Rightarrow \det \begin{bmatrix} \lambda + R/L & K/L & 0 \\ -K/J & \lambda + f/J & 0 \\ 0 & -1 & \lambda \end{bmatrix} = \lambda(\lambda + f/J)(\lambda + R/L) + \frac{K^2}{JL} \tag{4.51}$$

The eigenvalues (poles):

$\lambda_1 = 0, \lambda_2 = \text{-}8.1562, \lambda_3 = \text{-}25.5307$

Part 2: Linear Control System Analysis:

5.1 Introduction

The time-domain analysis of linear systems consists of determining the form of the output signal *y(t)* as a function of time when the system is subjected to a well-defined input signal. The time-domain response provides valuable insight into the system's stability, performance, and transient behavior.

The time response of an LTI system is typically divided into two parts:

a) **The Transient Response**
   This corresponds to the short-term behavior of the system immediately after a change in input or initial conditions. It reflects how the system transitions from its initial state to its final steady state. The transient response depends on the system's poles (eigenvalues of the system matrix or roots of the characteristic equation). Depending on the location of the poles in the complex plane, the response can be:
   - Exponentially decaying (stable),
   - Oscillatory,
   - Diverging (unstable).

b) **The Steady-State Response**
   This corresponds to the long-term behavior of the system once the transients have dissipated. It is typically determined by the nature of the input signal (e.g., step, ramp, sinusoid) and the system's gain.

5.1.1 Impulse Response:

The impulse response is the system's output when subjected to a Dirac delta function as the input $e(t) = \delta(t)$

5.1.2 Step Response:

The step response is the output of the system when the input is a unit step function $e(t) = u(t)$

5.2 Analysis of Fundamental Systems

The analysis of fundamental systems involves the study of first-order and second-order **systems**, which serve as the basic building blocks in control theory and provide essential insight into the dynamic behavior of more complex systems.

5.2.1 First-order system

A first-order system is a linear system whose behavior is governed by a first-order ordinary differential equation of the form:

$$\tau \frac{dy(t)}{dt} + y(t) = ke(t) \tag{5.1}$$

Applying the Laplace transform (assuming zero initial conditions), the transfer function of the system is given by:

$$G(s) = \frac{Y(s)}{E(s)} = \frac{K}{1+\tau s} \tag{5.2}$$

K and τ are positive constants, where, τ : is the time constant**,** *K: i*s the static gain of the system.

5.2.1.1 Impulse Response

We will study the response of the system to a Dirac delta input $e(t) = \delta(t)$. The Laplace transform of the input is then given by *E(s)=1*. The output is *Y(s)=G(s)*, thus :

$$Y(s) = \frac{K}{1+\tau s} = \frac{k/\tau}{1/\tau + s} \tag{5.3}$$

The inverse Laplace transform of the (5.3) is:

$$y(t) = \frac{K}{\tau} e^{-\frac{1}{\tau}t} \tag{5.4}$$

It can be observed (figure 5.1) that at time t=*0* the output is equal to K/τ and decreases exponentially, converging toward zero as $t \to +\infty$.

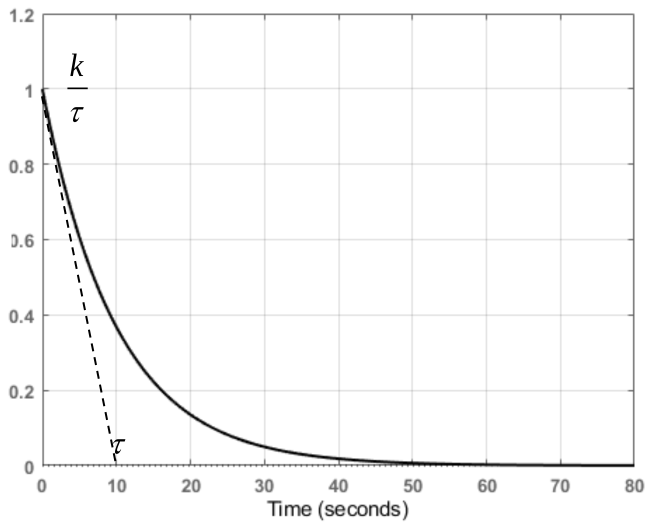


Figure 5.1: impulse response of first-order system

5.2.1.2 Step Response

We now study the response of the system to a unit step input. The Laplace transform of the input is given by E(s)=1/s. The output is given by *Y(s)=G(s). E(s)*, therefore, the output will be:

$$Y(s) = \frac{1}{s} \cdot \frac{\frac{k}{\tau}}{\frac{1}{\tau} + s} \tag{5.5}$$

The partial fraction decomposition of the output is:

$$Y(s) = \frac{A_1}{s} + \frac{A_2}{s + 1/\tau} \tag{5.6}$$

where $A_1=k$ et $A_2=-k$. The inverse Laplace transform of the output is:

$$y(t) = k - ke^{-\frac{1}{\tau}t} = k(1 - e^{-\frac{1}{\tau}t}) \tag{5.7}$$

Figure 5.2 illustrates the time evolution of the output in response to a unit step input. From the graph, one can define the response time, which is typically calculated as the time required for the output to reach and remain within **5%** of its final value.

$$y(t_r) = k(1 - e^{-t_r/\tau}) = 0.95k \Rightarrow t_r \simeq 3\tau \tag{5.8}$$

Note that at $t = \tau \Rightarrow y(\tau) = 0.63k$

5.2.1.3 Observations :

1. The output signal asymptotically approaches the value K but never reaches it exactly in a mathematical sense.
2. Two distinct phases can be identified:
    - **Transient regime**: This corresponds to the initial evolution of the output signal and lasts for a certain period of time.
    - **Steady-state regime**: From the moment when the output remains within 5% of its final value, the system is considered to be in a steady state.

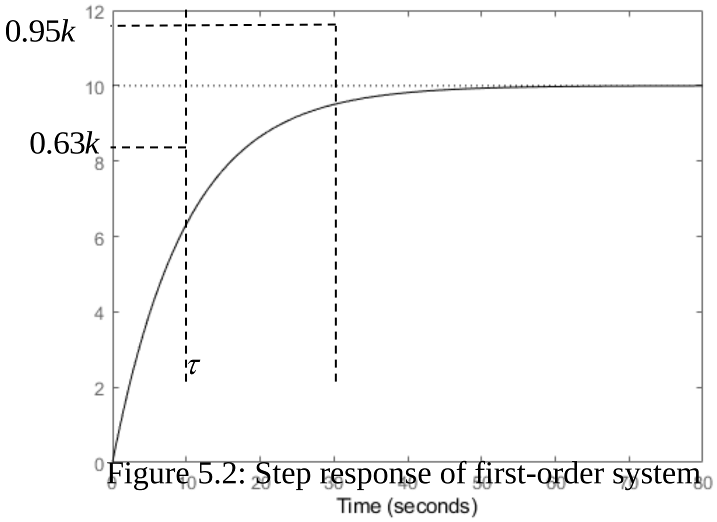


Figure 5.2: Step response of first-order system

5.2.1.4 Response to a Ramp Input

We will analyze the response of the system to a ramp input $e(t) = t^2$, The Laplace transform of the input is then given by $E(s)=1/s^2$. The output $Y(s)$ will be :

$$Y(s) = \frac{1}{s^2} \cdot \frac{\frac{k}{\tau}}{\frac{1}{\tau} + s} \tag{5.9}$$

The inverse Laplace transform of the output is:

$$y(t) = k(t - \tau) + k\tau e^{-t/\tau} \tag{5.10}$$

The form of this expression allows us to highlight an oblique asymptote of the curve. The term $k\tau e^{-t/\tau}$ tends toward zero as t approaches infinity (see Figure 5.3).

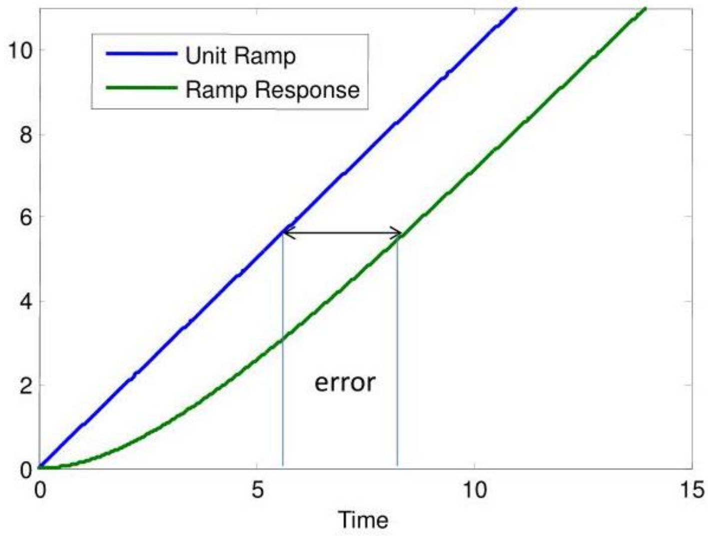


Figure 5.3: ramp response to first order system

5.2.2 Second ordre system

A second-order system is a dynamic system whose behavior is governed by a second-order linear differential equation with constant coefficients. The canonical form of such a system's differential equation is:

$$\frac{1}{\omega_n}\frac{d^2y(t)}{dt^2}+\frac{2\zeta}{\omega_n}\frac{dy(t)}{dt}+y(t)=ke(t) \tag{5.11}$$

The transfer function of a second-order system in canonical form is written as:

$$G(s)=\frac{Y(s)}{E(s)}=\frac{k}{\frac{1}{\omega_n}s^2+\frac{2\zeta}{\omega_n}s+1}=\frac{k\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2} \tag{5.12}$$

k :Static gain, ζ :damping coefficient, ω_n : natural undamped frequency.

5.2.2.1 Step Response of a Second-Order System:

We now study the response of the system to a unit step input. The Laplace transform of the input is given by E(s)=1/s. The output is given by *Y(s)=G(s). E(s)*, therefore, the output will be:

$$Y(s)=\frac{1}{s}\frac{k\omega_n}{s^2+2\zeta\omega_n s+\omega_n^2} \tag{5.13}$$

To study the impulse response, we calculate the poles of the transfer function, thus:

$s^2 + 2\zeta\omega_n s + \omega_n^2 = 0$. The determinant $\Delta = 4\zeta^2\omega_n^2 - 4\omega_n^2 = 4\omega_n^2(\zeta^2 - 1)$ and the poles will be:

$p_1 = -\zeta\omega_n - \omega_n\sqrt{\zeta^2 - 1}$, $p_2 = -\zeta\omega_n + \omega_n\sqrt{\zeta^2 - 1}$

Five cases are distinguished:

a. $\zeta > 1$: The poles are real and distinct, and the output can be written in the following form :

$$y(t) = k - k\frac{p_2 e^{-p_1\omega_n t} - p_1 e^{-p_2\omega_n t}}{p_2 - p_1} \tag{5.14}$$

The response is said to be aperiodic. The response exhibits no oscillation and is similar to the step response of a first-order system (Figure 5.4).

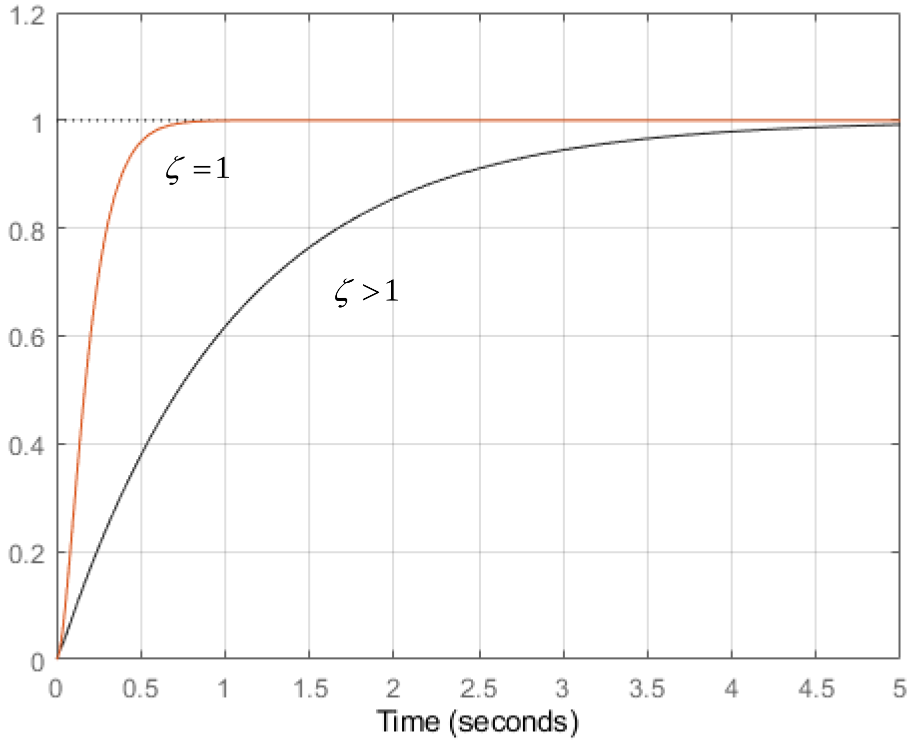


Figure 5.4: Aperiodic and critically aperiodic responses.

b. $\zeta = 1$: We have a double pole, and the output is of the following form:

$$y(t) = k - k(e^{-\omega_n t} - \omega_n t e^{-\omega_n t}) \tag{5.15}$$

The response is referred to as critically damped. It behaves similarly to the aperiodic response, with the key difference being that the critically damped response reaches the final value more quickly in the transient phase (Figure 5.4).

c. $0 < \zeta < 1$ The poles are complex and of the following form $p_{1,2} = -\zeta\omega_n \pm \omega_n j\sqrt{1-\zeta^2}$, and the output is of the following form:

$$y(t) = k - \frac{ke^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\sin(\omega_p t + \psi) \tag{5.16}$$

Where $\sin\psi = \sqrt{1-\zeta^2}$, $\cos\psi = \zeta$, et , $\omega_p = \omega_n\sqrt{1-\zeta^2}$.

The response is said to be damped oscillatory with a frequency ω_p , or pseudo-periodic (figure 5.5).

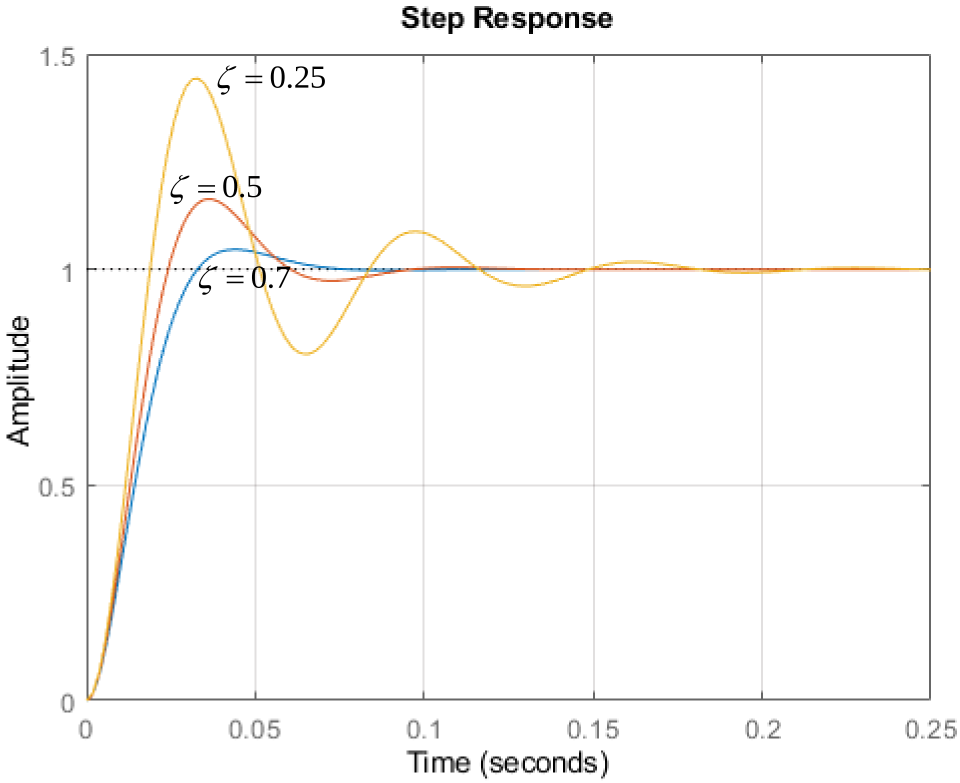


Figure 5.5: damped oscillatory response

5.2.2.2 Some Characteristics of the Pseudoperiodic Regime

- The time response at 5% of the final value is **:** $t_{r5\%} = \dfrac{3}{\zeta\omega_n}$
- The pseudo period is: $T_p = \dfrac{2\pi}{\omega_n\sqrt{1-\zeta^2}}$
- The first overshoot D is expressed as a percentage $D = 100e^{\frac{-\zeta\pi}{\sqrt{1-\zeta^2}}}$. For example, for a *D=5%*, $\zeta = 0.707$
- The peak time of the first overshoot $T_{peak} = \dfrac{\pi}{\omega_d}$, with $\omega_d = \omega_n\sqrt{1-\zeta^2}$

d. $\zeta = 0$ The poles of the system are purely imaginary $p_{1,2} = \pm j\omega_n$, and the response is simply sinusoidal with a frequency ω_n. It is said that the system is at the stability margin.

e. $\zeta < 0$ The response is undamped oscillatory and diverges. The system is unstable under these conditions.

5. 2.3 Influence of Poles and Zeros

5.2.3.1 Effect of Zero on the Step Response

In this part, we will consider the effects of adding zeros or additional poles to the second-order system under a step response.

Let the transfer function of the second-order system in canonical form with the addition of a zero be written as:

$$G_z(s) = \frac{(\frac{1}{z}s+1)\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n} \tag{5.17}$$

Note that writing the zero in the form $\frac{1}{z}s+1$, instead of $s+z$ is to maintain the final value as 1 or k (if $k \neq 1$).

$G_z(s)$ Can be written as :

$$G_z(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n} + \frac{1}{z}s\frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n} \tag{5.18}$$

$$G_z(s) = G(s) + \frac{1}{z}sG(s) \tag{5.19}$$

Where $G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n}$, The canonical form of a second-order system. The output is :

$$Y_z(s) = (G(s) + \frac{1}{z}G(s))\frac{1}{s} = \frac{1}{s}G(s) + \frac{1}{z}sG(s)\frac{1}{s} \tag{5.20}$$

$$Y_z(s) = Y(s) + \frac{1}{z} sY(s) \qquad (5.21)$$

The inverse Laplace transform of $sY(s)$ is $y'(t)$. Therefore, we obtain: $y_z(t) = y(t) + \frac{1}{z}\dot{y}(t)$.

Case $z > 0$: which corresponds to a zero in the left half-plane or left half-zero (LHP zero). Note that as z increases, the term 1/z becomes smaller, and the contribution of z diminishes and resembles the original response.

Adding a left half-plane (LHP) zero to a system's transfer function affects its step response as follows (figure (5.6)):

- **Increases overshoot**: The maximum peak of the system's response surpasses the desired final value to a greater extent.
- **Decreases peak time**: The time taken to reach the maximum peak is reduced, leading to a quicker response.
- **Decreases rise time**: The duration for the response to rise from a specified low value to a specified high value is shortened, resulting in a faster initial reaction.

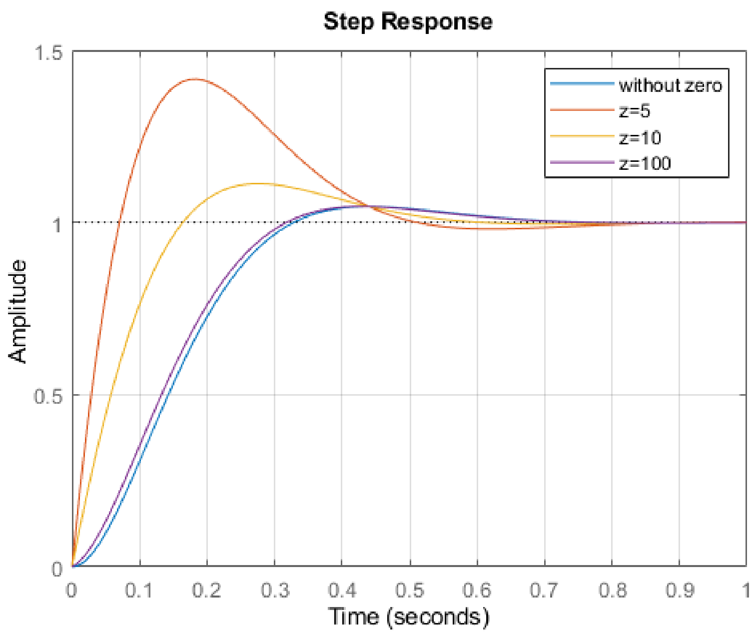


Figure 5.6:effect of LHP zero

Cas $z<0$: In this case, the derivative is subtracted from the input. As a result, the transient response initially moves in the opposite direction before eventually rising toward the steady-state value of 1 (see Figure 5.6). This phenomenon is known as undershoot. A right-half plane zero is associated with non-minimum phase behavior.

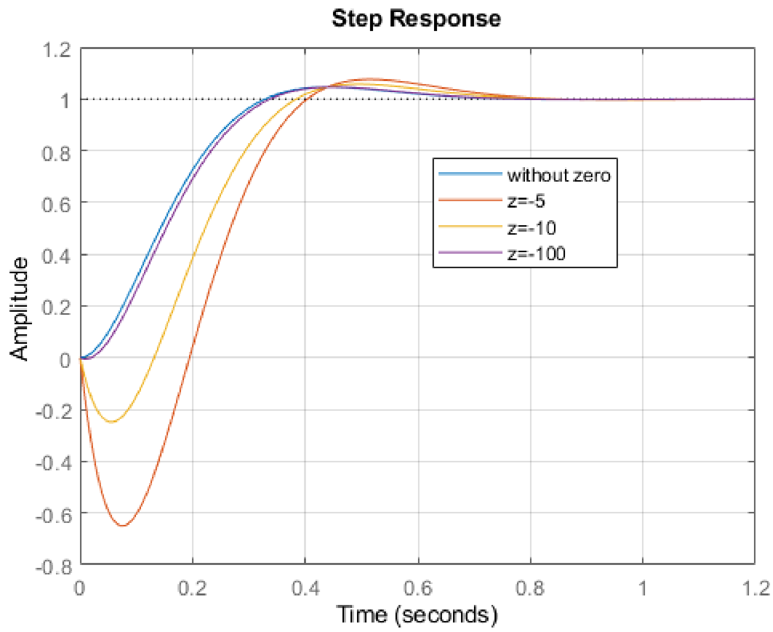


Figure 5.7:Effect of RHP zero

5.2.3.2 Effect of Poles on the Step Response

In general, regardless of the system's order, complex poles induce an oscillatory response. Note that poles with a positive real part lead to instability.

To analyze higher-order systems (n > 2), it can be advantageous to find a second-order model that provides a good approximation by considering the dominant poles. However, it is sufficient to decompose the transfer function into simple factors.

Dominant poles are poles whose real part is small and negative.

In the case of second-order systems operating in a pseudo-periodic regime, adding a pole such as:

$$G(s)=\frac{\omega_n^2}{(\frac{1}{p}s+1)s^2+2\zeta\omega_n s+\omega_n} \qquad (5.22)$$

The addition of a pole (with a negative real part) to the transfer function results in a slower response (see Figure 5.8).

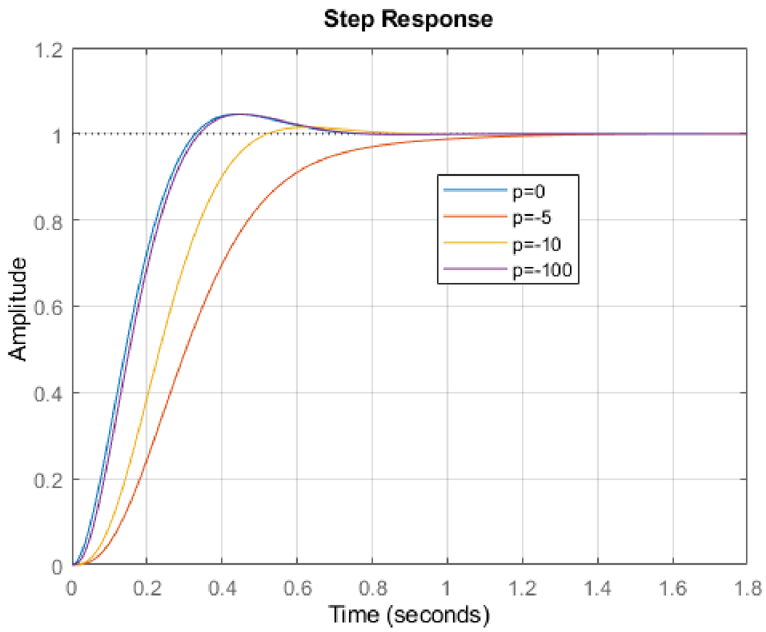
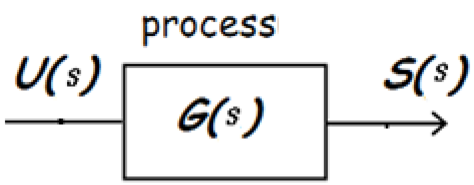


Figure 5.8: Effect of LHP pole

5.3 Exercises

Exercise 5.1: Identification of a First-Order System

A first-order system is subjected to a unit step input (Figure 1), and its step response is shown in Figure 2.



With $G(s)=\frac{K}{(1+\tau s)}$

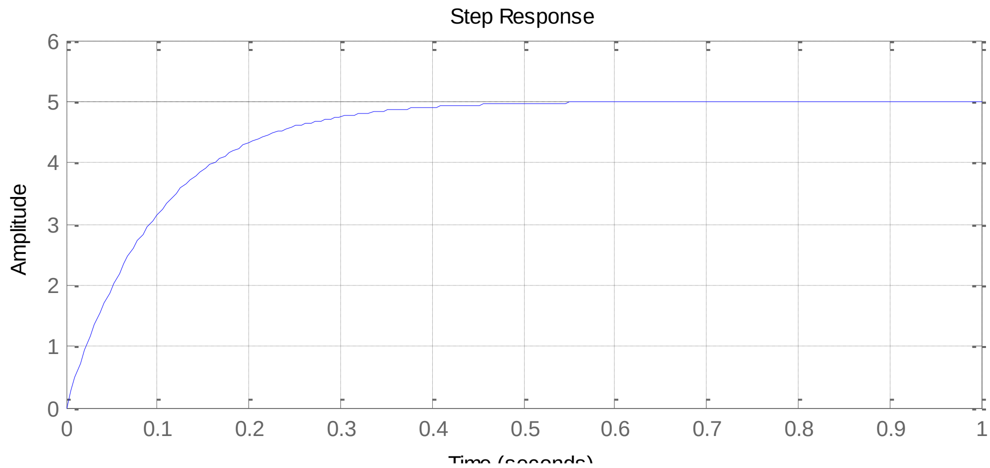


Figure 5.9: Step response of first-order system

1. Identify the system parameters: *K*: the static gain,τ: the time constant
2. Compute the *s(t)* if the input is *u(t)=t*
3. Plot s(t)

Exercise 5.2: Second-Order Control System

A physical process is modeled by a second-order transfer function:

$$G(s) = \frac{G_0}{(1+\tau_1 s)(1+\tau_2 s)} \quad , \; G_0 = 1, \; \tau_1 = 10s, \; \tau_2 = 2s$$

This process is placed in a closed-loop system with a proportional controller: *C(s)=K*

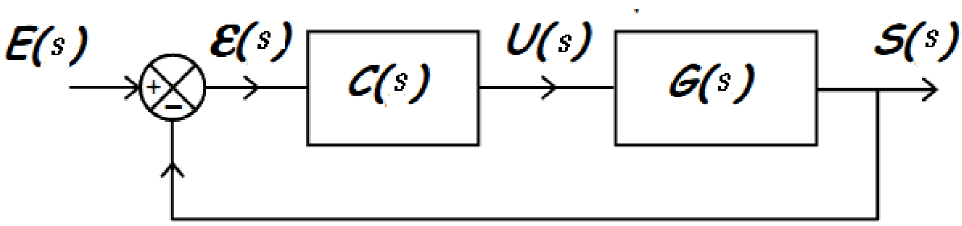


Figure 5.10: control system

1. a. Determine the closed-loop transfer function: *H(s) = S(s)/E(s)* and express it in its canonical form $H(s) = H_0 \dfrac{\omega_n^{\,2}}{s^2 + 2m\omega_n s + \omega_n^{\,2}}$

Deduce the expressions for the parameters of *H(s*): *H0* static gain, *m* damping coefficient, and *ω*n undamped natural frequency in terms of *τ1, τ2, G0, and K.*

b. Compute the value of *K* to obtain *m* = 0,7.

2. Now, consider a unit step reference input, and assume K is adjusted so that m=0.7.

a. Determine the steady-state value s(+∞) and compute its value.

b. Express and compute the steady-state error $\varepsilon_0(+\infty) = e(+\infty) - s(+\infty)$.

d. Sketch the response s(t).

3. To reduce the steady-state error, K is increased.

a. Compute the value of K required to achieve $\varepsilon_0(+\infty) = 0{,}05$ V.

b. Deduce the new value of the damping coefficient m.

c. Compute the relative overshoot D (in %).

d. Compute the new settling time $tr_{5\%}$.

e. Sketch the response s(t).

Exercise 5.2: First-Order PI Control System

Consider the R-C-R circuit of Figure 5.11 where the input is the supply voltage u(t), and the output is v(t)

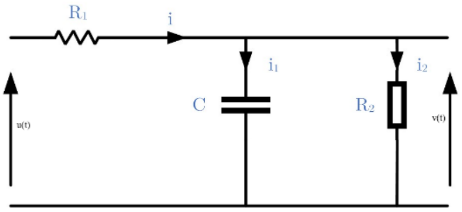


Figure 5.11: R-C-R circuit

a. Write the differential equation that relates the input to the output.
b. Determine the expression of the transfer function *G(s) = V(s)/U(s)* , and express it in its canonical form : $G(s) = \dfrac{K}{(1+\tau s)}$
c. Deduce the expressions of the parameters K and τ as functions of *R1, R2, C*
d. Calculate K and τ if R1=100 Ω , R2=1 $K\Omega$, C = 5500 μF

This process is inserted into a feedback control loop (figure 5.10) containing a proportional-integral (PI) controller such as: $C(s) = \dfrac{A}{s}(1+\tau s)$.

e. Determine the expression of the closed-loop transfer function *H(s) = V(s)/E(s)* and express it in the form: $H(s) = \dfrac{1}{(1+\tau_1 s)}$

f. Deduce the expression of the closed-loop time constant τ_1 as a function of *A* and *K*.

g. Evaluate *A* for τ_1=0.1

In the remainder of the exercise, the reference input is a unit step, and the gain *A* is set such that τ_1=0.1

h. Consider the steady-state regime. Determine the expression of v(+∞) and calculate its value.
i. Express the steady-state error ε0(+∞)=e(+∞)−v(+∞)
j. Deduce the expression of v(t)

5.4 Exercises solution

Exercise 5.1: Identification of a First-Order System

- From Figure 5.9, since the system is subjected to a unit step input, the output response is: $s(t) = K(1-e^{-\frac{1}{\tau}t})$, as $t \to \infty$, the output $s(t) \to K$, In this case, we observe that *K=5.*
- The time constant is determined from the graph at the point where *s(t)=0.63K.*At this point, the corresponding time is *t= τ.*
  Therefore, *τ=0.1.*
- The *s(t)* when the input is *u(t)=t,* The Laplace transform of the input is then given by *U(s)=*$1/s^2$. The output *Y(s)* will be :

$$S(s) = \frac{1}{s^2} \cdot \frac{\frac{K}{\tau}}{\frac{1}{\tau}+s} = \frac{A_1}{s} + \frac{A_2}{s^2} + \frac{A_3}{\frac{1}{\tau}+s} \tag{5.23}$$

Where :

$$\begin{aligned} A_1 &= -K\tau \\ A_2 &= Kt \\ A_3 &= K\tau \end{aligned} \tag{5.24}$$

The inverse Laplace transform of the output is:

$$s(t) = K(t-\tau) + K\tau e^{-t/\tau} \tag{5.25}$$

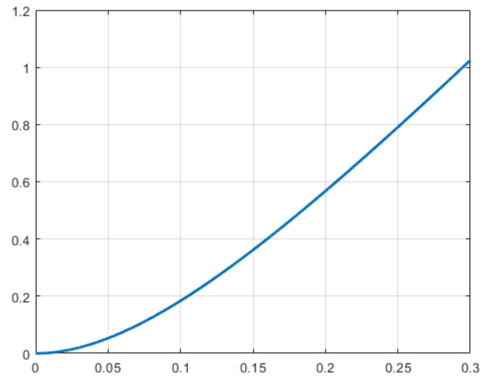


Figure 5.12: ramp response of first-order system

Exercise 5.2: Second-Order Control System

A physical process is modeled by a second-order transfer function:

$$G(s) = \frac{G_0}{(1+\tau_1 s)(1+\tau_2 s)} \quad , \; G_0 = 1, \; \tau_1 = 10s, \; \tau_2 = 2s$$

This process is placed in a closed-loop system with a proportional controller: *C(s)=K*

1. The closed-loop transfer function: *H(s) = S(s)/E(s)* in the canonical form

$$H(s) = H_0 \frac{\omega_n{}^2}{s^2 + 2m\omega_n s + \omega_n{}^2}$$

From the figure 5.9, the closed-loop transfer function is :

$$H(s) = \frac{C(s)G(s)}{1+C(s)G(s)} = \frac{K\dfrac{G_0}{(1+\tau_1 s)(1+\tau_2 s)}}{1+K\dfrac{G_0}{(1+\tau_1 s)(1+\tau_2 s)}} = K\frac{G_0}{(1+\tau_1 s)(1+\tau_2 s)+KG_0} \tag{5.26}$$

Developing the equation (5.26):

$$H(s) = \frac{KG_0}{\tau_1\tau_2 s^2 + (\tau_1+\tau_2)s + 1 + KG_0} = \frac{KG_0/\tau_1\tau_2}{s^2 + \dfrac{(\tau_1+\tau_2)}{\tau_1\tau_2}s + \dfrac{1+KG_0}{\tau_1\tau_2}} \tag{5.27}$$

By identification to the canonical form :

$$\omega_n = \sqrt{\frac{1+KG_0}{\tau_1\tau_2}}$$

$$H_0 = \frac{KG_0}{1+KG_0} \tag{5.28}$$

$$m = \frac{1}{2}\frac{\tau_1+\tau_2}{\sqrt{\tau_1\tau_2}\sqrt{1+KG_0}}$$

b. For $m = 0{,}7$, K= 2.6735

2. The input is step unit and $m = 0{,}7$, K= 2.6735

a. The steady-state value s(+∞) :

$$s(\infty)=\lim_{t\to\infty} s(t) = \lim_{s\to 0} sS(s) = H_0 = 0.7278$$

b. The steady-state error $\varepsilon_0(+\infty) = e(+\infty) - s(+\infty)$

$$\varepsilon_0(+\infty) = e(+\infty) - s(+\infty) = 1 - H_0 = 1 - 0.7278 = 0.2722$$

3. To reduce the steady-state error, *K* is increased. For $\varepsilon_0(+\infty)=0.05$, *K* will be

a. K=19

b. The new value of the damping coefficient m

m= 0.3

c. The relative overshoot *D* (in %).

$$D = 100 e^{\frac{-m\pi}{\sqrt{1-m^2}}} = 37.23\ \%$$

d. The new settling time $t_{r5\%}$

$$t_{r5\%} = \frac{3}{m\omega_n} = 10\text{s}$$

d. The figure 5.13 shows the response of the two previous cases :

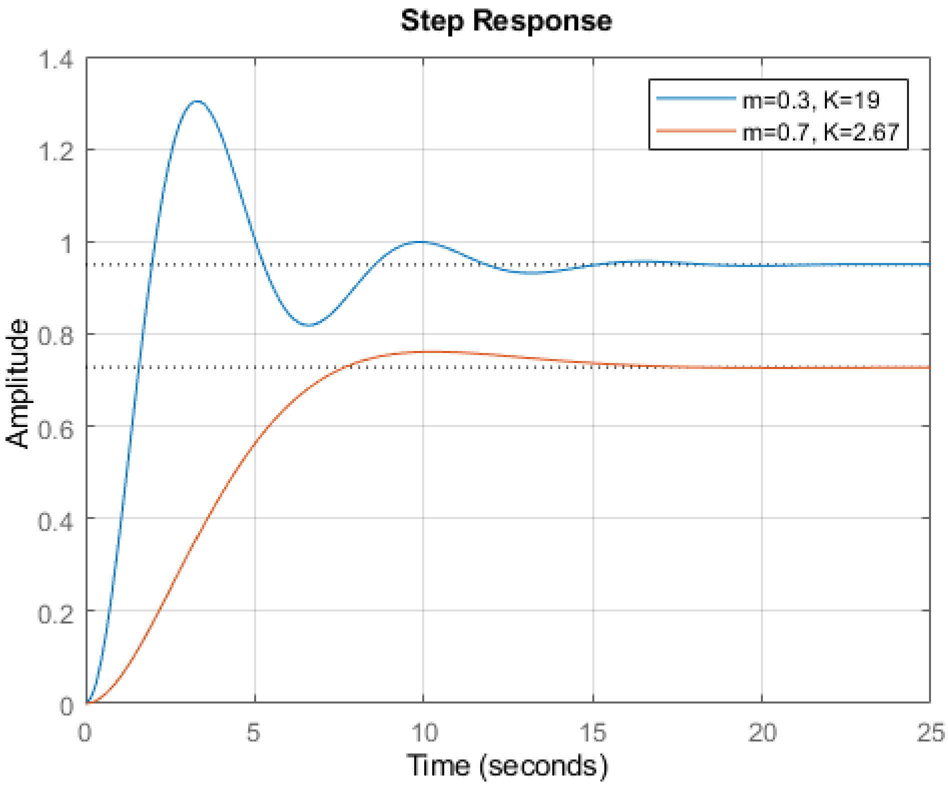


Figure 5.13: Step response of second order system

Exercise 5.2: First-Order PI Control System

a. The differential equation that relates the input to the output: applying Kirchhoff's law on the circuit RCR we can write:

$$\begin{cases} R_1 i + v(t) = u(t) \\ i = i_1 + i_2 \\ i_1 = C\dfrac{dv(t)}{dt}, i_2 = \dfrac{v(t)}{R_2} \end{cases} \tag{5.29}$$

Manipulation these equations, we obtain

$$R_1 i + v(t) = u(t) \Rightarrow R_1(i_1 + i_2) + v(t) = u(t)$$

$$\Rightarrow R_1 C\frac{dv(t)}{dt} + R_1\frac{v(t)}{R_2} + v(t) = u(t) \tag{5.30}$$

Thus, The differential equation that relates the input to the output is ;

$$R_1 C\frac{dv(t)}{dt} + \left(\frac{R_1 + R_2}{R_2}\right)v(t) = u(t) \tag{5.31}$$

b. The expression of the transfer function $G(s) = V(s)/U(s)$ in its canonical form :

Applying the Laplace transform we obtain the following transfer function :

$$G(s)=\frac{V(s)}{U(s)}=\frac{1}{R_1Cs+\frac{R_1+R_2}{R_2}} \quad (5.32)$$

The canonical form of *G(s)*:

$$G(s)=\frac{V(s)}{U(s)}=\frac{\frac{R_2}{R_2+R_1}}{1+\frac{R_1R_2C}{R_2+R_1}s} \quad (5.33)$$

c. $K=\frac{R_2}{R_2+R_1}, \tau=\frac{R_1R_2C}{R_2+R_1}$

d. R1=100 Ω , R2=1 $K\Omega$, C = 5500 μF
   $K=0.9091$
   $\tau=0.5$

This process is inserted into a feedback control loop (figure 5.10) containing a proportional-integral (PI) controller such as: $C(s)=\frac{A}{s}(1+\tau s)$.

e. The expression of the closed-loop transfer function *H(s) = V(s)/U(s* in the form:
   $$H(s)=\frac{1}{(1+\tau_1 s)}$$
   $$H(s)=\frac{C(s)G(s)}{1+C(s)G(s)}=\frac{KA/s}{1+KA/s}=\frac{KA}{s+KA} \quad (5.34)$$
   The canonical form:
   $$H(s)==\frac{1}{\frac{s}{KA}+1}$$

f. The expression of the closed-loop time constant τ_1 as a function of *A* and *K:*
   $$\tau_1=\frac{1}{KA}$$

g. Evaluate *A* for τ_1=0.1
   A=11

In the remainder of the exercise, the reference input is a unit step, and the gain *A* is set such that τ_1=0.1

h. the expression of v(+∞) :

$$v(\infty)=\lim_{t\to\infty} v(t) = \lim_{s\to 0} sV(s) = sH(s)U(s) = s\frac{1}{\tau_1 s+1}\frac{1}{s} = 1 \qquad (5.35)$$

i. The steady-state error ε0(+∞)=e(+∞)−v(+∞)

ε0(+∞)=e(+∞)−v(+∞)=1-1=0

j. The expression of v(t)

$$v(t) = 1 - e^{-\frac{1}{\tau_1}t} \qquad (5.36)$$

- Chapter 6: Frequency Analysis of Linear Systems

6.1 Introductions :

Frequency response analysis plays a fundamental role in understanding the behavior of linear time-invariant (LTI) systems. This analysis is particularly useful in characterizing how a system responds to sinusoidal inputs of varying frequencies, which is essential for both analysis and design in control engineering.

6.2 Definition:

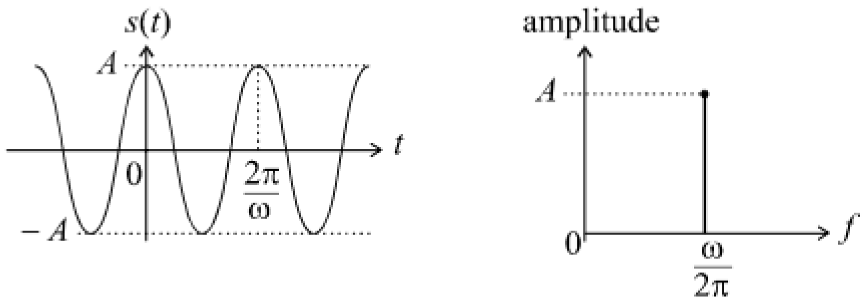
The frequency analysis refers to the steady-state output of a system when subjected to a sinusoidal input. Note that Any input signal can be represented through Fourier analysis as a finite or infinite sum of sinusoidal signals at different frequencies. Therefore, it is essential to understand how a linear system reacts to sinusoidal excitations across a range of frequencies, as this determines the system's performance in practical applications.

6.3 Spectrum concept:

The spectrum of a signal is its representation in the frequency domain that is, a description of how the signal can be decomposed into elementary components, typically sinusoidal functions.

Example 6.1 :

Consider the signal: $s(t) = A\sin(\omega t)$, The corresponding spectrum is illustrated in Figure 6.1.b



(a) (b)

Figure 6.1: Sinusoidal Signal Representations: (a)Time-domain,(b): frequency domain

Let us consider a signal: $s(t) = A_1\sin(\omega_1 t) + A_2\sin(\omega_2 t)$, this signal is a sum of two sinusoidal components with amplitudes A_1 and A_2, and frequencies ω_1 and ω_2, respectively. Each term represents a pure frequency component contributing to the overall shape of the signal in the time domain.

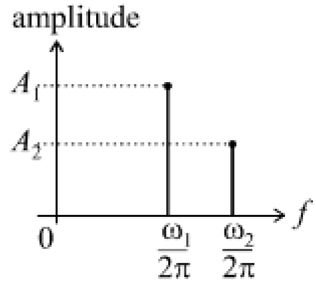


Figure 6.2: Frequency Representation of a Composite Signal

However, the phase information is missing in this type of representation when only the amplitude spectrum is displayed.

6.4 Frequency Response of a Linear System:

When a linear model is excited by a sinusoidal input $u(t) = U_m \sin(\omega t)$, the output of the system will also be sinusoidal $y(t) = A(\omega)U_m \sin(\omega t + \phi)$, but its amplitude and phase may be altered depending on the system's frequency response. The response is therefore sinusoidal with the same frequency as the input but phase-shifted relative to the input.

$$u(t) = U_m \sin(\omega t) \Rightarrow \boxed{G(s)} \Rightarrow y(t) = A(\omega)U_m \sin(\omega t + \phi(\omega))$$

Note that both the amplitude $A(\omega)$ and the phase $\phi(\omega)$ depend on the frequency**.**

6.4.1 Connection with the Transfer Function:

Let us consider the traditional block diagram of a system, translated into its Laplace domain representation**:**

$$Y(s) = G(s)U(s) \tag{6.1}$$

By assuming: $s = j\omega$, we obtain :

$$Y(j\omega) = G(j\omega)U(j\omega) \tag{6.2}$$

In the case of signals with finite energy $Y(j\omega), et\ U(j\omega)$ are the Fourier transforms of the input and output signals represent their frequency-domain representations.

Therefore, we can demonstrate

$$A(\omega) = |G(j\omega)| \text{ : Harmonic Gain}$$

$$\phi(\omega) = \arg(G(j\omega)) \text{ : Phase shift}$$

Thus, if we know the transfer function $G(s)$, we can deduce the complex value of the frequency response $G(j\omega)$ by substituting $s = j\omega$.

However, the information is more useful and easier to interpret when presented in graphical form. A graphical representation of the magnitude and phase of the frequency response provides a clear understanding of how the system modifies the amplitude and phase of different frequency components of the input signal.

6.5 Bode plot

A Bode plot consists of plotting two separate graphs corresponding respectively to the magnitude gain and the phase shift of a system's frequency response. For the magnitude plot, we do not plot the gain $G(j\omega)$ directly. Instead, we use a logarithmic scale and define the gain in decibels (dB) as: $\|G(j\omega)\|_{dB} = 20\log_{10}|G(j\omega)|$. The horizontal axis (frequency) is also represented on a logarithmic scale, which allows for a wide frequency range to be displayed compactly. The phase plot is typically expressed in degrees or radians.

As a general practice, an asymptotic Bode diagram of the transfer function is often drawn. This diagram provides a piecewise linear approximation of the actual plots and serves as a useful tool for quickly estimating the system's behavior, especially in the design and analysis of control systems.

6.6 Plotting Technique :

We will present a systematic method for constructing the asymptotic Bode diagram by first illustrating the plots of basic terms, which will then be generalized to handle any arbitrary transfer function. The advantage of using a logarithmic scale is that it transforms multiplicative terms in the transfer function into additive linear components in the Bode diagram.

6.6.1 Basic terms

Let the magnitude of the transfer function be given in the following form **:**

$$|G(j\omega)| = \left(\frac{\omega}{\omega_0}\right)^n = \left(\frac{2\pi f}{2\pi f_0}\right)^n = \left(\frac{f}{f_0}\right)^n, \tag{6.3}$$

where ω_0 is the cut-off frequency (also called the break frequency). Therefore :

$$\|G(j\omega)\|_{dB} = 20\log_{10}\left(\frac{\omega}{\omega_0}\right)^n = 20n\,\mathrm{og}_{10}\left(\frac{\omega}{\omega_0}\right) \tag{6.4}$$

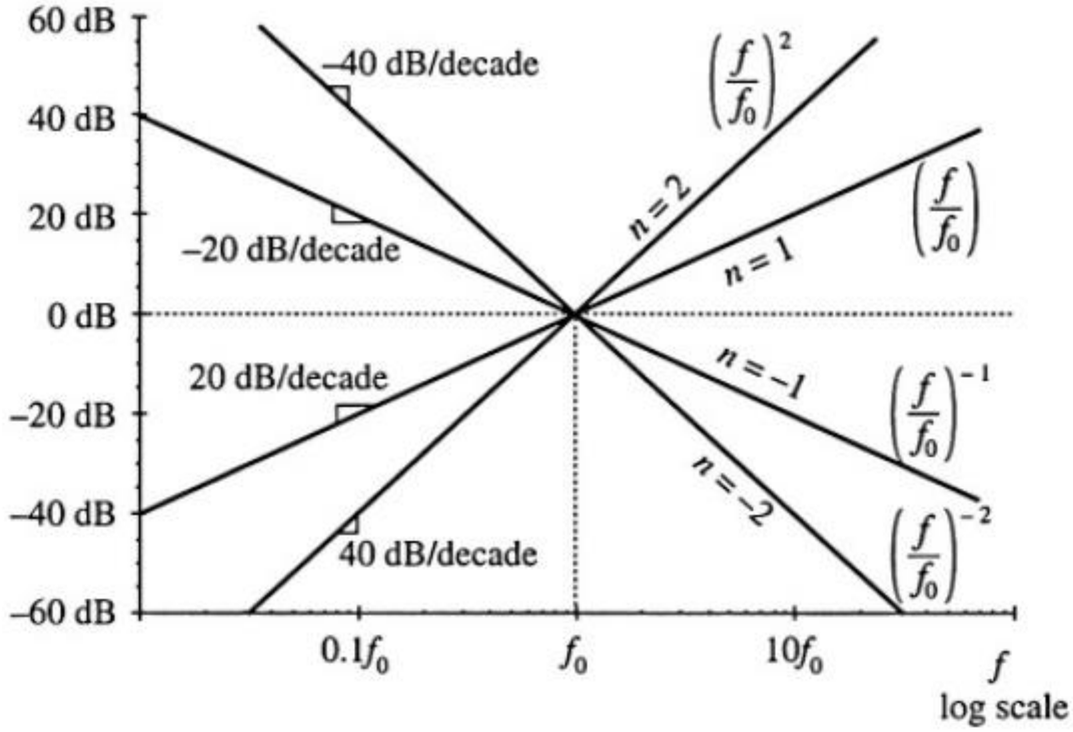


Figure 6.3: magnitude Bode plots of functions which vary as f^n

The equation (6.4) is plotted in Figure 6.3 for a several value of *n* which represents a linear function with a slope equal to 20n dB per decade.

6.6.1.1 First order system (single pole)

Consider the RC circuit shown in Figure 2.1 (chapter2), where the transfer function is :

$$G(s)=\frac{1/RC}{s+1/RC}=\frac{1}{RCs+1} \tag{6.5}$$

The transfer function can be written as :

$$G(s)=\frac{1}{\frac{s}{\omega_0}+1} \tag{6.6}$$

Where $\omega_0=\frac{1}{RC}=\frac{1}{\tau}$. Thus :

$$G(j\omega)=\frac{1}{1+\frac{j\omega}{\omega_0}} \tag{6.7}$$

The magnitude and the phase of the transfer function:

$$|G(j\omega)|=\frac{1}{\sqrt{1+(\frac{\omega}{\omega_0})^2}} \tag{6.8}$$

$$\phi(\omega)=-\tan^{-1}(\frac{\omega}{\omega_0}) \tag{6.9}$$

Here, we have assumed that ω_0 is real. In decibels, the magnitude is:

$$\|G(j\omega)\|_{db}=20\log_{10}\left(\frac{1}{\sqrt{1+(\frac{\omega}{\omega_0})^2}}\right)=-20\log\left(\sqrt{1+(\frac{\omega}{\omega_0})^2}\right) \tag{6.10}$$

The easy way to sketch the magnitude Bode plot of G is to investigate the asymptotic behavior for large and small frequencies.

For small frequency, $\omega<<\omega_0$ and $f<<f_0$: is true that

$$\left(\frac{\omega}{\omega_0}\right)<<1 \tag{6.11}$$

The term of $\left(\omega/\omega_0\right)^2$ the equation. (6.8) is therefore much smaller than 1, and hence equation (6.8) becomes:

$$|G(j\omega)| = \frac{1}{\sqrt{1}} = 1 \qquad (6.12)$$

In decibels, the magnitude is approximately zero.

The phase will be :

$$\phi(\omega) = 0 \qquad (6.13)$$

For high frequency, $\omega >> \omega_0$ and $f >> f_0$: is true that :

$$\left(\frac{\omega}{\omega_0}\right) >> 1 \qquad (6.14)$$

We can say that :

$$1 + \left(\frac{\omega}{\omega_0}\right)^2 \approx \left(\frac{\omega}{\omega_0}\right)^2 \qquad (6.15)$$

Hence equation (6.8) becomes:

$$|G(j\omega)| = \frac{1}{\sqrt{(\frac{\omega}{\omega_0})^2}} = \left(\frac{\omega}{\omega_0}\right)^{-1} \qquad (6.16)$$

In decibels, the magnitude is approximately.:

$$\|G(j\omega)\|_{db} = -20\log\left(\frac{\omega}{\omega_0}\right) \qquad (6.17)$$

The phase becomes :

$$\phi(\omega) = -90^\circ \qquad (6.18)$$

At $\omega = \omega_0$, $\|G(j\omega)\|_{db} = 20\log(\frac{1}{\sqrt{2}}) = -3dB$ and $\phi(\omega) = -45^\circ$

The asymptotes of $|G(j\omega)|$ are equal to 1 at low frequency, and at high frequency. The asymptotes intersect at ω_0 (Figure 6.4). The actual magnitude tends toward these asymptotes at very low frequency and very high frequency. In the vicinity of the corner frequency, the actual curve deviates somewhat from the asymptotes. In the case of phase, Since the high-frequency and low-frequency phase asymptotes do not intersect, we need a third asymptote to approximate the phase in the vicinity of the corner frequency One way to do this is illustrated in Figure 6.4, where the slope of the asymptote is chosen to be identical to the slope of the actual curve at It can be shown that, with this choice, the asymptote intersection frequencies and are given by:

$$\begin{aligned} \omega_a &= \omega_0 / 10 \\ \omega_b &= 10\omega_0 \end{aligned} \tag{6.19}$$

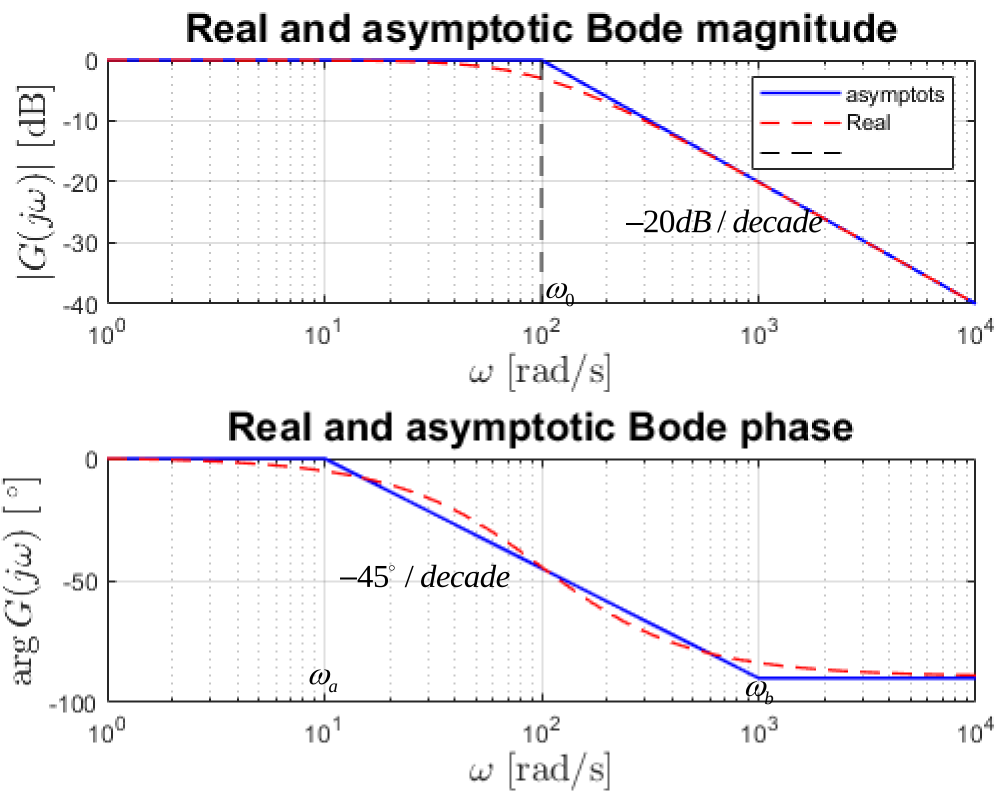


Figure 6.4: Bode plot of single pole (first order system)

Note: figure 6.4 is generated using the Matlab function presented in the appendix

6.6.1.2 Single zero

The transfer function of a single zero is :

$$G(s)=1+\frac{s}{\omega_0} \tag{6.20}$$

Therefore :

$$G(j\omega)=1+\frac{j\omega}{\omega_0} \tag{6.21}$$

The magnitude and the phase of a single zero are :

$$\begin{aligned} &|G(j\omega)|=\sqrt{1+\left(\frac{\omega}{\omega_0}\right)^2} \Rightarrow \|G(j\omega)\|_{dB}=20\log\sqrt{1+\left(\frac{\omega}{\omega_0}\right)^2} \\ &\phi(\omega)=\tan^{-1}\left(\frac{\omega}{\omega_0}\right) \end{aligned} \tag{6.22}$$

For small frequency, $\omega << \omega_0$:

$$\begin{aligned} &|G(j\omega)|=\frac{1}{\sqrt{1}}=1 \\ &\phi(\omega)=0 \end{aligned} \tag{6.23}$$

For high frequency $\omega >> \omega_0$

$$\begin{aligned} &|G(j\omega)|=\left(\frac{\omega}{\omega_0}\right) \\ &\phi(\omega)=90^{\circ} \end{aligned} \tag{6.24}$$

The bode plot of a single zero is presented in Figure 6.5 :

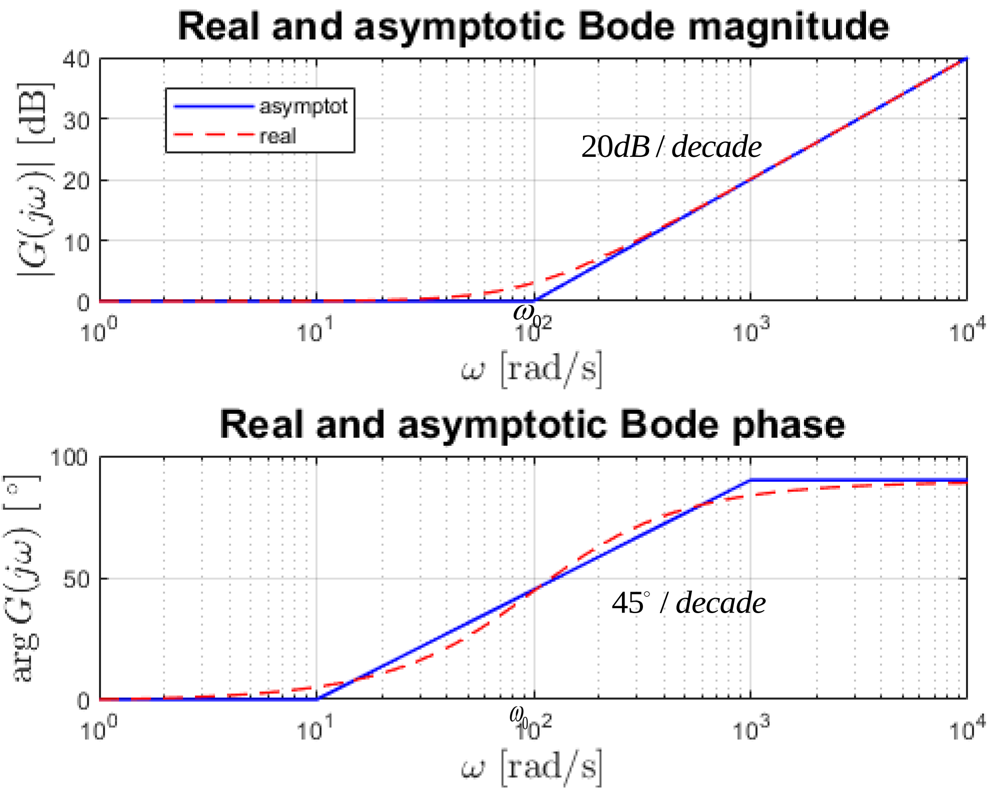


Figure 6.5: Bode plot of single zero

6.6.1.3 Right half plane zero

The transfer function of a right-half plane zero is :

$$G(s) = 1 - \frac{s}{\omega_0} \tag{6.25}$$

Therefore :

$$G(j\omega) = 1 - \frac{j\omega}{\omega_0} \tag{6.26}$$

The magnitude and the phase of a single zero are :

$$\begin{aligned} |G(j\omega)| &= \sqrt{1+\left(\frac{\omega}{\omega_0}\right)^2} \Rightarrow \|G(j\omega)\|_{dB} = 20\log\sqrt{1+\left(\frac{\omega}{\omega_0}\right)^2} \\ \phi(\omega) &= -\tan^{-1}\left(\frac{\omega}{\omega_0}\right) \end{aligned} \tag{6.27}$$

For small frequency, $\omega << \omega_0$:

$$|G(j\omega)| = \frac{1}{\sqrt{1}} = 1$$
$$\phi(\omega) = 0 \tag{6.28}$$

For high frequency $\omega >> \omega_0$

$$|G(j\omega)| = \left(\frac{\omega}{\omega_0}\right)$$
$$\phi(\omega) = -90^{\circ} \tag{6.29}$$

The bode plot of the magnitude of a right-half plane zero is the same as the case of a single zero, but the phase is inverted as shown in Figure 6.6

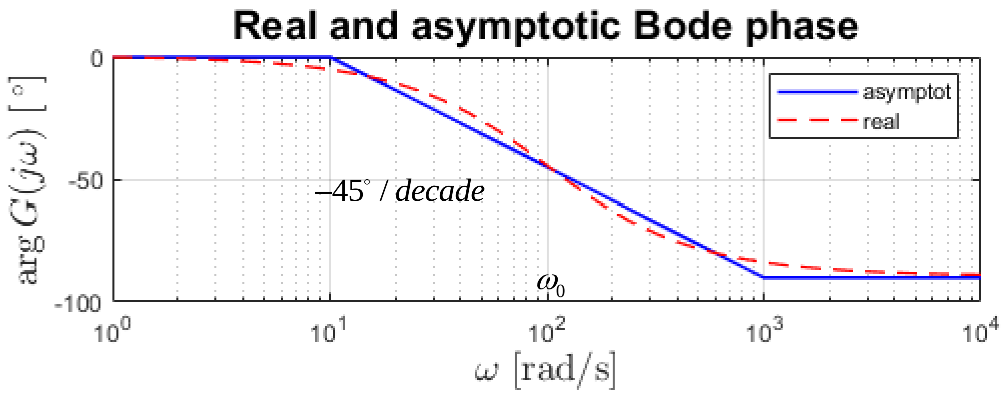


Figure 6.6: phase plot of RHP zero

6.6.1.4Combination of terms:

The Bode diagram of a transfer function composed of multiple poles, zeros, and gain terms can be constructed by superposition. At any given frequency, the magnitude (expressed in decibels) of the overall transfer function is equal to the sum of the magnitudes (in decibels) of its individual components. Similarly, the phase of the composite transfer function at that frequency is equal to the sum of the phases contributed by each individual term.

This additive property, which results from the logarithmic scale used in Bode plots, enables a straightforward and systematic construction of the overall diagram by analyzing and combining the elementary effects of basic components (constant gains, poles, zeros, etc.).

Let's consider the transfer function $G(s) = G_1(s)G_2(s)$, such as $G_1(s) = R_1(\omega)e^{j\theta_1}$ and $G_2(s) = R_2(\omega)e^{j\theta_2}$, therefore:

$$G(s) = R_1(\omega)R_2(\omega)e^{j(\theta_1+\theta_2)} \tag{6.30}$$

Hence, the composite phase is :

$$\theta(\omega) = \theta_1(\omega) + \theta_2(\omega) \tag{6.31}$$

The total magnitude is :

$$R(\omega) = R_1(\omega)R_2(\omega) \tag{6.32}$$

When expressed in decibels becomes

$$\|R(\omega)\|_{dB} = \|R_1(\omega)\|_{dB} + \|R_2(\omega)\|_{dB} \tag{6.33}$$

Thus, the composite phase of a transfer function is the sum of the individual phase contributions from each pole and zero. Likewise, when the magnitude is expressed in decibels, the composite magnitude is the sum of the individual magnitudes of the constituent terms. As a consequence, the slope of the composite magnitude plot (in dB per decade) is also the algebraic sum of the individual slopes (in dB/decade) contributed by each pole and zero.

Example; let's consider the transfer function :

$$G(s) = \frac{G_0}{(1+\frac{s}{\omega_1})(1+\frac{s}{\omega_2})} \tag{6.34}$$

Where $G_0 = 40, \omega_1 = 100 rad/\sec, \omega_2 = 1000 rad/\sec$

The bode plot of equation 6.34 is presented in the figure 6.7:

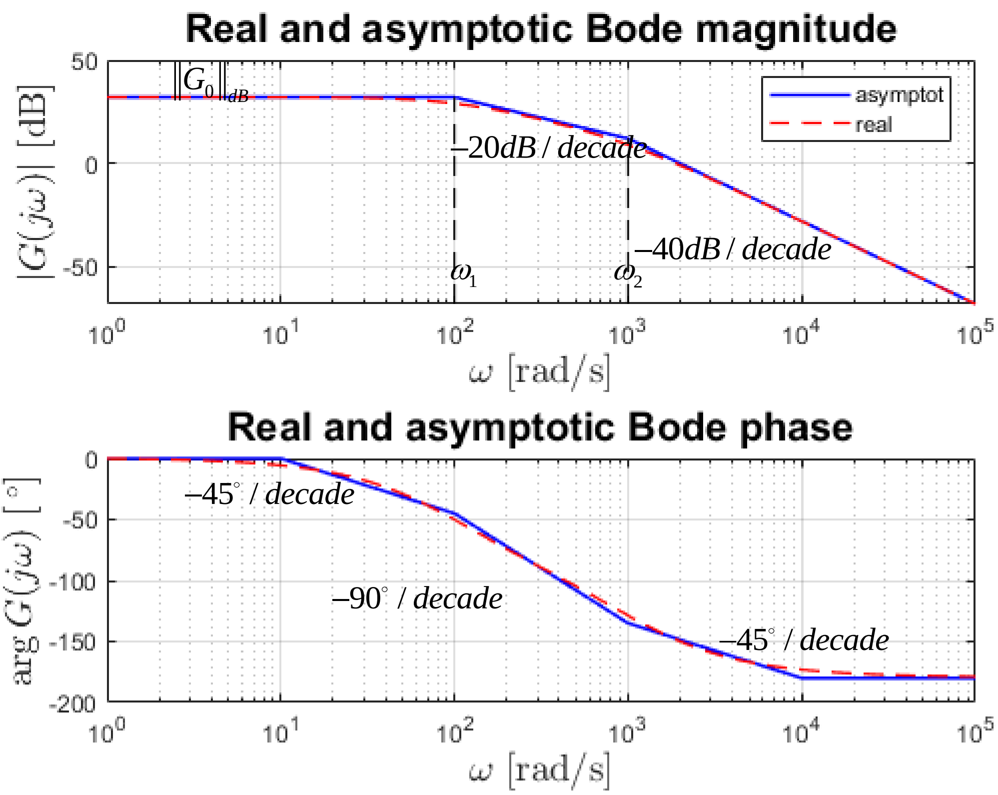


Figure 6.7: Bode plot of combination terms

6.6.1.4 Second order system

Let's consider a canonical form of second order transfer function:

$$G(s)=\frac{1}{1+2\zeta\frac{s}{\omega_0}+\left(\frac{s}{\omega_0}\right)^2}=\frac{1}{1+\frac{s}{Q\omega_0}+\left(\frac{s}{\omega_0}\right)^2} \tag{6.35}$$

Where $Q=\frac{1}{2\zeta}$ is the quality factor et its definition is :

$$Q=2\pi\frac{\text{peak energy}}{\text{dissipitated energy per cycle}}$$

As we have seen in Chapter 5, the type of roots depends on the value of ζ . Therefore, when the damping factor is greater than one, the roots are real and the bode plot is constructed as described in the previous section. When the damping factor is between $0<\zeta<1$, the poles are complexes and the magnitude of the transfer function is :

$$|G(j\omega)|=\left|\frac{1}{1+j\frac{\omega}{Q\omega_0}+\left(\frac{-\omega}{\omega_0}\right)^2}\right|=\frac{1}{\sqrt{\left(1-\left(\frac{\omega}{\omega_0}\right)^2\right)^2+\frac{1}{Q^2}\left(\frac{\omega}{\omega_0}\right)^2}} \tag{6.36}$$

The phase :

$$\phi(\omega) = -\tan^{-1}\left|\frac{\frac{1}{Q}\left(\frac{\omega}{\omega_0}\right)}{1-\left(\frac{\omega}{\omega_0}\right)^2}\right| \tag{6.37}$$

For small frequency, $\omega << \omega_0$:

$$\begin{aligned} |G(j\omega)| &= 1 \\ \phi(\omega) &= 0 \end{aligned} \tag{6.38}$$

For high frequency $\omega >> \omega_0$

$$\begin{aligned} |G(j\omega)| &= \left(\frac{\omega}{\omega_0}\right)^{-2} \\ \phi(\omega) &= -180^\circ \end{aligned} \tag{6.39}$$

When $\omega = \omega_0$,

$$|G(j\omega)| = Q \tag{6.40}$$

Figure 6.8 shows the real and the asymptotic bode plots of a second-order system with complex poles. The high-frequency asymptote has a slope of -40 dB/ decade. The asymptotes intersect at and are independent of *Q*. The parameter *Q* affects the deviation of the real curve from the asymptotes, in the neighborhood of the corner frequency ω_0. The exact transfer function has magnitude *Q* at the corner frequency ω_0. The phase tends to 0 degrees at low frequency and to -180 degrees at high frequency. At $\omega = \omega_0$ the phase is –90°.

6.6.2 Phase Margin and Gain Margin :

Phase margin and gain margin are fundamental stability indicators in the frequency domain, particularly when analyzing feedback control systems using Bode plots.

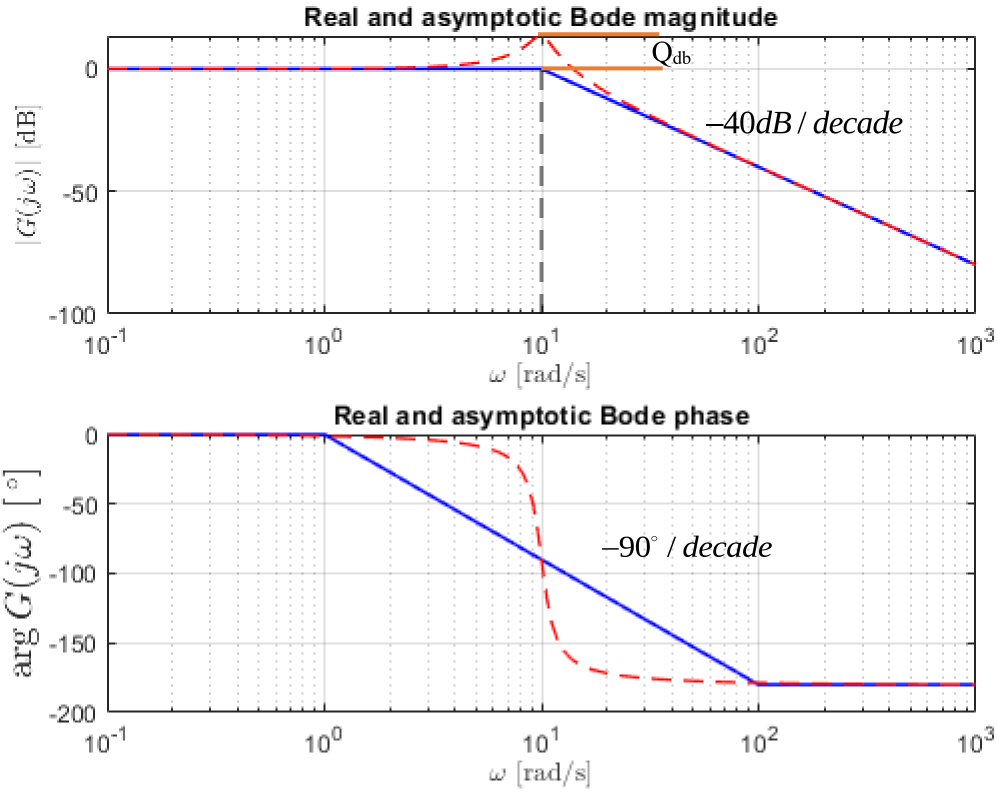


Figure 6.7: Bode plot of second-order system

- **Phase margin (PM)** is defined as the amount of additional phase lag required to bring the system to the verge of instability. It is measured at the **gain crossover frequency** ω_{gc}, which is the frequency at which the open-loop magnitude $|G(j\omega)|$ is equal to 1 (or 0 dB). Mathematically:

$$\text{Phase Margin} = 180^\circ + \arg(G(j\omega_{gc}))$$

  A larger positive phase margin indicates greater relative stability.

- **Gain margin (GM)** is the amount of gain increase (usually expressed in dB) required to make the system unstable. It is measured at the **phase crossover frequency** ω_{pc} which is the frequency at which the phase of the open-loop transfer function is -180°. Mathematically:

$$\text{Gain Margin (dB)} = -20\log_{10}|G(j\omega)|$$

  A larger gain margin implies the system can tolerate more gain before becoming unstable.

6.7 Nyquist Diagram

The Nyquist diagram is a graphical representation of a system's frequency response in the complex plane. It represents the transfer function in one graph.

The Nyquist plot is obtained by evaluating the open-loop transfer function $G(j\omega)$ over a range of frequencies ω∈[0,∞], and plotting the resulting complex values as points in the complex plane. Each point corresponds to a value $G(j\omega)$ with the horizontal axis representing the real part and the vertical axis the imaginary part.

Example:

The polar plot of a sinusoidal (figure (6.8))transfer function $G(j\omega)$ is a graphical representation in polar coordinates, where the magnitude $G(j\omega)$ is plotted as a function of the phase angle arg($G(j\omega)$) as the frequency ω varies from zero to infinity.

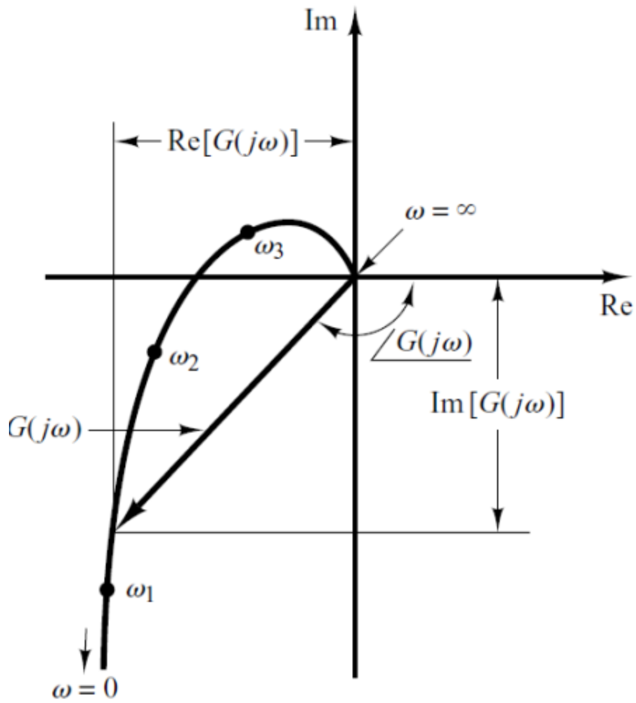


Figure6.8: Nyquit diagram of a sinusoidal function

6.7.1 Nyquist Plot of Integral and Derivative Factors

The polar plot of $G(j\omega) = \dfrac{1}{j\omega}$ is the negative imaginary as illustrated in Figure 6.9.a.

The polar plot of $G(j\omega) = j\omega$ is the positive imaginary as illustrated in Figure 6.9.b.

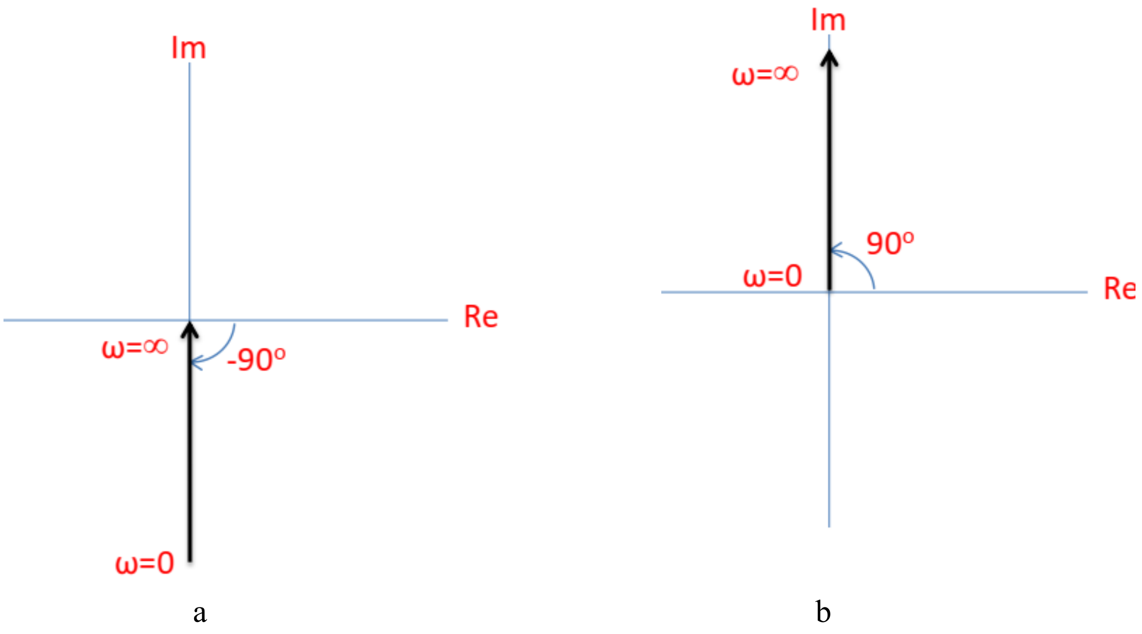


Figure 6.9: Nyquist diagram **of** Integral and Derivative Factors

6.7.2 Nyquist Plot of first-order system

Let's consider the transfer function of the first-order system as :

$$G(j\omega)=\frac{1}{j\omega+1} \tag{6.41}$$

$$G(j\omega)=\frac{1}{j\omega+1}.\frac{-j\omega+1}{-j\omega+1}=\frac{1}{1+\omega^2}-j\frac{\omega}{1+\omega^2} \tag{6.42}$$

The plot of this first-order system is shown in Figure 6.10

ω	Re	Im
0	1	0
0.5	0.8	-0.4
1	0.5	-0.5
2	0.2	-0.4
∞	0	undefined

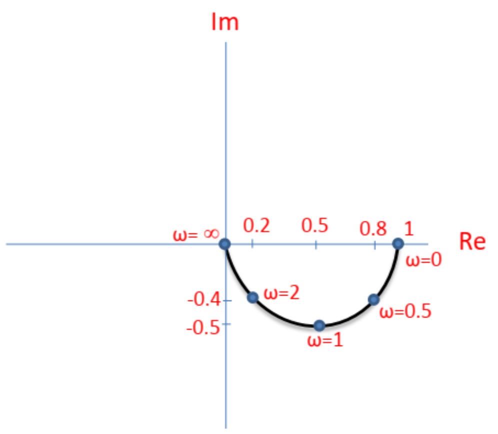


Figure 6.10: Nyquist plot of first-order system

6.7.3 Nyquist Plot transfer function

Let’s consider an arbitrary transfer function as :

$$G(s)=\frac{1}{s(s+1)} \tag{6.43}$$

Let's calculate $G(j\omega)$ by putting $s = j\omega$:

$$G(j\omega)=\frac{1}{j\omega(j\omega+1)} \tag{6.44}$$

$$G(j\omega)=\frac{1}{-\omega^2 + j\omega} \tag{6.45}$$

$$G(j\omega)=\frac{-\omega^2 - j\omega}{\omega^2+\omega^4} \tag{6.46}$$

The plot of this function transfer is shown in Figure 6.11

ω	Re	Im
0	-1	∞
0.1	-1	-10
0.5	-0.8	-1.6
1	-0.5	-0.5
2	-0.2	-0.1
3	-0.1	-0.03
∞	0	0

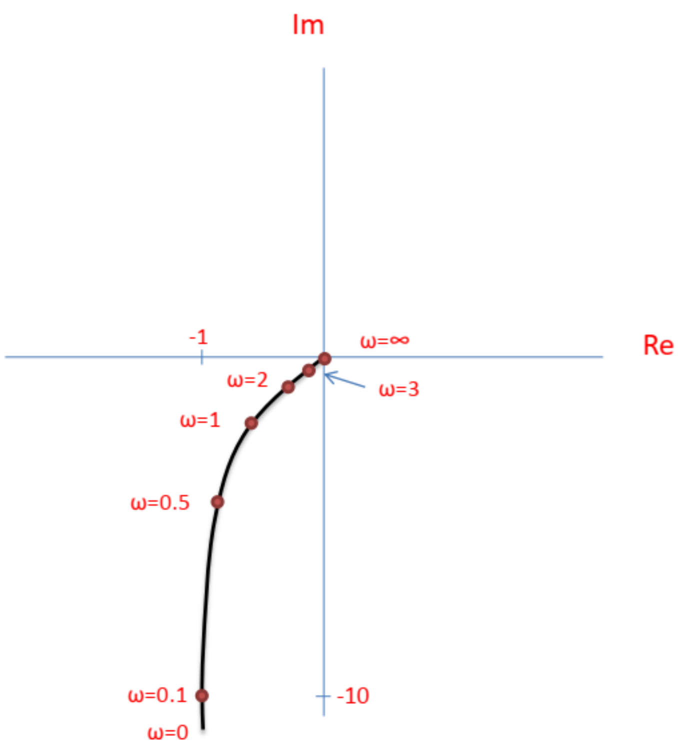


Figure 6.11: Nyquist plot of transfer function

Remark: The Nyquist plot is particularly useful for analyzing the stability of closed-loop systems using the Nyquist stability criterion.

6.8 Exercises

Exercise 6.1: inverted Frequency

- Plot the asymptotic Bode diagram for the following transfer functions**:**

$$G(s) = \frac{1}{(1+\frac{\omega_0}{s})} \qquad\qquad G(s) = (1+\frac{\omega_0}{s})$$

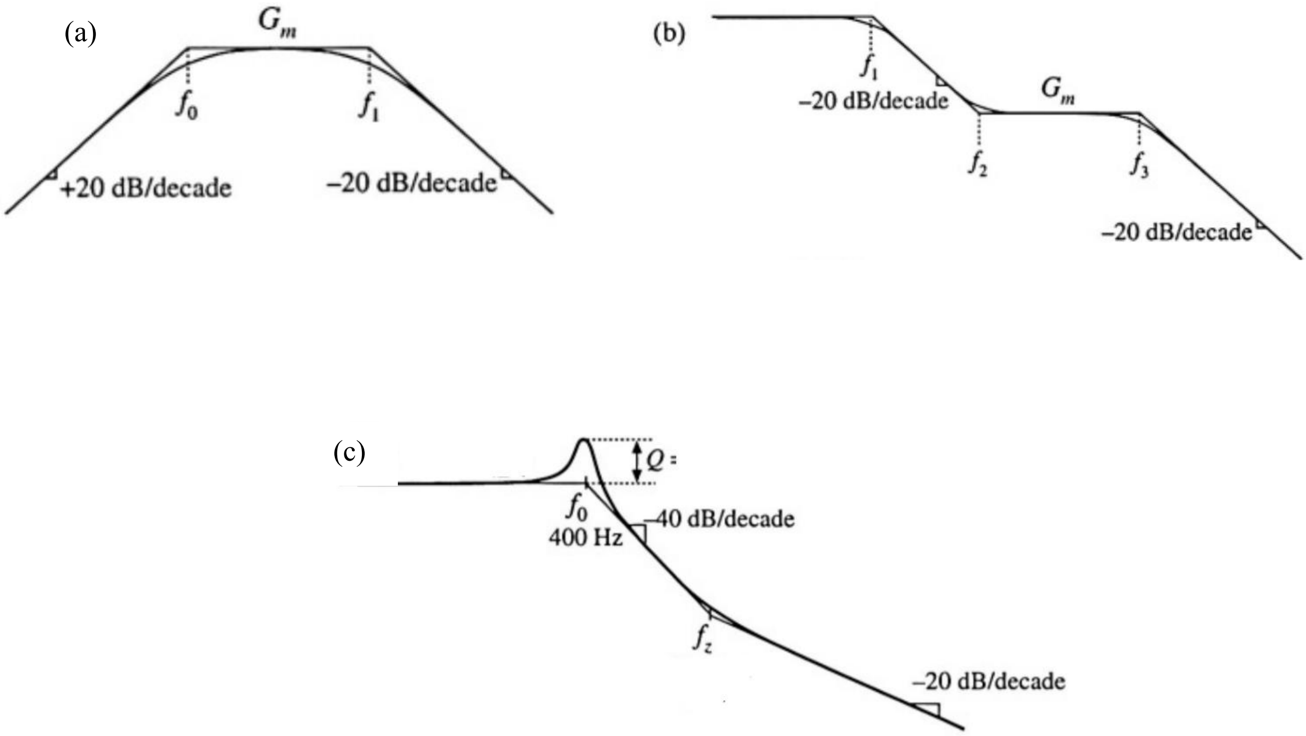
Exercise 6.2: Given the transfer function:

$$G(s) = G_0 \frac{1+\frac{s}{\omega_0}}{(1+\frac{s}{\omega_1})(1+\frac{s}{\omega_2})}$$

Plot the asymptotic Bode diagram for the transfer function in the following cases**:**

- $\omega_1 < \omega_0 < \omega_2$
- $\omega_0 < \omega_2 < \omega_1$

Exercise 6.3: Given the Bode magnitude plots of the following transfer functions:



- Express the transfer functions represented by the asymptotes in Figures (a), (b), and (c) in terms of factored pole-zero form. Assume that all poles and zeros have negative real parts.

6.9 Exercises solution :

Exercise 6.1:

The asymptotic Bode diagram for the following transfer functions:

a. $$G(s)=\frac{1}{(1+\frac{\omega_0}{s})}=\frac{\frac{s}{\omega_0}}{1+\frac{s}{\omega_0}} \tag{6.47}$$

So, we replace $s = j\omega$:

$$G(j\omega)=\frac{1}{1+\frac{\omega_0}{j\omega}}=\frac{1}{1-j\frac{\omega_0}{\omega}} \tag{6.48}$$

The magnitude and the phase of this inverted pole (equation 6.47) is:

$$|G(j\omega)|=\frac{1}{\sqrt{1+\left(\frac{\omega_0}{\omega}\right)^2}} \tag{6.49}$$

$$\phi(\omega)=-\tan^{-1}(-\frac{\omega_0}{\omega}) \tag{6.50}$$

For small frequency, $\omega << \omega_0$:

$$|G(j\omega)|=\frac{\omega}{\omega_0}\Rightarrow|G(j\omega)|_{dB}=20\log_{10}\left(\frac{\omega}{\omega_0}\right)$$
$$\phi(\omega)=90^{\circ} \tag{6.51}$$

For high frequency $\omega >> \omega_0$

$$|G(j\omega)|=1\Rightarrow|G(j\omega)|_{dB}=20\log_{10}(1)=0$$
$$\phi(\omega)=0 \tag{6.52}$$

The bode plot of an inverted pole is presented in Figure 6.12 :

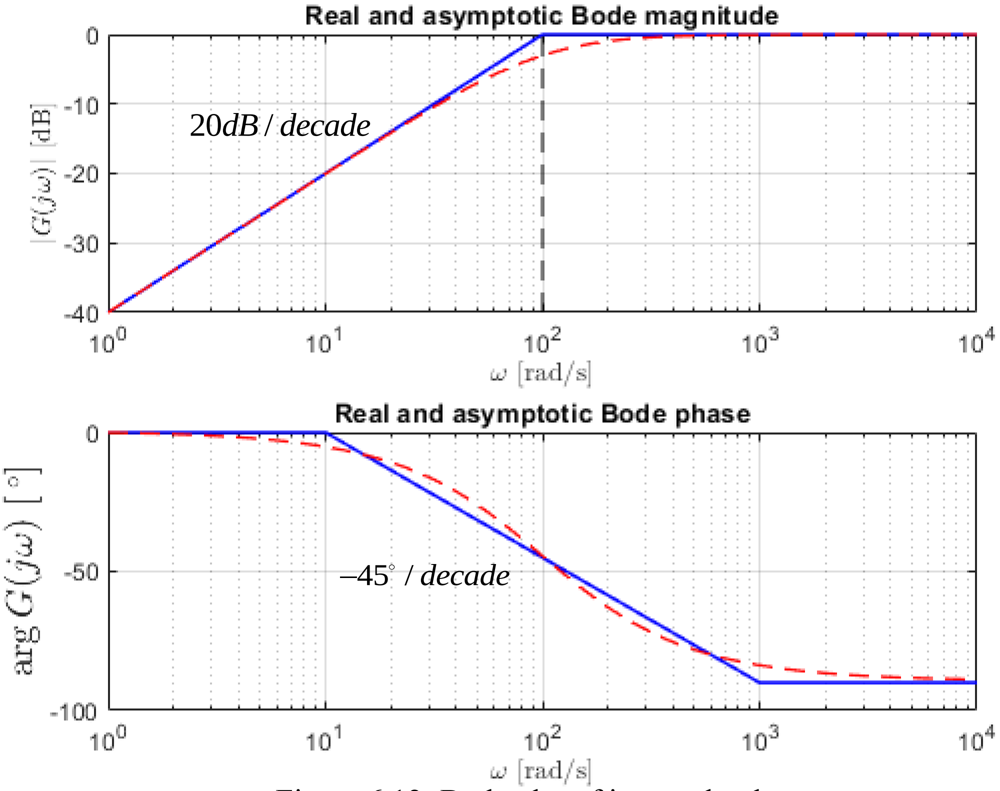


Figure 6.12: Bode plot of inverted pole

b. $G(s) = (1 + \frac{\omega_0}{s}) = \frac{(1 + \frac{s}{\omega_0})}{\frac{s}{\omega_0}}$ (6.53)

So, we replace $s = j\omega$:

$$G(j\omega) = 1 + \frac{\omega_0}{j\omega} = 1 - j\frac{\omega_0}{\omega} \tag{6.54}$$

The magnitude and the phase of this inverted zero (equation 6.47) is:

$$|G(j\omega)| = \sqrt{1 + \left(\frac{\omega_0}{\omega}\right)^2} \tag{6.55}$$

$$\phi(\omega) = \tan^{-1}(-\frac{\omega_0}{\omega}) \tag{6.56}$$

For small frequency, $\omega << \omega_0$:

$$|G(j\omega)| = \left(\frac{\omega}{\omega_0}\right)^{-1} \Rightarrow |G(j\omega)|_{dB} = -20\log_{10}\left(\frac{\omega}{\omega_0}\right) \qquad (6.57)$$
$$\phi(\omega) = -90^\circ$$

For high frequency $\omega >> \omega_0$

$$|G(j\omega)| = 1 \Rightarrow |G(j\omega)|_{dB} = 20\log_{10}(1) = 0 \qquad (6.58)$$
$$\phi(\omega) = 0$$

The bode plot of an inverted zero is presented in Figure 6.13 :

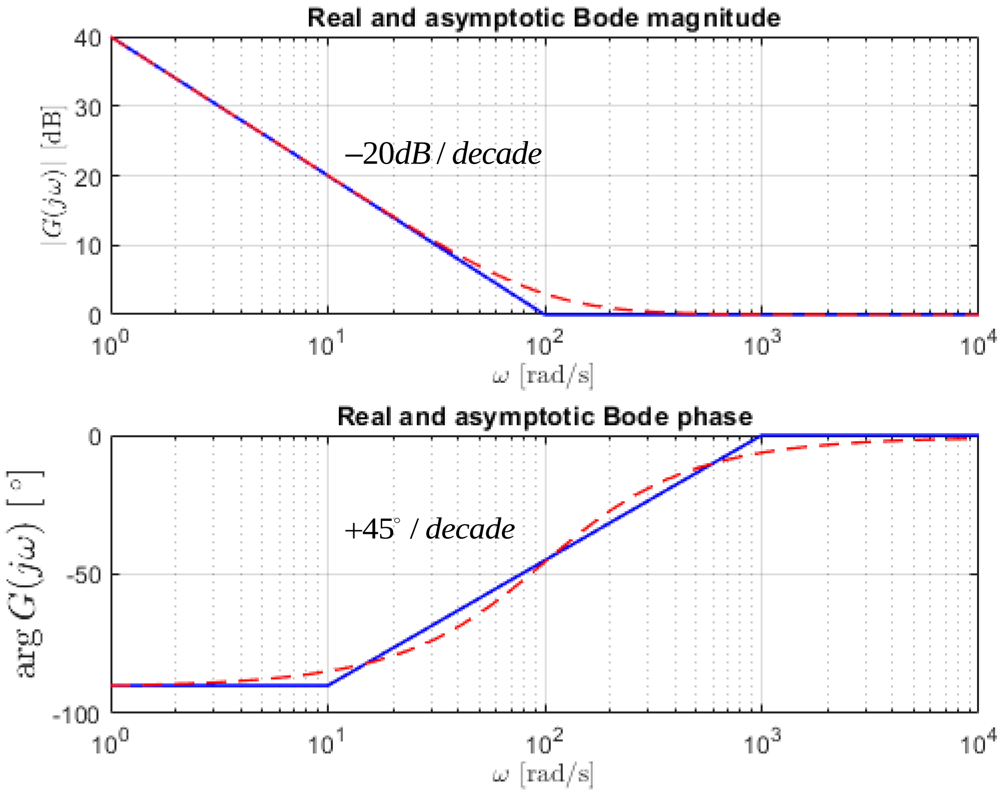


Figure 6.13: Bode plot of inverted zero

Exercise 6.2: Given the transfer function:

$$G(s) = G_0 \frac{1+\dfrac{s}{\omega_0}}{(1+\dfrac{s}{\omega_1})(1+\dfrac{s}{\omega_2})}$$

The asymptotic Bode diagram for the transfer function in the following cases**:**

- **Case 01 :** $\omega_1 < \omega_0 < \omega_2$

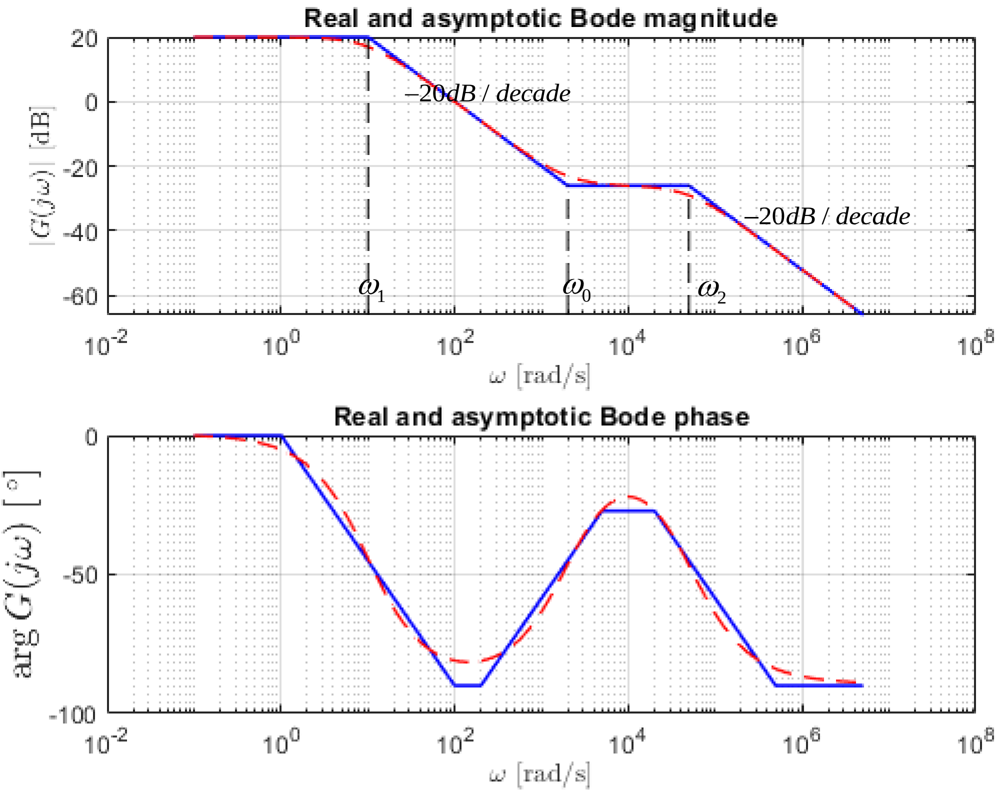


Figure 6.14: Bode plot of case1

- **Case2 :** $\omega_0 < \omega_2 < \omega_1$

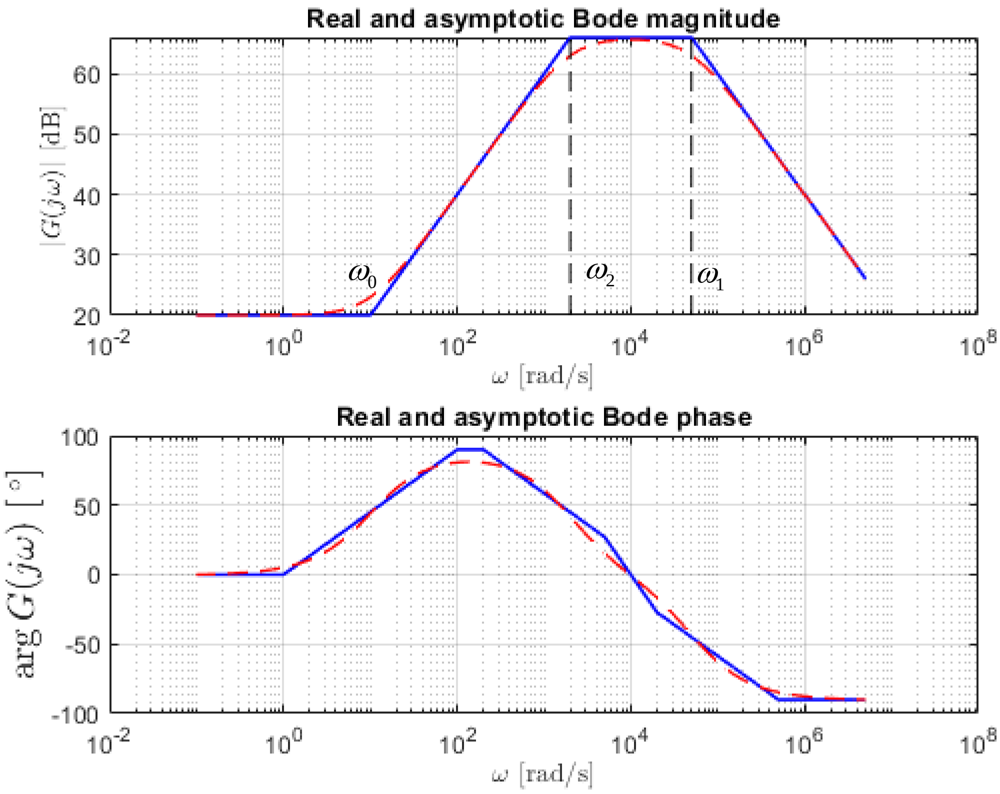
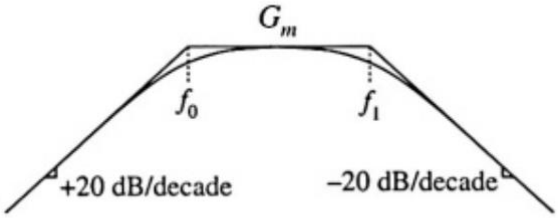


Figure 6.15: Bode plot of case2

Exercise 6.3 :

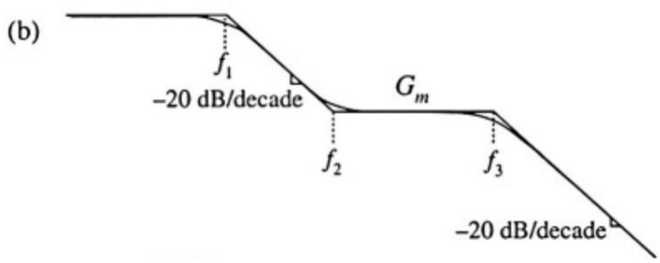
The transfer functions of the following asymptotic Bode plot

a.



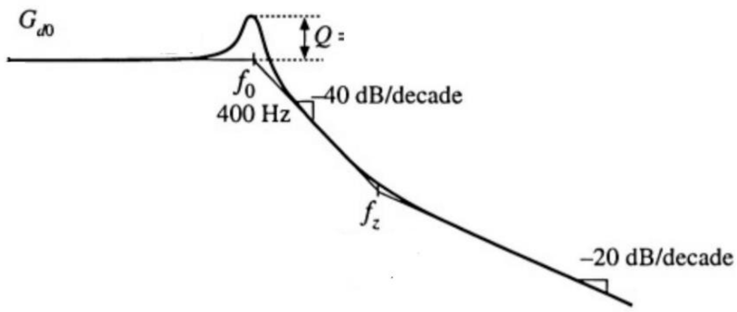
$$G(s)=\frac{G_m}{(1+\frac{\omega_0}{s})(1+\frac{s}{\omega_1})}=G_m\frac{(1+\frac{s}{\omega_0})}{(1+\frac{s}{\omega_0})(1+\frac{s}{\omega_1})}\text{ where }\omega_0=2\pi f_0\text{, and }\omega_1=2\pi f_1$$

b.



$$G(s)=G_m\frac{(1+\frac{s}{\omega_2})}{(1+\frac{s}{\omega_1})(1+\frac{s}{\omega_3})}\quad\text{where }\omega_i=2\pi f_i\text{, and }\ \omega_1<\omega_2<\omega_3$$

c.



$$G(s)=\frac{(1+\frac{s}{\omega_z})}{1+\frac{s}{Q\omega_0}+\left(\frac{s}{\omega_0}\right)^2}\text{ where }\omega_i=2\pi f_i$$

- Chapter 7: Stability of Linear Systems

7.1 Introduction :

In control theory, the stability of a linear system refers to its ability to produce a bounded output for every bounded input. This property is fundamental for the correct and safe operation of control systems. On the contrary, if the system is unstable, even a very small input signal can lead to an output that diverges toward infinity, which, in practice, may result in material damage and potentially cause harm to humans.

The analysis of stability depends on the system model, typically represented in the time domain (state-space representation) or the frequency domain (transfer function).

There are several criteria available to determine whether a linear system is stable or not. In what follows, we will present the most essential and widely used ones.

7.2 Root Location Criterion

The Root Location Criterion (also known as the Pole Location Criterion) is one of the most fundamental methods for assessing the stability of a linear time-invariant (LTI) system. A linear system is asymptotically stable if and only if all of its poles have strictly negative real parts. Recall that poles are the roots of the denominator of the system's transfer function. The presence of even one pole in the right half of the complex plane inevitably leads to instability. It has already been discussed that poles affect the system's response time and oscillatory behavior. Here, we see that their role is even more critical, as they fundamentally determine the stability of the system.

Note that If one or more poles lie exactly on the imaginary axis (i.e., have zero real part), and none of them are repeated, the system is marginally stable.

Proof: let's consider the transfer function

$$G(s)=\frac{(s-z_1)(s-z_2)............(s-z_{m-1})(s-z_m)}{(s-p_1)(s-p_2)............(s-p_{n-1})(s-p_n)} \qquad (7.1)$$

Where $n>m$

The transfer function $G(s)$ can be written as :

$$G(s)=\sum_{i=1}^{n}\frac{A_i}{(s-p_i)} \qquad (7.2)$$

The inverse transform of the equation (7.2) is :

$$g(t) = A_1 e^{p_1 t} + A_2 e^{p_2 t} + ... + A_n e^{p_n t} \tag{7.3}$$

From Equation (7.3), it is evident that if any pole has a positive real part, the time-domain response *g(t)* will grow unbounded, eventually diverging to infinity.

7.3 ROUTH'S stability Criterion

Consider a closed-loop transfer function:

$$G(s) = \frac{b_m s^m + b_{m-1} s^{m-1} + ... + b_1 s + b_0}{a_n s^n + a_{n-1} s^{n-1} + ... + a_1 s + a_0} = \frac{B(s)}{A(s)} \tag{7.4}$$

Where the *ai* s and *bi* s are real constants and $m \le n$. An alternative to factoring the denominator polynomial, Routh's stability criterion, determines the number of closed-loop poles in the right-half *s* plane.

7.3.1 Algorithm for applying Routh's stability criterion:

The algorithm described below, like the stability criterion, requires the order of *denominator of G(s)* to be finite.

- Characteristic polynomial:

$$A(s) = a_n s^n + a_{n-1} s^{n-1} + ... + a_1 s + a_0 \tag{7.5}$$

- Construct a table with $n + 1$ rows from the coefficients a_i of a polynomial $A(s)$ as:

s^n	a_n	a_{n-2}	a_{n-4}	….	a_0
s^{n-1}	a_{n-1}	a_{n-3}	a_{n-5}	….	0
s^{n-2}	$b_{n-1} = -\frac{\begin{vmatrix} a_n & a_{n-2} \\ a_{n-1} & a_{n-3} \end{vmatrix}}{a_{n-1}}$	$b_{n-3} = -\frac{\begin{vmatrix} a_n & a_{n-4} \\ a_{n-1} & a_{n-5} \end{vmatrix}}{a_{n-1}}$	b_{n-5}	….	0
s^{n-3}	$c_{n-1} = -\frac{\begin{vmatrix} a_{n-1} & a_{n-3} \\ b_{n-1} & b_{n-3} \end{vmatrix}}{b_{n-1}}$	$c_{n-3} = -\frac{\begin{vmatrix} a_{n-1} & a_{n-5} \\ b_{n-1} & b_{n-5} \end{vmatrix}}{b_{n-1}}$	c_{n-5}	….	0
……	…….	………	…….	….	….
s^0	a_0	0	0	…..	0

- Count the number of sign changes in the first column of the array. It can be shown that a necessary and sufficient condition for all roots of (2) to be located in the left-half plane is that all the *ai* are positive and all of the coefficients in the first column be positive.
- Relates the number of sign changes in the first column of the table to the number of roots in the closed right half-plane

Example 7.1: **Generic Cubic Polynomial.**

Consider the generic cubic polynomial:

$$A(s) = a_0 s^3 + a_1 s^2 + a_2 s + a_3 \tag{7.6}$$

Where all the a_i are positive. The Routh table is:

s^3	a_0	a_2
s^2	a_1	a_3
s^1	$\dfrac{a_1 a_2 - a_0 a_3}{a_1}$	0
s^0	a_3	0

So the condition that all roots have negative real parts is:

$$a_1 a_2 - a_0 a_3 > 0 \tag{7.7}$$

Example 7.2 :

So far, we have discussed only one primary application of the Routh–Hurwitz criterion, namely, determining the number of roots with nonnegative real parts, which directly indicates the stability of a linear system. However, the Routh criterion can also be used as a design tool specifically, to determine allowable ranges for system parameters to ensure stability. This makes it particularly useful in control system design and tuning.

Consider, for example, a system whose closed-loop transfer function is given by:

$$G(s) = \frac{K}{s(s^2 + s + 1)(s + 2) + K} \tag{7.8}$$

The characteristic equation is :

$$s^4 + 3s^3 + 3s^2 + 2s + K \tag{7.9}$$

The Routh array is :

s^4	1	3	K
s^3	3	2	0
s^2	$7/3$	K	0
s^1	2-9K/7	0	0
s^0	K	0	0

The condition for the system to be stable is that the parameter K must be:

$$0 < K < 14/9$$

7.4 Stability criterion based on the frequency response

There exists a stability criterion based on the frequency response of systems, known as the reverse criterion. Historically, it was introduced in the context of the Nyquist plot and is derived from the more comprehensive Nyquist stability criterion, which forms the basis of many developments in control theory.

7.4.1 Phase margin condition

A unity-feedback system is stable when, on the Bode plot of the corresponding open-loop transfer function, the following conditions are satisfied:

- At the gain crossover frequency (i.e., the frequency where the magnitude is 0 dB), the phase is greater than −180°. In other words, the phase margin is positive

7.4.2 Nyquist criterion

If the Nyquist plot of the open-loop transfer function *G(s)H(s)*of Figure 3.7 constructed using the standard Nyquist contour in the complex s-plane, encircles the critical point (−1+j0) in the counterclockwise direction as many times as there are right-half-plane (RHP) poles of *G(s)*, then the closed-loop system is stable**.**

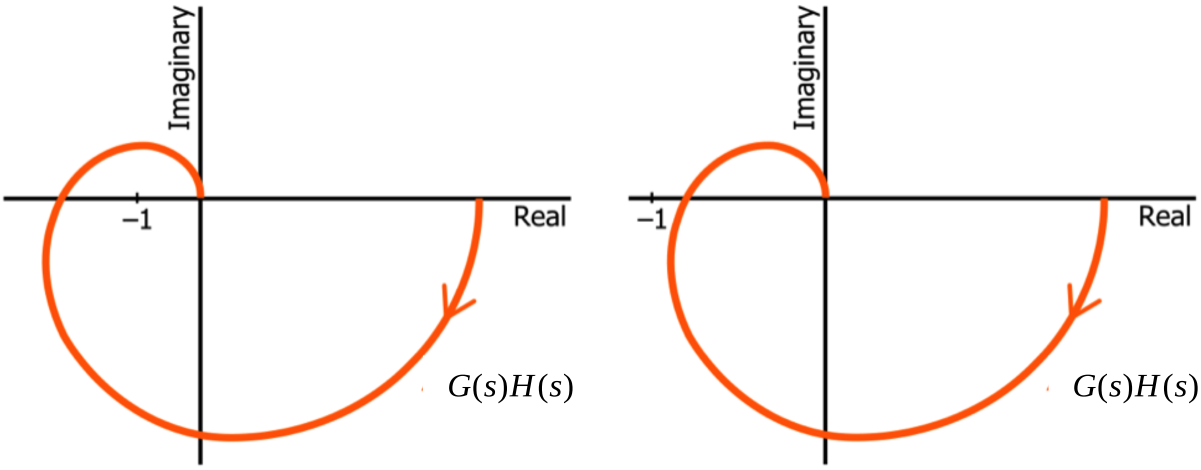


Figure 7.2: Nyquist stability criterion

7.4.2 .1Application of the Nyquist Criterion

To apply the Nyquist stability criterion and determine the stability of a closed-loop control system, the following systematic steps can be followed:

1. Determine the Open-Loop Transfer Function
   Identify the open-loop transfer function *G(s)H(s)* of the system.3
2. Plot the Frequency Response
   Evaluate the frequency response *G(jω)H(jω*) for positive frequencies, starting from ω=0 to ω=∞.

   For minimum-phase systems, the Nyquist plot typically starts on the real axis and moves toward the origin.

   If the transfer function contains integrator terms such as 1/s or 1/s, the plot may start at infinity on the real axis for ω=0, sweeping through ±90° (or ±180°) depending on the order of the pole at the origin.

3. Construct the mirror image of the positive-frequency locus with respect to the real axis. This represents the response for $\omega \in [-\infty, 0]$, completing the Nyquist contour.
4. Analyze Encirclements of the Critical Point (-1, 0)

   If the resulting Nyquist plot encircles the critical point $(-1+j0)$ a number of times equal to the number of right-half-plane poles of G(s)H(s), and in the counterclockwise direction, the closed-loop system is stable.

   If the plot passes through $(-1,0)$the system is marginally stable (i.e., on the boundary of stability).

   If the plot encircles $(-1,0$more or fewer times than the number of RHP poles, the system is unstable.

7.5 Exercises

Exercise 7.1 :

Let us consider a system with an open-loop transfer function *G(s)* defined by:

$$G(s)=\frac{K}{s(s+2)(s+4)} \quad K>0$$

- Using the Routh-Hurwitz criterion, determine the stability conditions of the closed-loop system when it is placed in a unity feedback configuration.

Exercise 7.2:

A closed-loop system with a proportional controller is shown in the figure 7.3 where ;

$$G(s)=\frac{k}{s(\tau s+1)^2} \quad \text{and} \quad C(s)=K_p$$

$$k=0.08, \ \tau = 20s$$

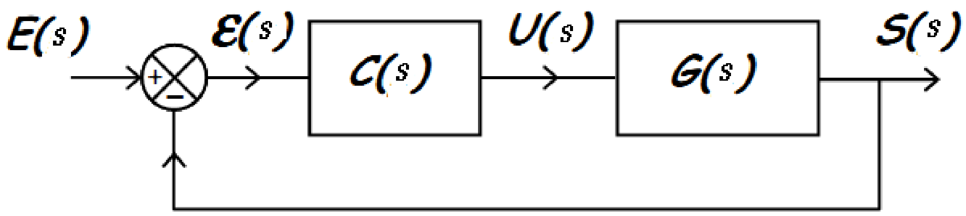


Figure 7.3: closed-loop control system

- Calculate the magnitude and phase of the open-loop transfer function, denoted by *A(s)*

- Plot the Nyquist diagram of *A(s)* for $K_p=1$.
- The system is stable? Why?
- Let' the gain $K_p=0.4$, Plot the Nyquist diagram of *A(s)*.

Exercise 7.3:

A closed-loop system with a PI (proportional integrator) controller is shown in the figure 7.3 where ;

$$G(s)=\frac{k}{(\tau s+1)^3} \text{ and } C(s)=K_p+\frac{K_i}{s}$$

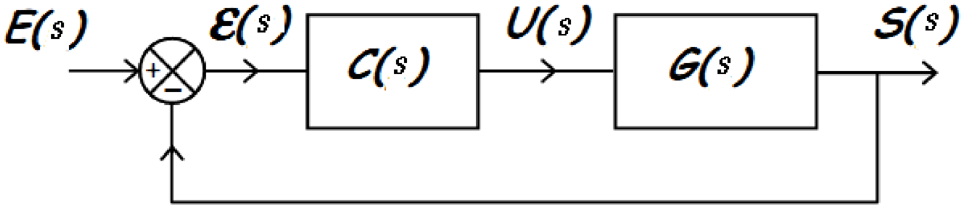


Figure 7.3: Control system with PI controller

- Using the Routh-Hurwitz stability criterion, determine the conditions of the proportional gain. K_p and the integral gain K_i ensure the stability of the closed-loop system.

7.6 Exercises Solution

Exercise 7.1 :

a. The closed-loop transfer function can be written as :

$$H(s)=\frac{G(s)}{1+G(s)}=\frac{\dfrac{k}{s(s+2)(s+4)}}{1+\dfrac{k}{s(s+2)(s+4)}} \tag{7.10}$$

Thus

$$H(s)=\frac{k}{s(s+2)(s+4)+k}=\frac{k}{s^3+6s^2+8s+k} \tag{7.11}$$

The Routh array is :

s^3	1	8	0
s^2	6	K	0
s^1	$\frac{6\text{x}8-k}{6}$	0	0
s^0	K	0	0

The condition for the system to be stable is that the parameter k must be:

$$0 < k < 48$$

Exercise 7.2:

a. The open loop transfer function *A(s)* of the block diagram of Figure (7.3):

$$A(s) == C(s)G(s) = K_p \frac{k}{s(\tau s+1)^2} \tag{7.12}$$

The magnitude and phase of *A(s):*

$$A(j\omega) = K_p \frac{k}{j\omega(j\omega\tau+1)^2} \tag{7.13}$$

$$\begin{cases} |A(j\omega)| = \dfrac{K_p k}{\omega(\tau^2\omega^2+1)} \\ \phi(A(j\omega)) = -\dfrac{\pi}{2} - 2\tan^1(\tau\omega) \end{cases} \tag{7.14}$$

b. Nyquist plot for $K_p=1$

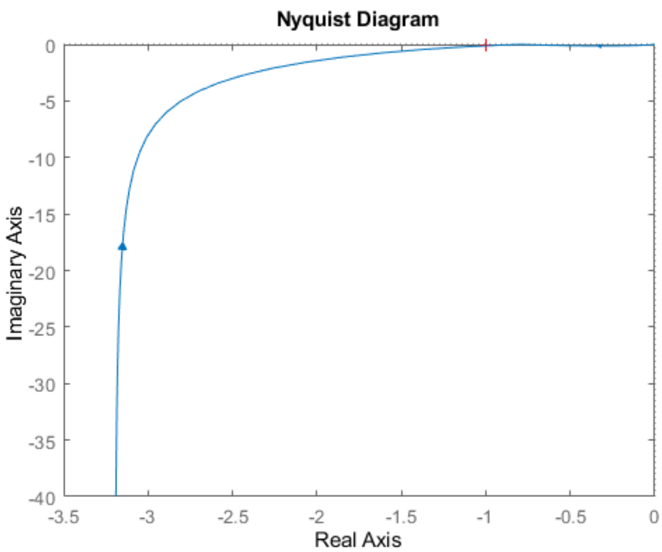


Figure 7.4: Nyquist plot for $K_p=1$

- The closed-loop system is stable because the Nyquist plot, traced in the direction of increasing frequency, leaves the critical point $(-1,0)$ on its left-hand side.

c. Nyquist plot for $K_p=0.4$

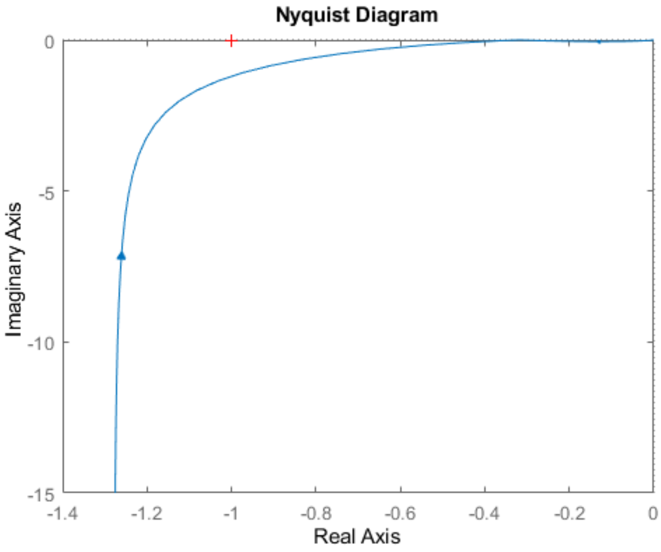


Figure 7.5: Nyquist plot for $K_p=0.4$

Exercise 7.3:

The open loop transfer function of the diagram of the figure 7.3;

$$A(s) == C(s)G(s) = (K_p + \frac{K_i}{s})\frac{k}{(\tau s+1)^3} \tag{7.15}$$

Therefore :

$$A(s) = \frac{(K_p s + K_i)k}{\tau^3 s^4 + 3\tau^2 s^3 + 3\tau s^2 + (K_p k+1)s + K_i k} \tag{7.16}$$

The Routh table :

s^4	τ^3	3τ	$K_i k$
s^3	$3\tau^2$	$K_p k+1$	0
s^2	α	$K_i k$	0
s^1	β	0	0
s^0	$K_i k$	0	0

Where: $\alpha = \dfrac{3\tau^2 \text{x} 3 - (K_p k + 1)\tau^3}{3\tau^2}$, $\beta = \dfrac{(3\tau - (K_p k + 1)\frac{\tau}{3})(K_p k + 1) - 3\tau^2 K_i k}{(3\tau - (K_p k + 1)\frac{\tau}{3})}$.

The condition for the system to be stable is that the parameters K_p *and* K_i must be:

$$\alpha = \frac{3\tau^2 \text{x} 3 - (K_p k + 1)\tau^3}{3\tau^2} > 0 \Rightarrow K_p < \frac{8}{k} \tag{7.17}$$

$$\beta > 0 \Rightarrow K_i < \frac{3(K_p k + 1)}{\tau k} - \frac{(K_p k + 1)^2}{3\tau k} \tag{7.18}$$

Appendix

Asymptotic Bode Plot:

The following script defines a MATLAB function named asym_bode that generates an asymptotic Bode diagram, as presented in the course. This function takes as input a transfer function defined using MATLAB's tf function.

```
function asym_bode(obj)
% ASYM_BODE takes as an input a tf,zpk,ss or symbolic object and outputs an
% asymptotic Bode plot of its frequency response. Poles and zeros at the
% origin are accounted for, as well as time delays. The phase is
% interpolated according to 0.1w=0, w=45[deg], 10w=90[deg] (adjusted for
% multiplicity). At the moment purely imaginary poles and zeroes are not
% supported.
switch class(obj)
    case 'tf'
        num=obj.num{:};
        den=obj.den{:};
        zroots=roots(num);
        proots=roots(den);
    case 'zpk'
        zroots=obj.Z{:};
        proots=obj.P{:};
    case 'ss'
        obj=zpk(obj);
        zroots=obj.Z{:};
        proots=obj.P{:};
    case 'sym'
        [symNum,symDen] = numden(obj);
        num = sym2poly(symNum);
        den = sym2poly(symDen);
        obj=tf(num,den);
        zroots=roots(num);
        proots=roots(den);
    otherwise
        error('Please input either a zpk or tf object')
        return
end
%Find static gain
K=zpk(obj).k;
% Find and remove imaginary roots
im_p=proots(find(real(proots)==0));
proots(find(real(proots)==0))=[];
im_z=zroots(find(real(zroots)==0));
zroots(find(real(zroots)==0))=[];
for ii=1:length(proots)
    w_p(1,ii)=abs(proots(ii));
    w_p(2,ii)=-1*abs(sign(real(proots(ii))));
    w_p(3,ii)=sign(real(proots(ii)));
end
for ii=1:length(zroots)
    w_z(1,ii)=abs(zroots(ii));
    w_z(2,ii)=abs(sign(real(zroots(ii))));
    w_z(3,ii)=-1*sign(real(zroots(ii)));
end
```

Appendix

```
try
    w_t=[w_p,w_z];
catch
    try
        w_t=[w_p];
    catch
        w_t=[w_z];
    end
end
interval=sort([0.01*w_t(1,:), 0.05*w_t(1,:), 0.1*w_t(1,:), 0.2*w_t(1,:), w_t(1,:),
2*w_t(1,:), 5*w_t(1,:), 10*w_t(1,:),100*w_t(1,:)]);
mag=zeros(size(w_t,2),length(interval));
Phi=zeros(size(w_t,2),length(interval));
% calculate magnitude of each pole and zero
for ii=1:size(mag,1)
        m=20*w_t(2,ii);
        n=-m*log10(w_t(1,ii));
    for jj=1:size(mag,2)
        if jj < find(interval==w_t(1,ii))+1
            mag(ii,jj)=0;
        else
            mag(ii,jj)=m*log10(interval(jj))+n;
        end
    end
end
mag=sum(mag,1);
% Account for integrators and differentiators (mag)
if length(find(imag(im_p)==0))>0
    mag_imp=zeros(length(find(imag(im_p)==0)),length(interval));
    m=-20;
    n=0;
    for ii=1:size(mag_imp,1)
        for jj=1:size(mag_imp,2)
             mag_imp(ii,jj)=m*log10(interval(jj))+n;
        end
    end
    mag=mag+sum(mag_imp,1);
end
if length(find(imag(im_z)==0))>0
    mag_imz=zeros(length(find(imag(im_z)==0)),length(interval));
    m=20;
    n=0;
    for ii=1:size(mag_imz,1)
        for jj=1:size(mag_imz,2)
             mag_imz(ii,jj)=m*log10(interval(jj))+n;
        end
    end
mag=mag+sum(mag_imz,1);
end
% Account for static gain
mag=mag+mag2db(prod(abs(zroots))/prod(abs(proots)))+mag2db(K);
% Calculate phase
for ii=1:size(Phi,1)
        m=(pi/4)*w_t(3,ii);
        n=-m*log10(interval(find(interval==0.1*w_t(1,ii))));%m;
        n=n(1);
    for jj=1:size(Phi,2)
        if jj < find(interval==0.1*w_t(1,ii))+1
            Phi(ii,jj)=0;
```

```
        elseif jj < find(interval==10*w_t(1,ii))+1
            Phi(ii,jj)=m*log10(interval(jj))+n;
        else
            Phi(ii,jj)=(pi/2)*w_t(3,ii);
        end
    end
end
Phi=sum(Phi,1);
%integrators and differentiators (phase)
if length(find(imag(im_p)==0))>0
    phase_imp=(-pi/2)*ones(length(find(imag(im_p)==0)),length(interval));
    Phi=Phi+sum(phase_imp,1);
end
if length(find(imag(im_z)==0))>0
    phase_imz=zeros(length(find(imag(im_z)==0)),length(interval));
    Phi=Phi+sum(phase_imz,1);
end
% linear term for deadtime
phase_dt=-obj.InputDelay*interval;
% real curves
[gain_vec,phase_vec,w_vec]=bode(obj,{interval(1),interval(end)});
gain_vec=squeeze(gain_vec); phase_vec=squeeze(phase_vec); w_vec=squeeze(w_vec);
pp=phase_vec(1);
% Adjust real plot to account for NMP
phase_vec=phase_vec-pp+rad2deg(Phi(1));
% plots
figure
% magnitude
subplot(2,1,1)
semilogx(interval,mag,'b','LineWidth',1.2);
hold on
% title
title('Real and asymptotic Bode magnitude','FontSize',10)
xlabel('$\omega$ [rad/s]','Interpreter','latex','FontSize',10)
ylabel('$|G(j\omega)|$ [dB]','Interpreter','latex','FontSize',10)
% real plot
plot(w_vec,mag2db(gain_vec),'--r','LineWidth',1);
grid on
% mark frequencies
y0=get(gca,'ylim');
y0=y0(1);
Ind=unique(mod(find(interval(:)==w_t(1,:)),length(interval)),'stable');
tx = [w_t(1,:);w_t(1,:);nan(1,length(w_t(1,:)))];
ty = [y0*ones(1,length(w_t(1,:)));mag(Ind);nan(1,length(w_t(1,:)))];
plot(tx(:),ty(:),'--k','LineWidth',0.75);
% phase
subplot(2,1,2)
semilogx(interval,rad2deg(Phi+phase_dt),'b','LineWidth',1.2);

title('Real and asymptotic Bode phase','FontSize',10)
xlabel('$\omega$ [rad/s]','Interpreter','latex','FontSize',10)
ylabel('$\mathrm{arg}\,G(j\omega)$
[$\,^\circ$]','Interpreter','latex','FontSize',14)
hold on
semilogx(w_vec,phase_vec,'--r','LineWidth',1);grid on
grid on
end
```

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