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Handout

Point Particle Mechanics

Course

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Introduction:

This document provides a comprehensive foundation in the mathematical and physical principles essential for understanding classical mechanics and related physical phenomena. It covers the fundamental concepts that link mathematics and physics to describe the behavior of the physical world. This document is an introductory reference for first-year science and technology students in general, and engineering students in particular. This document was written according to the curriculum for first-year science and technology students.

The first chapter of this document is devoted to reviewing the mathematical concepts students need, in addition to studying physical quantities, their dimensions, and dimensional analysis, which ensures the consistency of physical equations. The first chapter also discusses advanced mathematical tools, such as functions of several variables, partial derivatives, and vector operations, which are essential tools for modeling physical systems. Coordinate systems—Cartesian, polar, cylindrical, and spherical—are introduced to describe positions and motions in various contexts.

Other chapters focus on kinematics and dynamics, detailing the motion of point particles in various frames of reference, including uniform, uniformly accelerated, sinusoidal, and curvilinear motion. Newton's laws of motion, forces (such as friction, tension, buoyancy, and spring forces), momentum, work, energy, and rotational dynamics are also explored, providing a robust framework for analyzing mechanical systems.

Through clear definitions, mathematical derivations, examples, and problem sets, this document provides readers with the tools to systematically analyze physical systems, making it a valuable resource for the study of mechanics and related fields of physics.

Analysis dimensional and Mathematical background

I.1 Analysis dimensional:

I.1.1 Introduction:

Human beings have realized the importance of measurements in daily life and the development of societies since the dawn of history. This prompted ancient civilizations to develop simple measurement systems necessary to simplify life. At first, the human being relied on parts of his body to measure lengths and distances. For example, the cubit was a commonly used unit of measurement. This measurement tool was first used by the Egyptians around 3000 BC. The size of the cubit in Egyptian civilization is equal to the length between the elbow and the tip of the middle finger. This unit of measurement played an important role in Pharaonic engineering, especially in the construction of the pyramids. Other ancient civilizations such as the Babylonian, Roman, and Greek civilizations developed other measurement systems such as the foot and the inch. Despite this, the development of trade systems between peoples with the beginning of the Industrial Revolution and the European Renaissance led to the need to unify measurement systems between societies. The French Revolution was one of the most significant turning points in the history of measurement systems. In 1799, France officially adopted the metric system. The meter was then defined as one ten-millionth of the distance from the North Pole to the Equator via Paris. This system remains the cornerstone of modern measurement systems.

I.1.2 Physical quantity:

A physical quantity "P" is any property or characteristic of an object or a physical phenomenon that can be measured or quantified using numbers and units. Examples include mass, length, time, temperature, electric current, force, and volume.

I.1.2 a) Types of physical quantities:

There are two types of physical quantities:

Fundamental (base) quantities, such as length, mass, and time, are defined independently and measured directly.

Derived quantities, these quantities are defined in terms of base quantities.

I.1.2 b) International system IS:

The International System of Units, known also by the abbreviation SI (from the French language Système international d'unités), is the modern form of the metric system and the world's most widely used system for measurement. The SI is coordinated by the International Bureau of Weights and Measures (BIPM)(Mills et al., 2011).

The basic dimensions in IS:

Length (L) is measured by the meter.

Mass (M) is measured by the kilogram.

Time (T) is measured by the second.

Electric current (I) is measured by amper.

Temperature (θ) is measured by Kelvin.

Amount of substance (N) is measured by mol.

Luminous intensity (J).

Figure 1 Fundamental unit

Derived Units: All other units in the SI are derived from these base units through multiplication, division, and exponentiation.

Table 1 Example of derived units:

Derived Unit	Symbol	Physical Quantity	Base Unit Expression
Newton	N	Force	$kg \cdot m/s^2$
Joule	J	Energy, Work, Heat	$kg \cdot m^2/s^2$
Watt	W	Power	$kg \cdot m^2/s^3$
Pascal	Pa	Pressure	$kg/m \cdot s^2$

Coulomb	C	Electric charge	$A \cdot s$
Volt	V	Electric potential	$kg \cdot m^2/s^3 \cdot A^{-1}$
Ohm	Ω	Electrical resistance	$kg \cdot m^2/s^3 \cdot A^{-2}$
Siemens	S	Electrical conductance	$s^3 \cdot A^2/kg \cdot m^2$
Farad	F	Capacitance	$s^4 \cdot A^2/kg \cdot m^2$
Hertz	Hz	Frequency	s^{-1}

I. 1.3 Prefixes:

The SI uses prefixes (such as kilo-, milli-, micro-) to denote multiples or fractions of the base units, making it suitable for expressing very large or very small quantities depending on the applications.

Table 2 Prefixes

Prefix	Symbol	Factor	Scientific Notation
tera	T	1,000,000,000,000	10^{12}
giga	G	1,000,000,000	10^{9}
mega	M	1,000,000	10^{6}
kilo	k	1	10^{3}
hecto	h	100	10^{2}
deca	da	10	10^{1}
(base)	—	1	10^{0}
deci	d	0.1	10^{-1}
centi	c	0.01	10^{-2}
milli	m	0.001	10^{-3}
micro	µ	0.000001	10^{-6}
nano	n	0.000000001	10^{-9}
pico	p	0.000000000001	10^{-12}

femto	f	0.000000000000001	10^{-15}
atto	a	0.000000000000000001	10^{-18}

I.1.4 Definition of some unit in IS:

The definitions of the SI units are based on fundamental constants of nature such as the speed of light, the Planck constant, and the cesium frequency.

I.1.4. a) Meter:

The meter is the most common unit for measuring lengths; the definition of this unit has undergone major changes since its development until the past few years. The meter was initially defined as a unit of measurement equal to one ten-millionth of the distance from the equator to the North Pole via Paris. The first standard meter was created in 1799 in the form of a platinum rod. This platinum rod was changed to a platinum-iridium rod due to the fact that this alloy was more resistant to corrosion compared to the platinum rod. With the emergence of the quantum revolution at the beginning of the twentieth century and the development of technological systems, there was a need to change the definition of the meter. The definition of the meter was modified in 1960 to become as follows:

"*The meter is a length equal to 1,650,763.73 times the wavelength in a vacuum of the radiation emitted by a krypton atom when electrons transit between the atomic levels.*"(Brzhezinskii et al., 1970)

I.1.4. b) Second

The second is the basic unit of time in the International System of Units (SI). The definition of this unit has undergone various developments over time. Its current official definition, in effect since 1967, is based on a fundamental property of the cesium-133 atom.

The second is defined by taking the constant numerical value of the cesium frequency, the frequency of the unperturbed hyperfine transition in the ground state of the cesium-133 atom, to be exactly 9,192,631,770 when expressed in hertz (Hz), which is equal to s^{-1}.

This definition means that "*one second is equal to the duration of 9,192,631,770 cycles of radiation corresponding to the transition between the two hyperfine levels of the ground state of the cesium-133 atom*".(Gill, 2011)

I.1.4. c) Kilogram:

The kilogram is the base unit of mass in the International System of Units (SI). It is the last SI unit to be defined by linking it to a cosmological constant, just as the second and the meter were redefined. Since 2019, the kilogram has been defined by taking the numerical value of Planck's constant, h, to be exactly $6.62607015 \times 10^{-34}$ when expressed in joule-seconds (J·s), which is equivalent to kg·m²·s⁻¹. This definition links the kilogram to the fundamental constants of nature, specifically Planck's constant, the speed of light, and the frequency of cesium.(Wood and Bettin, 2019)

I.1.5 Dimensions:

The physical nature of a quantity P is characterized by its dimension [P].

I.1.5. a) The Dimensions of the fundamental quantities:

Table 3 The fundamental unit

Physical Quantity	SI Unit	Dimension Symbol
Length	meter (m)	**L**
Mass	kilogram (kg)	**M**
Time	second (s)	**T**
Electric Current	ampere (A)	**I**
Thermodynamic Temperature	Kelvin (K)	$\boldsymbol{\theta}$
Amount of Substance	mole (mol)	**N**
Luminous Intensity	candela (cd)	**J**

I.1.5. b) Equation of Dimensions

The dimension of any physical quantity, whatever is fundamental or derived, [P] can be expressed by a combination of the seven basic dimensions. This combination is called the equation of dimensions and can be formulated as follows:

$$[P] = M^a L^b T^c I^d \theta^e N^f J^h$$

Where a,b,c,d,e,f, and g are real numbers.

I.1.5. c) Homogeneity of a formula:

A formula: A = B is said to be homogeneous if the two physical quantities A and B have the same dimensions.

I.1.5. d) Fundamental Rules of Dimensional Analysis:

1- Every term in a physical equation must have the same dimensions.
2- In the dimensional analysis we use only fundamental dimensions when expressing physical quantities.

3- The dimension of any real number and functions is one.

4- We cannot add or subtract quantities of different dimensions.

Example:

Find the dimension of velocity, acceleration, force, work, pressure, power, gravitational potential energy, electrical potential, gravitational constant, momentum, impulse, density, angular momentum, surface tension, dynamic viscosity, magnetic field, electrical field, resistivity, and heat capacity

Velocity:

$$v = \frac{d}{t}, \quad [v] = \frac{L}{T} = LT^{-1}$$

Acceleration:

$$a = \frac{v}{t}, \quad [a] = \frac{LT^{-1}}{T} = LT^{-2}$$

Force

$$F = m \cdot a, \quad [F] = M \cdot LT^{-2}$$

Work

$$W = F \cdot d, \quad [W] = M \cdot LT^{-2} \cdot L = ML^2T^{-2}$$

Pressure:

$$P = \frac{F}{A}, \quad [P] = \frac{MLT^{-2}}{L^2} = ML^{-1}T^{-2}$$

Power:

$$P = \frac{W}{t}, \quad [P] = \frac{ML^2T^{-2}}{T} = ML^2T^{-3}$$

Gravitational potential energy

$$U = mgh, \quad [U] = M \cdot (LT^{-2}) \cdot L = ML^2T^{-2}$$

Electrical potential

$$V = \frac{W}{Q}, \quad [V] = \frac{ML^2T^{-2}}{IT} = ML^2T^{-3}I^{-1}$$

Gravitational constant

$$F = G\frac{m_1 m_2}{r^2} \Rightarrow G = \frac{Fr^2}{m_1 m_2}$$

$$[G] = \frac{MLT^{-2} \cdot L^2}{M^2} = M^{-1}L^3T^{-2}$$

Momentum

$$p = mv, \quad [p] = M \cdot LT^{-1}$$

Impulse

$$J = Ft, \quad [J] = MLT^{-2} \cdot T = MLT^{-1}$$

Density

$$\rho = \frac{m}{V}, \quad [\rho] = \frac{M}{L^3} = ML^{-3}$$

Angular momentum

$$L = I\omega, \quad [L] = (ML^2) \cdot T^{-1} = ML^2T^{-1}$$

Surface tension

$$T = \frac{F}{L}, \quad [T] = \frac{MLT^{-2}}{L} = MT^{-2}$$

Dynamic viscosity

$$\eta = \frac{Ft}{Ax}, \quad [\eta] = \frac{MLT^{-2} \cdot T}{L^2 \cdot L} = ML^{-1}T^{-1}$$

Magnetic field

$$B = \frac{F}{qv}, \quad [B] = \frac{MLT^{-2}}{IT \cdot LT^{-1}} = MT^{-2}I^{-1}$$

Electric field

$$E = \frac{F}{q}, \quad [E] = \frac{MLT^{-2}}{IT} = MLT^{-3}I^{-1}$$

Resistivity

$$\rho = R\frac{A}{L}, \quad [\rho] = ML^2T^{-3}I^{-2} \cdot \frac{L^2}{L} = ML^3T^{-3}I^{-2}$$

Heat capacity

$$C = \frac{Q}{\Delta T}, \quad [C] = \frac{ML^2T^{-2}}{\Theta} = ML^2T^{-2}\Theta^{-1}$$

I.2 Mathematical background

I.2.1 a) Functions of several variables:

A function f is said to be of several variables if it takes inputs from a domain in Rn, where n is the number of variables, and maps to a value in R for scalar-valued functions or Rm for vector-valued functions, where m is a natural number.

<u>Examples:</u>

Paraboloid Function:

$$f(x, y) = x^2 + y^2$$

Temperature Distribution in 3D Space:

$$T(x, y, z) = x^2 + y^2 + z^2$$

Electric Potential in Cylindrical Coordinates:

$$V(r, \theta, z) = \frac{1}{\sqrt{r^2 + z^2}}$$

I.2.1 b) Partial derivatives of a function of several variables:

The partial derivative measures how a mathematical function of multiple variables (x_1, x_2, ..., x_n) changes when only one variable is varied, for example, x_1, keeping the others constant (x_2, ..., x_n).

Notation

Several notations are used:

$$\frac{\partial f(x, y, z)}{\partial x}$$

$$f_x(x, y, z)$$

$$\partial f_x(x, y, z)$$

We will use the first notation in this course.

Example:

Find the partial derivatives of the following functions:

$$f(x,y) = x^2y + 3y^2)$$

$$\frac{\partial f}{\partial x} = 2xy, \quad \frac{\partial f}{\partial y} = x^2 + 6y$$

$$f(x,y) \ = \ e^{xy}$$

$$\frac{\partial f}{\partial x} = ye^{xy}, \quad \frac{\partial f}{\partial y} = xe^{xy}$$

$$f(x,y,z) = xyz + x^2z$$

$$\frac{\partial f}{\partial x} = yz + 2xz, \quad \frac{\partial f}{\partial y} = xz, \quad \frac{\partial f}{\partial z} = xy + x^2$$

$$f(x,y) = \ln(x^2 + y^2)$$

$$\frac{\partial f}{\partial x} = \frac{2x}{x^2 + y^2}, \quad \frac{\partial f}{\partial y} = \frac{2y}{x^2 + y^2}$$

$$f(x,y) = \sin(xy)$$

$$\frac{\partial f}{\partial x} = y\cos(xy)\,, \quad \frac{\partial f}{\partial y} = x\cos(xy)$$

$$f(x,y) = x^3 + y^3 + 3xy$$

$$\frac{\partial f}{\partial x} = 3x^2 + 3y, \quad \frac{\partial f}{\partial y} = 3y^2 + 3x$$

$$f(x,y) = \tan^{-1}\left(\frac{y}{x}\right)$$

$$\frac{\partial f}{\partial x} = -\frac{y}{x^2 + y^2}, \quad \frac{\partial f}{\partial y} = \frac{x}{x^2 + y^2}$$

$$f(x,y) = x^y$$

$$\frac{\partial f}{\partial x} = yx^{y-1}, \quad \frac{\partial f}{\partial y} = x^y \ln x$$

$$f(x,y) = \ln(xy + 1)$$

$$\frac{\partial f}{\partial x} = \frac{y}{xy+1}, \quad \frac{\partial f}{\partial y} = \frac{x}{xy+1}$$

I.2.1 c) Total differential of a function of several variables:

The total differential of a function of several variables (x_1, x_2, ..., x_n) gives the approximate change in the function due to small changes in all its variables.

$$df\ (x,y) = \frac{df(x,y)}{dx}\,dx + \frac{df(x,y)}{dy}\,dy$$

I.2.2 Vectors:

I.2.2.a) Definition

A vector is a mathematical quantity that has magnitude, sense, and direction. It is represented geometrically as an arrow and algebraically as an ordered set of components.

For example, in 3D space:

$$\vec{V} = x\,\vec{\imath} + y\,\vec{\jmath} + z\,\vec{k}$$

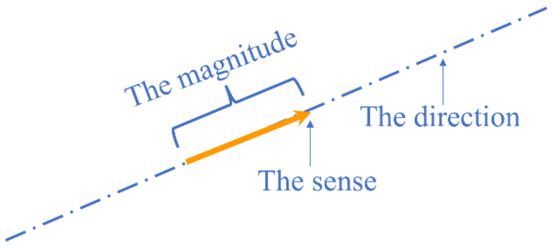

Figure 2 Vector characteristics

Table 4 Characteristics of vectors

Type	**Description**
Zero vector	Magnitude = 0, direction undefined
Unit vector	Magnitude = 1, represents direction only
Position vector	Points from the origin to a location in space
Equal vectors	Same magnitude and direction
Opposite vectors	Same magnitude, opposite direction

I.2.2 b) Vector Magnitude:

The magnitude or length of a vector is how long the vector is, regardless of its direction. The mathematical expression of the magnitude represents as:

$$|V| = \sqrt{x^2 + y^2 + z^2}$$

Examples:

$$\overrightarrow{v_1} = 3\,\vec{i} + 4\,\vec{j}$$

$$|\overrightarrow{v_1}| = \sqrt{(3)^2 + (4)^2} = \sqrt{9 + 16} = \sqrt{25} = 5$$

$$\overrightarrow{v_1} = -\,\vec{i} + 2\,\vec{j} + 2\,\vec{k}$$

$$|\overrightarrow{v_2}| = \sqrt{(-1)^2 + 2^2 + 2^2} = \sqrt{1 + 4 + 4} = \sqrt{9} = 3$$

I.2.2 c) Cartesian coordinate system:

The Cartesian coordinate system is the first system developed historically, successfully linking both branches of geometry and algebra. It was developed in the 17th century by René Descartes, the famous mathematician who proposed the idea of using algebra to describe geometry. In his 1637 book, Geometry, Descartes (Descartes, 1954) demonstrated how points on a plane can be represented by ordered pairs of numbers (x, y), measured along two perpendicular axes. At roughly the same time, Pierre de Fermat independently developed similar ideas, and their combined work laid the foundations of analytic geometry. This system revolutionized mathematics by allowing geometric problems to be solved using algebraic equations. Over time, the system was expanded to include three dimensions and became fundamental in fields such as physics, engineering, computer science, and economics.

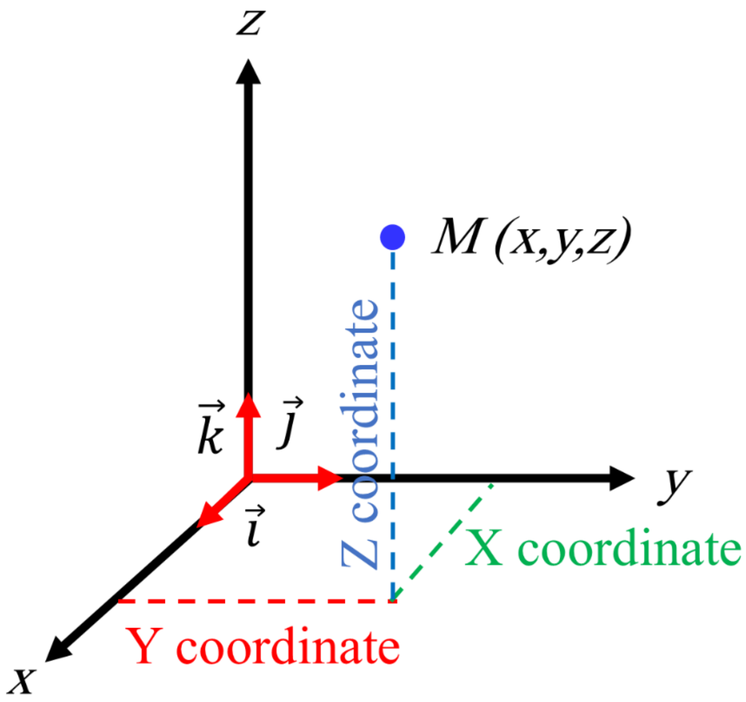

Figure 3 Position of point M in Cartesian coordinates

1.2.2. d) Cartesian reference:

In a three-dimensional space, the Cartesian coordinate system consists of an origin (point O) and directed and orthogonal (perpendicular to each other) axes passing through this origin.

x-axis (denoted Ox);

y-axis (denoted Oy);

z-axis (denoted Oz);

<u>Orthonormal Cartesian Basis:</u>

In a three-dimensional space, the Cartesian coordinate system has an orthonormal vector basis consisting of three pairwise orthogonal unit vectors denoted as follows:

$\vec{\imath}$: carried by the Ox axis and oriented along its orientation.

$\vec{\jmath}$: carried by the Oy axis and oriented along its orientation.

$\vec{k}$: carried by the Oz axis and oriented along its orientation.

<u>Vector Representation of the Displacement from Point A to Point B in Cartesian basis:</u>

Let A, and B tow point in the space where A (x_a, y_a, z_a) and B (x_b, y_b, z_b). The vector ($\overrightarrow{AB}$), representing the displacement from point A to point B, is given by:

$$\overrightarrow{AB} = (x_b - x_a)\,\vec{\imath} + (y_b - y_a)\,\vec{\jmath} + (z_b - z_a)\,\vec{k}$$

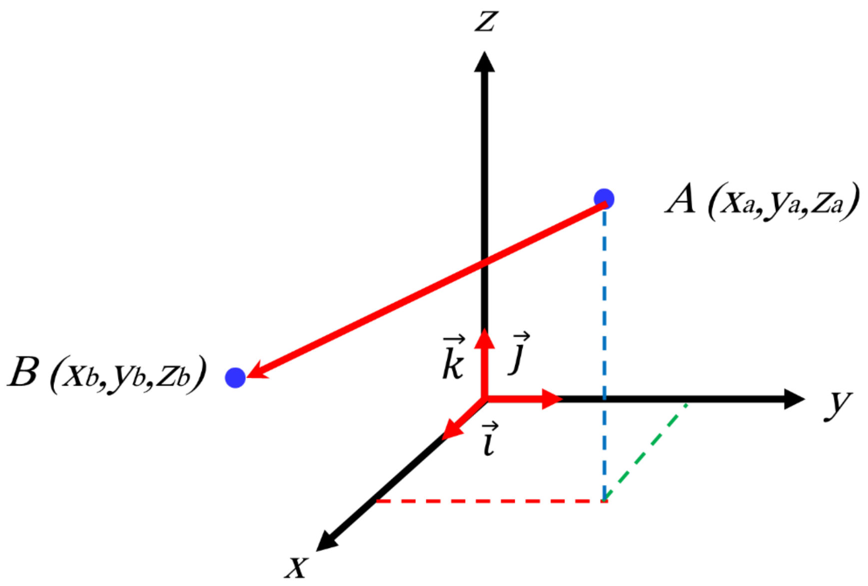

Figure 4 Displacement

<u>Examples:</u>

Let $A(1,2,3)$ and $B(4,6,8)$.

$\overrightarrow{AB} = 3\vec{\imath} + 4\vec{\jmath} + 5\vec{k}$

Let A(-2, 0, 5) and B(1, -3, 2).

$\overrightarrow{AB} = 3\vec{\imath} - 3\vec{\jmath} - 3\vec{k}$

1.2.2 e) Vector operation:

Let $\vec{u}$ and $\vec{v}$ two vectors:

$$\vec{u} = u_1\vec{\imath} + u_2\vec{\jmath} + u_3\vec{k}, \quad \vec{v} = v_1\vec{\imath} + v_2\vec{\jmath} + v_3\vec{k}$$

The sum of these two vectors is:

$$\vec{u} + \vec{v} = (u_1 + v_1)\vec{\imath} + (u_2 + v_2)\vec{\jmath} + (u_3 + v_3)\vec{k}$$

Vector Subtraction:

$$\vec{u} - \vec{v} = (u_1 - v_1)\vec{\imath} + (u_2 - v_2)\vec{\jmath} + (u_3 - v_3)\vec{k}$$

<u>Examples:</u>

Let $\vec{u}$ and $\vec{v}$ two vectors:

$$\vec{u} = 2\vec{\imath} + 3\vec{\jmath} - \vec{k}, \quad \vec{v} = \vec{\imath} - 4\vec{\jmath} + 2\vec{k}$$

$$\vec{u} + \vec{v} = (2+1)\vec{\imath} + (3-4)\vec{\jmath} + (-1+2)\vec{k} = 3\vec{\imath} - \vec{\jmath} + \vec{k}$$

$$\vec{a} = -\vec{\imath} + 2\vec{\jmath} + 4\vec{k}, \quad \vec{b} = 3\vec{\imath} - \vec{\jmath} + \vec{k}$$

$$\vec{a} - \vec{b} = (-1-3)\vec{\imath} + (2+1)\vec{\jmath} + (4-1)\vec{k} = -4\vec{\imath} + 3\vec{\jmath} + 3\vec{k}$$

Properties of Vector Addition:

Existence of Additive Inverse:

The result of adding a vector to its opposite is nil vector:

$$\vec{v} + (-\vec{v}) = \vec{0}$$

Commutativity:

The order in which vectors are added does not affect the result.

$$(\vec{A} + \vec{B}) + \vec{C} = \vec{A} + (\vec{B} + \vec{C})$$

Associativity:

When adding three or more vectors, the grouping of the vectors does not affect the result.

$$(\vec{A} + \vec{B}) + \vec{C} = \vec{A} + (\vec{B} + \vec{C})$$

Existence of Zero Vector:

There exists a zero vector $\vec{0}$ such that adding it to any vector $\vec{A}$ does not change the vector $\vec{A}$:

$$\vec{A} + \vec{0} = \vec{A}$$

Distributivity of Scalar Multiplication over Vector Addition:

Scalar multiplication distributes over vector addition. If ccc is a scalar and $\vec{A}$, $\vec{B}$ are vectors:

$$c(\vec{A} + \vec{B}) = c\vec{A} + c\vec{B}$$

Distributivity of Scalar Addition over Vector Addition:

Scalar addition distributes over vector addition:

$$c(\vec{A} + \vec{B}) = c\vec{A} + c\vec{B}$$

Examples:

$$\vec{A} = \vec{\imath} + 2\vec{\jmath}, \quad \vec{B} = 3\vec{\imath} - \vec{\jmath}$$

$$\vec{A}+\vec{B}=(1+3)\vec{\imath}+(2-1)\vec{\jmath}=4\vec{\imath}+\vec{\jmath}$$

$$\vec{A}+\vec{B}=(1+3)\vec{\imath}+(2-1)\vec{\jmath}=4\vec{\imath}+\vec{\jmath}$$

$$\vec{B}+\vec{A}=(3+1)\vec{\imath}+(-1+2)\vec{\jmath}=4\vec{\imath}+\vec{\jmath}$$

$$\vec{A}=\vec{\imath},\quad \vec{B}=2\vec{\imath}+\vec{\jmath},\quad \vec{C}=-\vec{\imath}+3\vec{\jmath}$$

$$\left(\vec{A}+\vec{B}\right)+\vec{C}=(3\vec{\imath}+\vec{\jmath})+(-\vec{\imath}+3\vec{\jmath})=2\vec{\imath}+4\vec{\jmath}$$

$$\vec{A}+\left(\vec{B}+\vec{C}\right)=\vec{\imath}+(\vec{\imath}+4\vec{\jmath})=2\vec{\imath}+4\vec{\jmath}$$

$$\vec{A}=4\vec{\imath}-2\vec{\jmath},\quad \vec{0}=0\vec{\imath}+0\vec{\jmath}$$

$$\vec{A}+\vec{0}=(4+0)\vec{\imath}+(-2+0)\vec{\jmath}=4\vec{\imath}-2\vec{\jmath}$$

$$\vec{A}=2\vec{\imath}-5\vec{\jmath},\quad -\vec{A}=-2\vec{\imath}+5\vec{\jmath}$$

$$\vec{A}+\left(-\vec{A}\right)=(2-2)\vec{\imath}+(-5+5)\vec{\jmath}=0\vec{\imath}+0\vec{\jmath}=\vec{0}$$

$$\vec{A}=\vec{\imath}+\vec{\jmath},\quad \vec{B}=2\vec{\imath}-\vec{\jmath},\quad c=3$$

$$\vec{A}+\vec{B}=(1+2)\vec{\imath}+(1-1)\vec{\jmath}=3\vec{\imath}$$

$$c\left(\vec{A}+\vec{B}\right)=3\cdot 3\vec{\imath}=9\vec{\imath}$$

$$c\vec{A}+c\vec{B}=3(\vec{\imath}+\vec{\jmath})+3(2\vec{\imath}-\vec{\jmath})=(3\vec{\imath}+3\vec{\jmath})+(6\vec{\imath}-3\vec{\jmath})=9\vec{\imath}$$

Example: $\vec{A}=\vec{\imath}+2\vec{\jmath},\quad c=2,\quad d=5$

$$(c+d)\vec{A}=7(\vec{\imath}+2\vec{\jmath})=7\vec{\imath}+14\vec{\jmath}$$

$$c\vec{A}+d\vec{A}=2(\vec{\imath}+2\vec{\jmath})+5(\vec{\imath}+2\vec{\jmath})=(2\vec{\imath}+4\vec{\jmath})+(5\vec{\imath}+10\vec{\jmath})=7\vec{\imath}+14\vec{\jmath}$$

<u>The dot product:</u>

The dot product, also known as the scalar product, is an algebraic operation that takes two vectors and gives a single scalar, a real number. ***This product measures how much one vector extends in the direction of another.***

<u>The analytic expression:</u>

Let $\vec{a}$ and $\vec{b}$ two vectors were,

$$\vec{a}=a_1\vec{\imath}+a_2\vec{\jmath}+a_3\vec{k},$$

$$\vec{b} = b_1\vec{\imath} + b_2\vec{\jmath} + b_3\vec{k}$$

The analytic expression of the dot product of these two vectors is:

$$\vec{a} \cdot \vec{b} = a_1b_1 + a_2b_2 + a_3b_3$$

Alternatively, we can use magnitudes and angle θ between the vectors to find the dot product:

$$\vec{a} \cdot \vec{b} = |\vec{a}|\,|\vec{b}|\cos\theta$$

Examples:

$$\vec{a} = 1\vec{\imath} + 2\vec{\jmath}, \quad \vec{b} = 3\vec{\imath} + 4\vec{\jmath}$$

$$\vec{a} \cdot \vec{b} = (1)(3) + (2)(4) = 3 + 8 = 11$$

$$\vec{a} = 1\vec{k}, \quad \vec{b} = 1\vec{\imath} + 2\vec{\jmath} + 3\vec{k}$$

$$\vec{a} \cdot \vec{b} = 0 + 0 + 3 = 3$$

$$\vec{a} = 1\vec{\imath} - 1\vec{\jmath}, \quad \vec{b} = -1\vec{\imath} + 1\vec{\jmath}$$

$$\vec{a} \cdot \vec{b} = (1)(-1) + (-1)(1) = -1 - 1 = -2$$

$$\vec{a} = 2\vec{\imath} + 2\vec{\jmath}, \quad \vec{b} = 2\vec{\imath} + 2\vec{\jmath}$$

$$\vec{a} \cdot \vec{b} = 4 + 4 = 8$$

The cross product

The cross product ($\vec{a} \times \vec{b}$) of two vectors ($\vec{a}$) and ($\vec{b}$) is a vector perpendicular to both ($\vec{a}$) and ($\vec{b}$), ***with magnitude equal to the area of the parallelogram formed by*** ($\vec{a}$) and ($\vec{b}$), and direction given by the right-hand rule.

Analytic expression in terms of basis vectors:

Let $\vec{a}$, and $\vec{b}$

$$\vec{a} = a_1\vec{\imath} + a_2\vec{\jmath} + a_3\vec{k},$$

$$\vec{b} = b_1\vec{\imath} + b_2\vec{\jmath} + b_3\vec{k}$$

$$\vec{a} \times \vec{b} = (a_2b_3 - a_3b_2)\vec{\imath} - (a_1b_3 - a_3b_1)\vec{\jmath} + (a_1b_2 - a_2b_1)\vec{k}$$

Also, we can calculate the magnitude of $\vec{a} \times \vec{b}$ as:

$$|\vec{a} \times \vec{b}| = |\vec{a}|\,|\vec{b}|\sin\theta$$

Examples:

$$\vec{a} = \vec{\imath} + 2\vec{\jmath}, \quad \vec{b} = 3\vec{\imath} + 4\vec{\jmath}$$

$$\vec{a} \times \vec{b} = (2 \cdot 0 - 0 \cdot 4)\vec{\imath} - (1 \cdot 0 - 0 \cdot 3)\vec{\jmath} + (1 \cdot 4 - 2 \cdot 3)\vec{k} = -2\vec{k}$$

$$\vec{a} = \vec{k}, \quad \vec{b} = \vec{\imath} + \vec{\jmath} + \vec{k}$$

$$\vec{a} \times \vec{b} = (0 \cdot 1 - 1 \cdot 1)\vec{\imath} - (0 \cdot 1 - 1 \cdot 1)\vec{\jmath} + (0 \cdot 1 - 0 \cdot 1)\vec{k} = -\vec{\imath} + \vec{\jmath}$$

I.2.3 Coordinate systems:

1.2.3. a) The Cartesian coordinates:

In the Cartesian coordinate system, the position vector of a point M is given as:

$$\overrightarrow{OM} = x\vec{\imath} + y\vec{\jmath} + z\vec{k}$$

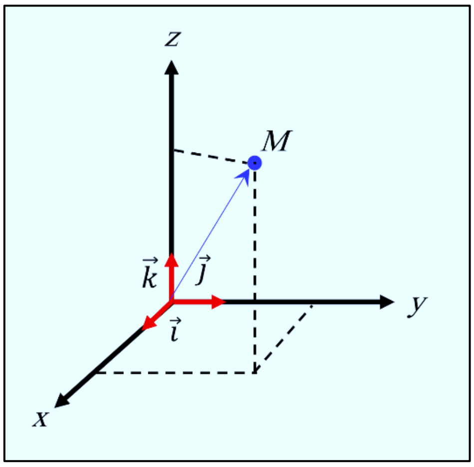

Figure 5 Position in Cartesian coordinates

The magnitude of the position vector:

$$\overrightarrow{OM} = \sqrt{x^2 + y^2 + z^2}$$

The direction $\overrightarrow{OM}$ est le segment de droite M O, and the sens of $\overrightarrow{OM}$ is from the point O to the point M.

1.2.3. b) The Polar coordinates:

Polar coordinate is a two-dimensional coordinate system in which the position of a point M in the plane is described by:

A distance from a fixed point called the origin, denoted by ρ, and an angle from a fixed direction, typically the positive x-axis, denoted by θ.

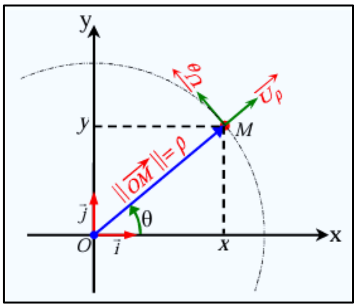

Figure 6 Position in polar coordinates

As shown in FIG. $|\overrightarrow{OM}| = \rho$ which is the distance between M and O.

ρ is the radial coordinate, and θ is the angular coordinate.

The base of the polar coordinates is $\overrightarrow{u_\rho}$ and $\overrightarrow{u_\theta}$, here $\overrightarrow{u_\rho}$ is the unit vector of the vector $\overrightarrow{OM}$

$$\overrightarrow{u_\rho} = \frac{\overrightarrow{OM}}{|\overrightarrow{OM}|}$$

The relation between the polar coordinates and the Cartesian coordinates:

Based on the FIG, we can find that the relation between the Cartesian coordinates (x, y) is:

$$\mathrm{x} = \rho \cos\theta$$

$$y = \rho \sin\theta$$

Therefore,

$$\overrightarrow{OM} = \rho\cos\theta\,\vec{\imath} + \rho\sin\theta\,\vec{\jmath}$$

$$\overrightarrow{OM} = \rho\,\overrightarrow{u_\rho}$$

$$\rho = \sqrt{x^2 + y^2}$$

$$\theta = \arctan(\frac{y}{x})$$

The unit vector of $\overrightarrow{OM}$ becomes

$$\overrightarrow{u_\rho} = \frac{\overrightarrow{OM}}{\rho} = \cos\theta\,\vec{\imath} + \sin\theta\,\vec{\jmath}$$

Also,

$$\overrightarrow{u_\theta} = -\sin\theta\,\vec{\imath} + \cos\theta\,\vec{\jmath}$$

Unlike the fixed unit vectors in the Cartesian coordinate system, the unit vectors in the polar coordinate system are position-dependent; thus, as the point M moves, the polar unit vectors also change their direction accordingly. The derivation with respect to the polar angle of the unit vectors $\overrightarrow{u_\rho}$ and $\overrightarrow{u_\theta}$, gives us:

$$\frac{d\overrightarrow{u_\rho}}{d\theta}$$

$$= \frac{d}{d\theta}\left(\cos\theta\, \vec{\imath} + \sin\theta\, \vec{\jmath} \right)$$

$$= -\sin\theta\, \vec{\imath} + \cos\theta\, \vec{\jmath}$$

$$= \overrightarrow{u_\theta}$$

$$\frac{d\overrightarrow{u_\theta}}{d\theta}$$

$$= \frac{d}{d\theta}(-\sin\theta\, \vec{\imath} + \cos\theta\, \vec{\jmath})$$

$$= -\cos\theta\, \vec{\imath} - \sin\theta\, \vec{\jmath}$$

$$= -\overrightarrow{u_\rho}$$

1.2.3. c) cylindrical coordinates:

In the cylindrical coordinate system, the position of a point M is identified by 3D coordinate system that extends polar coordinates by adding a vertical height component z.

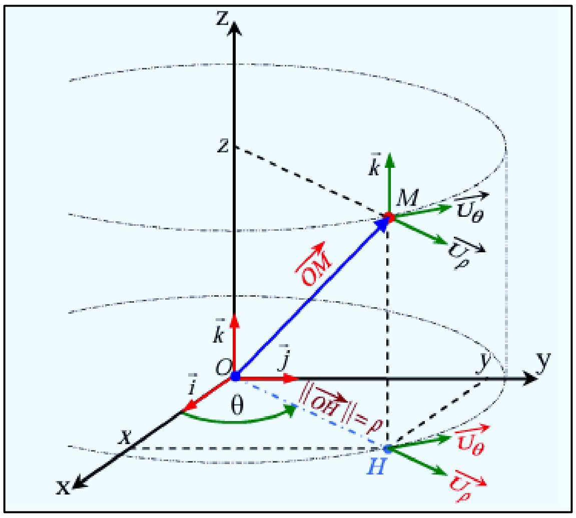

Figure 7 position in cylindrical coordinates

The relation between cylindrical coordinates and Cartesian coordinates:

$$x = \rho\cos\theta$$

$$y = \rho \sin \theta$$

$$Z = z$$

From Cylindrical to Cartesian:

$$\rho = \sqrt{x^2 + y^2}$$

$$\theta = \arctan(\frac{y}{x})$$

$$Z = z$$

The unit vector of the cylindrical coordinates:

$$\overrightarrow{u_\rho} = \cos \theta \, \vec{\imath} + \sin \theta \, \vec{\jmath}$$

$$\overrightarrow{u_\theta} = -\sin \theta \, \vec{\imath} + \cos \theta \, \vec{\jmath}$$

$$\vec{k} = \vec{k}$$

The position vector:

$$\overrightarrow{OM} = \rho \, \overrightarrow{u_\rho} + z \, \vec{k}$$

Example:

Convert the point (x, y, z) = (3, 4, 5) to cylindrical coordinates:

$$\rho = \sqrt{x^2 + y^2} = \sqrt{3^2 + 4^2} = \sqrt{9 + 16} = \sqrt{25} = 5$$

$$\theta = \tan^{-1}\left(\frac{y}{x}\right) = \tan^{-1}\left(\frac{4}{3}\right) \approx 0.93 \text{ radians}$$

$$Z = 5$$

1.2.3. d) Spherical coordinates:

In three-dimensional space, the position of a point M is located in the spherical coordinate system by:

The radial distance r is given by:

$$r = \left|\overrightarrow{OM}\right|$$

r is the distance between O and M.

The angle θ is the angle between the vector $\overrightarrow{OM}$ and the axe Oz.

The angle φ is the angle between the projection of $\overrightarrow{OM}$ on the plane (x,y) and the axis Ox.

The spherical coordinates use radial distance and two angles to locate point M.

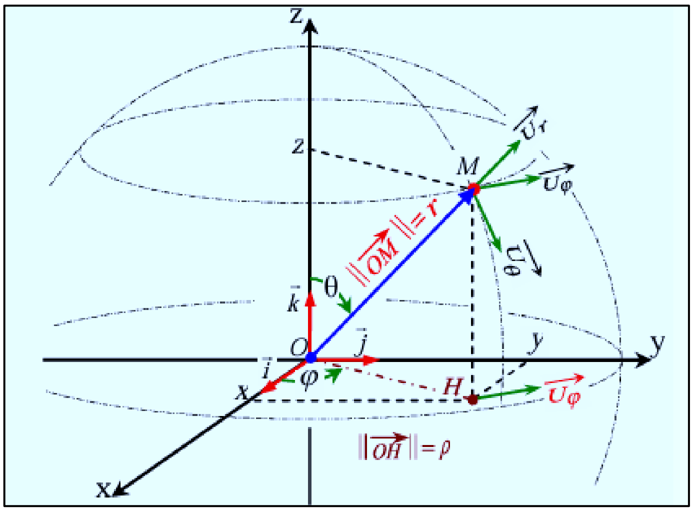

Figure 8 Spherical coordinates

The relation between the spherical coordinates and the Cartesian coordinates:

From Cartesian to spherical coordinates:

$$x = r\,sin\,\theta\ \cos\varphi$$

$$y = r\,sin\,\theta\ \sin\varphi$$

$$z\ =\ r\,cos\,\theta$$

From the spherical to Cartesian coordinates:

$$r = \sqrt{x^2 + y^2 + z^2}$$

$$\theta = \arccos\left(\frac{z}{r}\right)$$

$$\varphi = \arctan\left(\frac{y}{x}\right) \quad \text{(with quadrant adjustment)}$$

The unit vector of the spherical coordinates:

Radial Unit Vector:

$$\overrightarrow{U_r} = \sin\theta\cos\varphi\ \vec{\imath} + \sin\theta\sin\varphi\ \vec{\jmath} + \cos\theta\ \vec{k}$$

Polar Angle Unit Vector:

$$\overrightarrow{U_\theta} = \cos\theta\cos\varphi\ \vec{\imath} + \cos\theta\sin\varphi\ \vec{\jmath} - \sin\theta\ \vec{k}$$

Azimuthal Angle Unit Vector:

$$\overrightarrow{U_\varphi} = -\sin\varphi\ \vec{\imath} + \cos\varphi\ \vec{\jmath}$$

Algebraic Properties

Orthonormality:

$$\overrightarrow{U_r} \cdot \overrightarrow{U_\theta} = \overrightarrow{U_\theta} \cdot \overrightarrow{U_\varphi} = \overrightarrow{U_\varphi} \cdot \overrightarrow{U_r} = 0$$

$$\overrightarrow{U_r} \times \overrightarrow{U_\theta} = \overrightarrow{U_\varphi}$$

Cross Product Relations:

$$\overrightarrow{U_r} \times \overrightarrow{U_\theta} = \overrightarrow{U_\varphi}$$

$$\overrightarrow{U_\theta} \times \overrightarrow{U_\varphi} = \overrightarrow{U_r}$$

$$\overrightarrow{U_\varphi} \times \overrightarrow{U_r} = \overrightarrow{U_\theta}$$

<u>Examples</u>**:**

Convert the following point coordinates from Cartesian to Spherical system:

M (2, 0, 0)

P (3, 4, 0)

B (1, 1, $\sqrt{2}$)

Point M:

$$r = \sqrt{2^2 + 0^2 + 0^2} = 2$$

$$\theta = \arccos\left(\frac{0}{2}\right) = \frac{\pi}{2}$$

$$\varphi = \arctan\left(\frac{0}{2}\right) = 0$$

Point P:

$$r = \sqrt{3^2 + 4^2 + 0^2} = 5$$

$$\theta = \arccos\left(\frac{0}{5}\right) = \frac{\pi}{2}$$

$$\varphi = \arctan\left(\frac{4}{3}\right) \approx 0.927 \text{ rad}$$

Point B:

$$r = \sqrt{1^2 + 1^2 + \left(\sqrt{2}\right)^2} = 2$$

$$\theta = \arccos\left(\frac{\sqrt{2}}{2}\right) = \frac{\pi}{4}$$

$$\varphi = \arctan\left(\frac{1}{1}\right) = \frac{\pi}{4}$$

Kinematics of the Material Point

Kinematics of the material point

II.1 Introduction:

Kinematics is one of the fundamental branches of classical mechanics. It describes the motion of objects, focusing solely on geometric and temporal aspects without considering the forces causing the motion. The Kinematics is based on some principal physical quantities: position, time, velocity, acceleration, and trajectory.

II.2 Definition:

II.2.1 Point particle:

In kinematics, motion problems are simplified by approximating the dimensions of a body into a point particle. A point particle is a point-like mass with no volume, internal structure, or dimensions. This simplification is valid when the size and rotation of the body are negligible compared to its overall motion.

II.2.2 Reference Frame:

To describe the motion of a point particle, we need a reference frame or coordinate system. This is a set of axes, such as x, y, and z axes in Cartesian coordinates, or r, θ, and z in cylindrical coordinates, with a defined origin (0,0,0), from which all positions and motions are measured. Motion and rest are relative to the chosen reference frame.

II.2.3 Position:

The location of a point particle in space at a specific time, relative to the chosen origin of the reference frame. It is a vector quantity, which means it has both magnitude; its distance from the origin, and a direction.

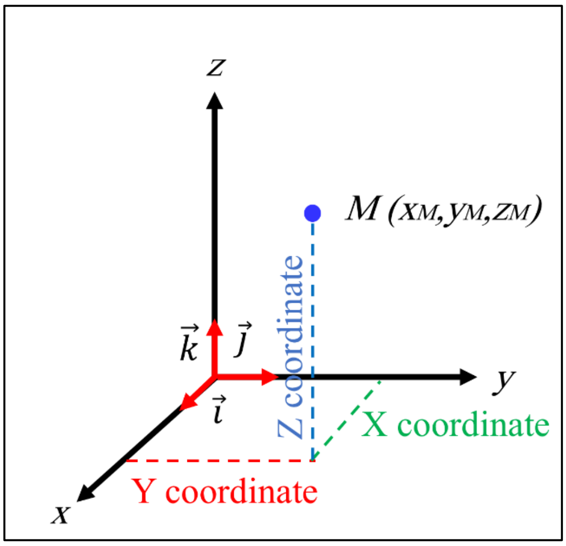

Figure 9 Position

Mathematical expression of the position in Cartesian coordinates:

$$\vec{r}(t) = \overrightarrow{OM}(t) = x_M(t)\,\vec{i} + y_M(t)\,\vec{j} + z_M(t)\,\vec{k}$$

Mathematical expression of the position in Polar coordinates:

$$\overrightarrow{OM} = \rho\, \overrightarrow{u_\rho}$$

Mathematical expression of the position in cylindrical coordinates:

$$\overrightarrow{OM} = \rho\, \overrightarrow{u_\rho} + z\, \vec{k}$$

Mathematical expression of the position in spherical coordinates:

$$\overrightarrow{OM} = r\, \overrightarrow{u_r}$$

II.2.4 Displacement $\boldsymbol{\Delta\vec{r}(t)}$:

It is the change in the particle's point position. It's the straight-line vector drawn from the initial position of the particle's point to its final position. the displacement is a vector quantity.

$$\Delta\vec{r}(t) = \vec{r}(t_2) - \vec{r}(t_1)$$

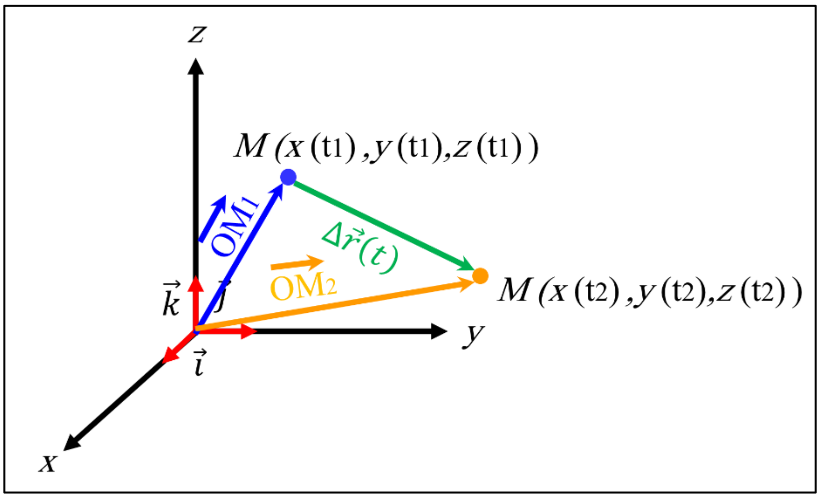

Figure 10 Displacement

II.2.5 Velocity ($\vec{\mathbf{v}}$):

It is the rate of change of a particle's displacement with respect to time. It's a vector quantity, which means it has both magnitude (speed) and direction.

Average velocity:

$$\overrightarrow{v_{\text{avg}}} = \frac{\Delta\vec{r}}{\Delta t} = \frac{\vec{r}(t_2) - \vec{r}(t_1)}{t_2 - t_1}$$

$$\vec{v}(t) = \frac{d\vec{r}}{dt}$$

II.2.5.a) Vector velocity in Cartesian coordinates:

The vector velocity in the Cartesian coordinate system is expressed as a vector with components along the *x*, *y*, and *z* axes. It describes the rate of change of the position vector $\overrightarrow{OM}$ with respect to time, as follows.

$$\vec{v}(t) = \frac{d\overrightarrow{OM}(t)}{dt}$$

Velocity vector in component form (Leibniz notation):

$$\vec{v}(t) = \frac{dx}{dt}\vec{\imath} + \frac{dy}{dt}\vec{\jmath} + \frac{dz}{dt}\vec{k}$$

Velocity vector in component form (Newton notation):

$$\vec{v}(t) = \dot{x}(t)\vec{\imath} + \dot{y}(t)\vec{\jmath} + \dot{z}(t)\vec{k}$$

We note here that the base $(\vec{\imath}, \vec{\jmath}, \vec{k})$ is characterized by:

$$\frac{d\vec{\imath}}{dt} = \vec{0}, \quad \frac{d\vec{\jmath}}{dt} = \vec{0}, \quad \frac{d\vec{k}}{dt} = \vec{0}$$

<u>Example:</u>

Let a point *M* move in space such that its position vector (relative to origin O) is:

$$\overrightarrow{OM}(t) = (2t^2)\vec{\imath} + (3t)\vec{\jmath} + (5)\vec{k}$$

Calculate the velocity vector.

Solution

$$\frac{dx}{dt} = 4t, \quad \frac{dy}{dt} = 3, \quad \frac{dz}{dt} = 0$$

$$\vec{v}(t) = \frac{d\overrightarrow{OM}(t)}{dt} = 4t\vec{\imath} + 3\vec{\jmath}$$

Let the position vector of a point *M* be:

$$\overrightarrow{OM}(t) = R\cos(\omega t)\,\vec{\imath} + R\sin(\omega t)\,\vec{\jmath}$$

Calculate the velocity vector.

Solution

$$\vec{v}(t) = \frac{d\overrightarrow{OM}(t)}{dt} = -R\omega\sin(\omega t)\,\vec{\imath} + R\omega\cos(\omega t)\,\vec{\jmath}$$

II.2.5 b) Vector velocity in polar coordinates:

Vector velocity in polar coordinates is the rate of change of a particle point position with respect to time, expressed using radial and angular components.

The relation of the velocity is given as:

$$\vec{v} = \frac{d}{dt}(\rho\,\overrightarrow{u_\rho})$$

$$= \frac{d\rho}{dt}\overrightarrow{u_\rho} + \rho\frac{d\overrightarrow{u_\rho}}{dt} \quad \text{(Product Rule)}$$

$$= \dot{\rho}\,\overrightarrow{u_\rho} + \rho\frac{d\overrightarrow{u_\rho}}{dt}$$

The unit vector derivative is:

$$\frac{d\overrightarrow{u_\rho}}{dt} = \frac{d\theta}{dt}\overrightarrow{u_\theta} = \dot{\theta}\overrightarrow{u_\theta}$$

Therefore,

$$\vec{v} = \underbrace{\dot{\rho}\overrightarrow{u_\rho}}_{\text{Radial}} + \underbrace{\rho\dot{\theta}\overrightarrow{u_\theta}}_{\text{Tangential}}$$

$$\boxed{v_\rho = \dot{\rho}}$$

$$\boxed{v_\theta = \rho\dot{\theta}}$$

Conversion to Cartesian coordinates:

$$v_x = \dot{\rho}\cos\theta - \rho\dot{\theta}\sin\theta$$

$$v_y = \dot{\rho}\sin\theta + \rho\dot{\theta}\cos\theta$$

II.2.5.c) Vector velocity in cylindrical coordinates:

$$\vec{v} = \frac{d\vec{r}}{dt} = \frac{d}{dt}(\rho\overrightarrow{u_\rho} + z\vec{k})$$

$$\frac{d\rho}{dt}\overrightarrow{u_\rho} + \rho\frac{d\overrightarrow{u_\rho}}{dt} + \frac{dz}{dt}\vec{k} \quad \text{(Product Rule)}$$

The unit vector derivative is:

$$\frac{d\overrightarrow{u_\rho}}{dt} = \frac{d\theta}{dt}\overrightarrow{u_\theta} = \dot{\theta}\overrightarrow{u_\theta}$$

Therefore,

$$\vec{v} = \underbrace{\dot{\rho}\overrightarrow{u_\rho}}_{\text{Radial}} + \underbrace{\rho\dot{\theta}\overrightarrow{e_\theta}}_{\text{Angular}} + \underbrace{\dot{z}\vec{k}}_{\text{Vertical}}$$

$$\boxed{v_\rho = \dot{\rho}}$$

$$\boxed{v_\theta = \rho\dot{\theta}}$$

$$\boxed{v_z = \dot{z}}$$

Conversion to Cartesian:

$$v_x = \dot{\rho}\cos\theta - \rho\dot{\theta}\sin\theta$$

$$v_y = \dot{\rho}\sin\theta + \rho\dot{\theta}\cos\theta$$

$$v_z = \dot{z}$$

Example:

A drone moves with the following cylindrical coordinates as functions of time:

$$\rho(t) = 2t\ \mathrm{m}$$

$$\theta(t) = \frac{\pi}{4}t\ \mathrm{rad}$$

$$z(t) = 3t^2\ \mathrm{m}$$

Find its velocity vector in cylindrical and Cartesian coordinates at (t = 2s).

Solution:

$$\rho(t) = 2t\ \mathrm{m}$$

$$\dot{\rho} = 2\ \mathrm{m/s}$$

$$\theta(t) = \frac{\pi}{4}t\ \mathrm{rad}$$

$$\dot{\theta} = \frac{\pi}{4}\ \mathrm{rad/s}$$

$$z(t) = 3t^2\ \mathrm{m}$$

$$\dot{z} = 6(2) = 12\ \mathrm{m/s}$$

Using the fundamental equation of the velocity:

$$\vec{v} = \dot{\rho}\overrightarrow{u_\rho} + \rho\dot{\theta}\overrightarrow{u_\theta} + \dot{z}\vec{k}$$

$$\boxed{\vec{v} = 2\overrightarrow{u_\rho} + \pi\overrightarrow{u_\theta} + 12\vec{k}\ \mathrm{m/s}}$$

In cartesian coordinates:

$$\boxed{\vec{v} = -\pi\vec{\imath} + 2\vec{\jmath} + 12\vec{k}\ \mathrm{m/s} \approx -3.14\vec{\imath} + 2\vec{\jmath} + 12\vec{k}\ \mathrm{m/s}}$$

II.2.5.d) Vector velocity in spherical coordinates:

$$\vec{v} = \frac{d\vec{r}}{dt} = \frac{d}{dt}(r\,\overrightarrow{u_r})$$

$$= \dot{r}\overrightarrow{U_r} + r\frac{d\overrightarrow{U_r}}{dt}$$

The unit vector derivative is:

$$\frac{d\overrightarrow{U_r}}{dt} = \dot{\theta}\,\overrightarrow{U_\theta} + \dot{\varphi}\sin\theta\,\overrightarrow{U_\varphi}$$

$\overrightarrow{U_\theta}$: Polar unit vector

$$\overrightarrow{U_\varphi}\text{: Azimuthal unit vector}$$

$$\vec{v} = \underbrace{\dot{r}\,\overrightarrow{U_r}}_{\text{Radial}} + \underbrace{r\dot{\theta}\,\overrightarrow{U_\theta}}_{\text{Polar}} + \underbrace{r\dot{\varphi}\sin\theta\,\overrightarrow{U_\varphi}}_{\text{Azimuthal}}$$

$$\boxed{v_r = \dot{r}}$$

$$\boxed{v_\theta = r\dot{\theta}}$$

$$\boxed{v_\varphi = r\dot{\varphi}\sin\theta}$$

Conversion to Cartesian:

$$v_x = \dot{r}\sin\theta\cos\varphi + r\dot{\theta}\cos\theta\cos\varphi - r\dot{\varphi}\sin\theta\sin\varphi$$

$$v_y = \dot{r}\sin\theta\sin\varphi + r\dot{\theta}\cos\theta\sin\varphi + r\dot{\varphi}\sin\theta\cos\varphi$$

$$v_z = \dot{r}\cos\theta - r\dot{\theta}\sin\theta$$

<u>Example:</u>

A weather balloon moves with the following spherical coordinates as functions of time:

$$r(t) = 3t^2 + 1\text{ m}$$

$$\theta(t) = \frac{\pi}{12}t\text{ rad}$$

$$\varphi(t) = \frac{\pi}{6}t\text{ rad}$$

Find its velocity vector in spherical and Cartesian coordinates at t = 2 s.

Solution:

$$r = 3(2)^2 + 1 = 13\text{ m}$$

$$\dot{r} = 6(2) = 12\text{ m/s}$$

$$\theta = \frac{\pi}{12}(2) = \frac{\pi}{6}\text{ rad}$$

$$\dot{\theta} = \frac{\pi}{12}\text{ rad/s}$$

$$\varphi = \frac{\pi}{6}(2) = \frac{\pi}{3}\text{ rad}$$

$$\dot{\varphi} = \frac{\pi}{6}\text{ rad/s}$$

$$\vec{v} = \dot{r}\,\overrightarrow{U_r} + r\dot{\theta}\,\overrightarrow{U_\theta} + r\dot{\varphi}\sin\theta\,\overrightarrow{U_\varphi}$$

$$\boxed{\vec{v} = 12\overrightarrow{U_r} + 3.40\overrightarrow{U_\theta} + 3.40\overrightarrow{U_\varphi}\text{ m/s}}$$

II.2.6 Acceleration ($\vec{a}$):

This is the rate of change of an object's velocity with respect to time. Mathematically, it is the second derivative of a particle's position vector with respect to time, therefore, the acceleration is also a vector quantity.

II.2.6.a) Acceleration vector in Cartesian coordinates:

The acceleration can be defined as the second derivative of the position:

$$\vec{a}(t) = \frac{d\vec{v}}{dt} = \frac{d^2\vec{r}}{dt^2}$$

For a position vector :

$$\vec{r}(t) = x(t)\vec{\imath} + y(t)\vec{\jmath} + z(t)\vec{k}$$

The acceleration vector can be given as:

$$\vec{a}(t) = \frac{d^2x}{dt^2}\vec{\imath} + \frac{d^2y}{dt^2}\vec{\jmath} + \frac{d^2z}{dt^2}\vec{k}$$

$$= \ddot{x}\vec{\imath} + \ddot{y}\vec{\jmath} + \ddot{z}\vec{k}$$

Example:

Given the position functions:

$$x(t) = 2t^3 - 4t$$

$$y(t) = 5cos(\pi\, t)$$

$$z(t) = 3e^{0.2t}$$

Find the acceleration vector at t = 1s.

Solution :

The second derivatives

$$\ddot{x}(t) = \frac{d^2}{dt^2}(2t^3 - 4t) = 12t$$

$$\ddot{y}(t) = \frac{d^2}{dt^2}(5\cos\pi\, t) = -5\pi^2\cos\pi\, t$$

$$\ddot{z}(t) = \frac{d^2}{dt^2}(3e^{0.2t}) = 3(0.2)^2 e^{0.2t} = 0.12e^{0.2t}$$

At t = 1s.

$$\ddot{x}(1) = 12(1) = 12\ \mathrm{m/s^2}$$

$$\ddot{y}(1) = -5\pi^2\cos(\pi\cdot 1) = 5\pi^2 \approx 49.35\ \mathrm{m/s^2}$$

$$\ddot{z}(1) = 0.12e^{0.2} \approx 0.146\ \mathrm{m/s^2}$$

Therefore, the acceleration vector is:

$$\boxed{\vec{a}(1) = 12\vec{\imath} + 49.35\vec{\jmath} + 0.146\vec{k}\ \mathrm{m/s^2}}$$

II.2.6. b) Acceleration polar coordinates:

The acceleration vector in polar coordinates is derived from the second time derivative of the position vector.

$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{d^2\vec{r}}{dt^2}$$

From the velocity vector in polar coordinates

$$\vec{a} = \frac{d}{dt}\left(\dot{\rho}\overrightarrow{U_\rho} + \rho\dot{\theta}\overrightarrow{U_\theta}\right)$$

$$= \ddot{\rho}\overrightarrow{U_\rho} + \dot{\rho}\frac{d\overrightarrow{U_\rho}}{dt} + \frac{d}{dt}(\rho\dot{\theta})\overrightarrow{U_\theta} + \rho\dot{\theta}\frac{d\overrightarrow{U_\theta}}{dt}$$

Using unit vector derivatives:

$$\frac{d\overrightarrow{U_\rho}}{dt} = \dot{\theta}\,\overrightarrow{U_\theta}$$

$$\frac{d\overrightarrow{U_\theta}}{dt} = -\dot{\theta}\,\overrightarrow{U_\rho}$$

Therefore,

$$\vec{a} = \underbrace{\left(\ddot{\rho} - \rho\dot{\theta}^2\right)\overrightarrow{U_\rho}}_{\text{Radial}} + \underbrace{\left(\rho\ddot{\theta} + 2\dot{\rho}\dot{\theta}\right)\overrightarrow{U_\theta}}_{\text{Angular}}$$

For uniform circular motion

$$\dot{\rho} = 0, \ddot{\theta} = 0$$

$$a_\rho = -\rho\dot{\theta}^2 \quad \text{(Centripetal)}$$

$$a_\theta = 0$$

$$\boxed{\vec{a} = -\rho\omega^2\overrightarrow{U_\rho}} \quad \text{where } \omega = \dot{\theta}$$

II.2.6.c) Acceleration in cylindrical coordinates:

The acceleration vector in cylindrical coordinates is derived from the second time derivative of the position vector.

$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{d^2\vec{r}}{dt^2}$$

$$\vec{r} = \rho\,\overrightarrow{U_\rho} + z\,\vec{k}$$

$$\vec{a} = \frac{d}{dt}\left(\dot{\rho}\overrightarrow{U_\rho} + \rho\dot{\theta}\overrightarrow{U_\theta} + \dot{z}\,\overrightarrow{U_z}\right)$$

$$= \ddot{\rho}\overrightarrow{U_\rho} + \dot{\rho}\frac{d\overrightarrow{U_\rho}}{dt} + \frac{d}{dt}(\rho\dot{\theta})\overrightarrow{U_\theta} + \rho\dot{\theta}\frac{d\overrightarrow{U_\theta}}{dt} + \ddot{z}\,\vec{k}$$

$$\frac{d\overrightarrow{U_\rho}}{dt} = \dot{\theta}\,\overrightarrow{U_\theta}$$

$$\frac{d\overrightarrow{U_\theta}}{dt} = -\dot{\theta}\;\overrightarrow{U_\rho}$$

$$\frac{d\vec{k}}{dt} = 0$$

Therefore,

$$\vec{a} = \left(\ddot{\rho} - \rho\dot{\theta}^2\right)\overrightarrow{U_\rho} + (\rho\ddot{\theta} + 2\dot{\rho}\,\dot{\theta})\,\overrightarrow{U_\theta} + +\ddot{z}\,\vec{k}$$

II.2.6.d) Acceleration in spherical coordinates:

In spherical coordinates, the position vector is:

$$\vec{r} = r\overrightarrow{U_r}$$

The acceleration vector in spherical coordinates is derived from the second time derivative of the position vector.

$$\vec{a}(t) = \frac{\mathrm{d}\vec{V}(t)}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\dot{\rho}\cdot\overrightarrow{U_\rho} + \rho\cdot\dot{\theta}\cdot\overrightarrow{U_\theta}\right) = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathrm{d}\rho}{\mathrm{d}t}\cdot\overrightarrow{U_\rho} + \rho\cdot\frac{\mathrm{d}\theta}{\mathrm{d}t}\cdot\overrightarrow{U_\theta}\right)$$

$$= \left[\frac{\mathrm{d}^2\rho}{\mathrm{d}t^2}\cdot\overrightarrow{U_\rho} + \frac{\mathrm{d}\rho}{\mathrm{d}t}\cdot\frac{\mathrm{d}\overrightarrow{U_\rho}}{\mathrm{d}t}\right] + \left[\frac{\mathrm{d}\rho}{\mathrm{d}t}\cdot\frac{\mathrm{d}\theta}{\mathrm{d}t}\cdot\overrightarrow{U_\theta} + \rho\cdot\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2}\cdot\overrightarrow{U_\theta} + \rho\cdot\frac{\mathrm{d}\theta}{\mathrm{d}t}\cdot\frac{\mathrm{d}\overrightarrow{U_\theta}}{\mathrm{d}t}\right]$$

II.3. Movement (motion) of a particle point:

The motion or the movement of a particle point ‘material point’ refers to a phenomenon in which the change in the position of a material mass of negligible dimensions relative to a frame of reference over time is studied(Chow, 2024).

II.3.1. Type of movement (motion) of a particle point:

The types of motion of a material point can be classified based on path, velocity, acceleration, and force.

II.3.1.a) Rectilinear Motion (Straight Line)

In the rectilinear motion, the particle moves along a straight path.

The vector position: For motion along the (x) −axis, the position vector of a material point at time (t) is:

$$\vec{r}(t) = x(t)\,\vec{\imath}$$

$x(t)$ is the position function, $\vec{\imath}$, is the unit vector in x axis.

In this case, the motion is purely along the ($\vec{\imath}$)-direction: therefore, $y(t) = 0, and\ z(t) = 0$.

The vector velocity: The vector velocity is the time derivative of the position.

$$\vec{v}(t) = \dot{x}(t)\,\vec{\imath}$$

The vector acceleration: The vector acceleration is the time derivative of the velocity.

$$\vec{a}(t) = \ddot{x}(t)\,\vec{\imath}$$

From acceleration to velocity:

$$\vec{v}(t) = \overrightarrow{v_0} + \int_{t_0}^{t} \vec{a}(\mathrm{t})\ dt$$

Where, $\overrightarrow{v_0}$ is the initial velocity of the particle.

$If\,(\vec{a}(t) = a_0\,\vec{\imath})\ is\ constant$:

$\vec{v}(t) = \overrightarrow{v_0} + a_0(t - t_0)\,\vec{\imath}$

From the velocity to position:

$$\vec{r}(t) = \overrightarrow{r_0} + \int_{t_0}^{t} \vec{v}(\mathrm{t})\ dt$$

Where, $\overrightarrow{r_0}\ is\ the\ initial\ position\ of\ the\ particle.$

$$\vec{r}(t) = \overrightarrow{r_0} + \overrightarrow{v_0}(t - t_0) + \frac{1}{2}a_0(t - t_0)^2\,\vec{\imath}$$

II.3.1.b) Uniform rectilinear motion:

Uniform Rectilinear Motion (URM) is the motion of a material point along a straight line with the following characteristics:

1- A constant velocity ($\vec{v}(t) = \overrightarrow{v_0}$).
2- A zero acceleration $\vec{a}(t) = 0$.
3- A linear trajectory along one axis ($e.g., the\ (\ x\ ) - axis$).

Analysis:

Let the motion be along the $(\ x\ )$ −axis. The position vector at time $(\ t\ )$ is:

$$\vec{r}(t) = x(t)\,\vec{\imath}$$

Since the velocity is constant:

$$\vec{v}(t) = \dot{x}(t)\,\vec{\imath} = v_0\,\vec{\imath}$$

If the initial position is ($\overrightarrow{r_0} = x_0\,\vec{\imath}$), then:

$$\vec{r}(t) = x_0\,\vec{\imath} + v_0(t - t_0)\,\vec{\imath} = [x_0 + v_0(t - t_0)]\vec{\imath}$$

Therefore,

$$x(t) = x_0 + v_0(t - t_0)$$

<u>The acceleration:</u>

$$\vec{a}(t) = \dot{v}_0\,\vec{\imath} = \vec{0}$$

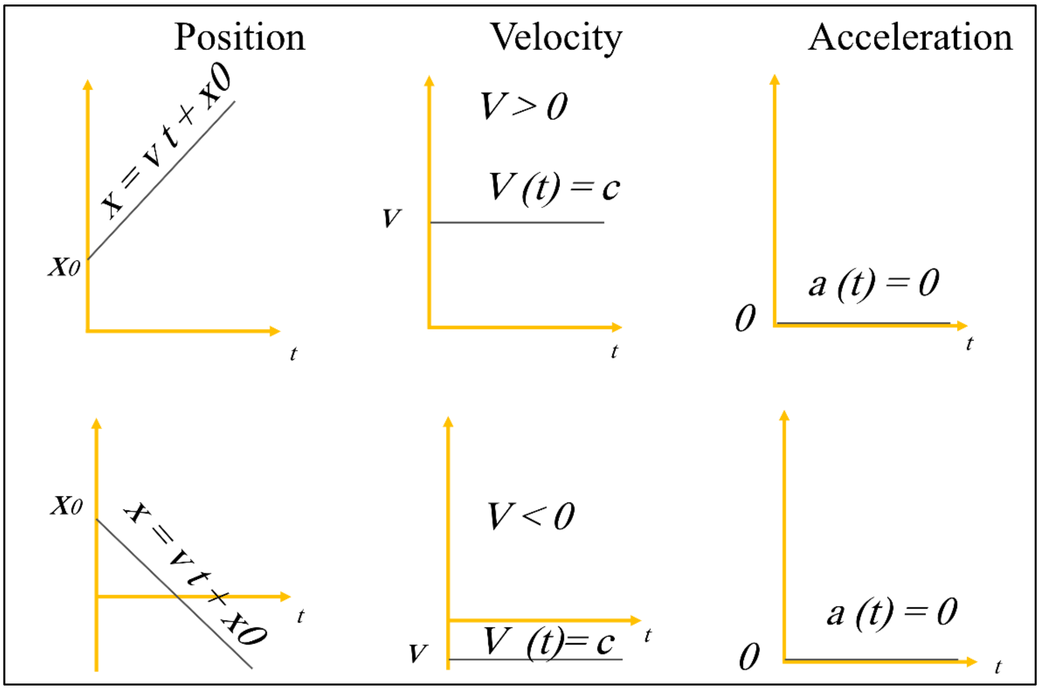

Figure 11 Uniform rectilinear motion

II.3.1.c) Uniformly varied rectilinear movement:

Uniformly varying rectilinear motion refers to the motion of a material point along a straight line under the following conditions:

1- Constant acceleration ($\vec{a}(t) = \overrightarrow{a_0}$)
2- A changing velocity linearly over time.
3- A motion along a single direction, assumed here to be the (x) −axis.

Analysis:

We consider motion only along the (x) −axis:

$$\vec{r}(t) = x(t)\,\vec{\imath}$$

The acceleration vector given as :

$$\vec{a}(t) = \overrightarrow{a_0} = a_0\,\vec{\imath}$$

Velocity is the time integral of acceleration:

$$\vec{v}(t) = \overrightarrow{v_0} + \int_{t_0}^{t} \vec{a}(\mathrm{t})\ dt$$

Therefore,

$$\vec{v}(t) = \overrightarrow{v_0} + \overrightarrow{a_0}(t - t_0)$$

$$If\,(\overrightarrow{v_0} = v_0\,\vec{\imath})and(\overrightarrow{a_0} = a_0\,\vec{\imath}), then:$$

$$\vec{v}(t) = [v_0 + a_0(t - t_0)]\vec{\imath}$$

Position is the integral of velocity

$\vec{r}(t) = \overrightarrow{r_0} + \int_{t_0}^{t} \vec{v}(\mathrm{t})\ d\mathrm{t}$

Substitute $(\vec{v}(\mathrm{t}) = \overrightarrow{v_0} + \overrightarrow{a_0}(\mathrm{t} - t_0))$

$$\vec{r}(t) = \overrightarrow{r_0} + \int_{t_0}^{t} [\overrightarrow{v_0} + \overrightarrow{a_0}(\mathrm{t} - t_0)]d\mathrm{t}$$

$$Thus, if\,(\overrightarrow{r_0} = x_0\,\vec{\imath}), (\overrightarrow{v_0} = v_0\,\vec{\imath}), t_0\ =\ 0\ s\ and(\overrightarrow{a_0} = a_0\,\vec{\imath}), then:$$

$$\vec{r}(t) = \left[x_0 + v_0(t) + \frac{1}{2}a_0(t)^2\right]\vec{\imath}$$

<u>Example:</u>

The initial conditions of the motion of a particle are:

$$x_0 = 0 \quad \mathrm{m},$$

$$v_0 = 5 \quad \mathrm{m/s},$$

$$a_0 = 2 \quad \mathrm{m/s^2},$$

$$t_0 = 0 \quad \mathrm{s}$$

So:

$$x(t) = x_0 + v_0(t) + \frac{1}{2}a_0(t)^2 = 5t + t^2$$

$$v(t) = v_0 + a_0(t - t_0) = 5 + 2t$$

$$a(t) = a_0 = 2\ \mathrm{m/s^2},$$

The variation of the position function, the velocity function, and the acceleration function with time presented in the following figure.

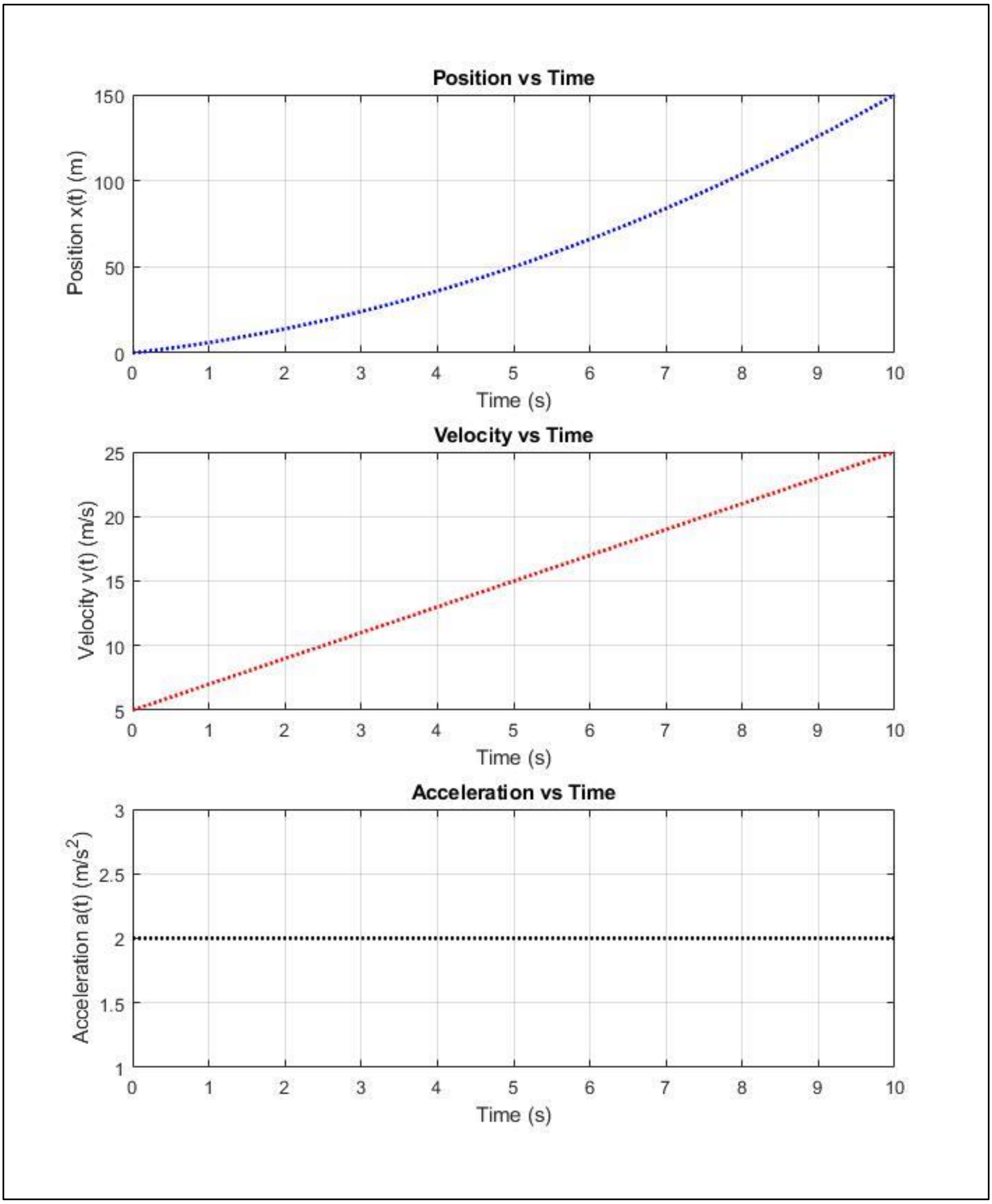

Figure 12 Uniformly varied rectilinear movement

II.3.1.d) Sinusoidal rectilinear motion :

Sinusoidal Rectilinear Motion refers to the oscillatory movement of a particle point along a straight line where its displacement "position", velocity, and acceleration follow a sinusoidal pattern which means sine or cosine functions.

Analysis:

The displacement of a particle M moving along a straight line be given by:

$$\vec{x}(t) = A\sin(\omega t + \phi)\ \vec{\imath}$$

Here,

A is the amplitude of the motion of the particle M, which is defined as the maximum displacement from equilibrium.

ϕ is the phase angle.

t is the time.

The velocity:

The velocity is the first derivative of the position:

$$\vec{v}(t) = A\omega\cos(\omega t + \phi)\ \vec{\imath}$$

The acceleration :

The acceleration is the time derivative of velocity:

$$\vec{a}(t) = -A\omega^2\sin(\omega t + \phi)\ \vec{\imath}$$

This relation can be given as :

$$\vec{a}(t) = -\omega^2\vec{x}(t)$$

Characteristics of the motion :

1- $T = \frac{2\pi}{\omega}$ Period
2- $f = \frac{1}{T} = \frac{\omega}{2\pi}$ Frequency
3- $v_{\max} = A\omega$ maximum velocity
4- $a_{\max} = A\omega^2$ maximum acceleration

Example:

Let's take a particle that moves according to the following equation :

$$\vec{x}(t) = A\sin(\omega t + \phi)\ \vec{\imath}$$

Were A is the amplitude A = 3 m and ω is the angular frequency $\omega = 2$ and $\phi = 0$

1-Find the velocity and acceleration of the particle

2- plot the variation of the position, velocity, and acceleration as a function of time.

Solution:

The velocity is the first derivative of the position:

$$\vec{v}(t) = 6\cos(2t)\ \vec{\imath}$$

The acceleration is the time derivative of velocity:

$$\vec{a}(t) = -12\sin(2t)\ \vec{\imath}$$

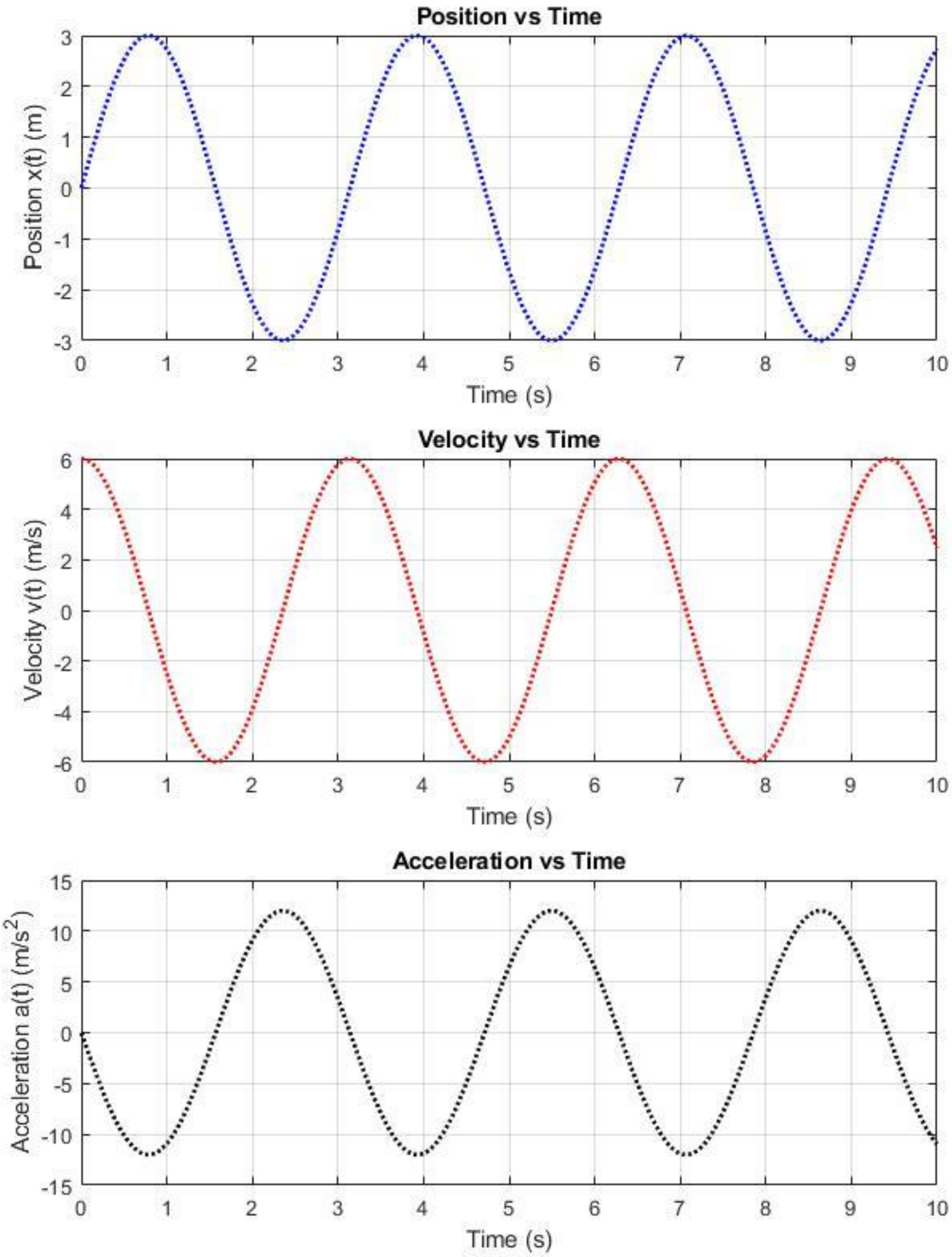

Figure 13 Sinusoidal rectilinear motion

II.3.2. Curvilinear Motion

Curvilinear Motion refers to the movement of an object (in our study, a particle point) along a curved path in two or three dimensions. Contrary to rectilinear motion, where the motion of the particle is described as a straight-line motion, the trajectory in curvilinear motion is characterized by continuous changes in direction of the object.

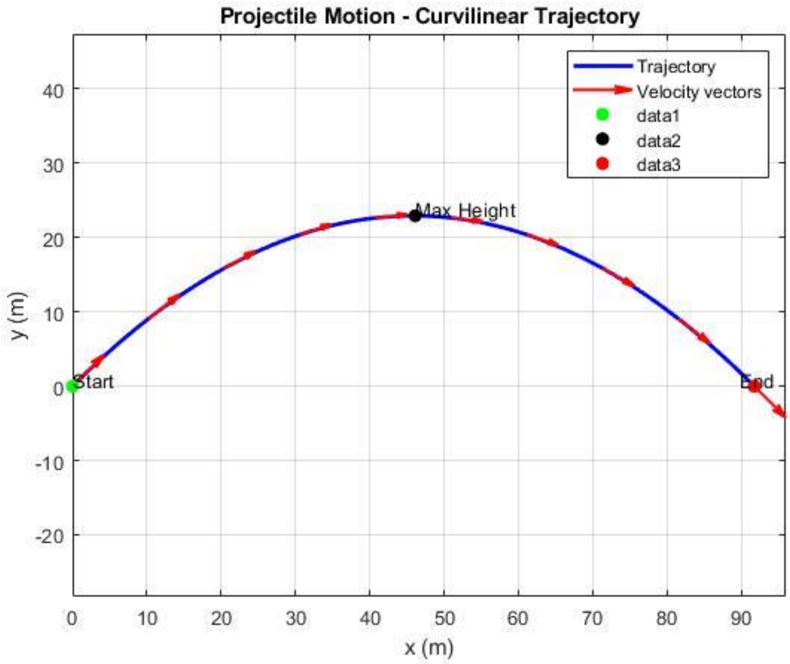

Figure 14 Curvilinear Motion

To analyze the motion of a particle that moves along a curved path, we use the intrinsic 'Frenet-Serret' coordinates. These coordinates are defined by the following unit vector:

II.3.2.a) Curvilinear abscissa:

Let *M* be a material point that moves along a curvilinear trajectory (C). The intrinsic position of M at time t, relative to an initial position M_0, is defined by the curvilinear abscissa.

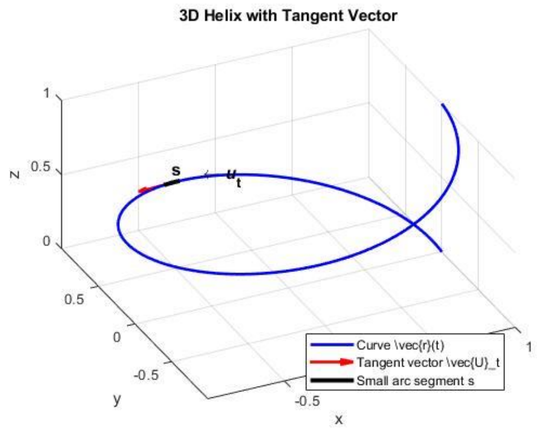

Figure 15 Curvilinear abscissa

II.3.2.b) The unit vectors:

The tangential unit vector:

The tangential unit vector is a vector that is tangent to the trajectory at a given point and oriented in the direction of increasing arc length along the curve.

$$\overrightarrow{U_t} = \frac{d\vec{r}}{ds}$$

Here, the arc length s is a scalar quantity that measures the distance traveled along a curve from a fixed reference point (curvilinear abscissa), and the differential ds represents an infinitesimal element of arc length along the curve.

Normal unit vector:

The normal unit vector is a vector that points in the direction of the tangent vector $\overrightarrow{U_t}$ is changing. It is perpendicular to the tangent vector and lies in the osculating plane of the curve.

$$\overrightarrow{U_n} = \frac{\frac{d\overrightarrow{U_t}}{ds}}{\left|\frac{d\overrightarrow{U_t}}{ds}\right|}$$

The binormal vector:

The binormal vector is the vector product (cross product $\times$) of the tangent vector $\overrightarrow{U_t}$ and the normal vector $\overrightarrow{U_n}$.

Properties of the Unit Vectors:

$$\overrightarrow{U_t} \cdot \overrightarrow{U_n} = 0,$$

$$\overrightarrow{U_n} \cdot \overrightarrow{U_b} = 0,$$

$$\overrightarrow{U_b} \cdot \overrightarrow{U_t} = 0$$

$$\overrightarrow{U_t} \times \overrightarrow{U_n} = \overrightarrow{U_b}$$

II.3.2. c) Velocity and Acceleration in Intrinsic Coordinates:

Let the position of a particle moving along a space curve be given by the vector function $\vec{r}(t)$. The velocity and acceleration vectors can be expressed in terms of intrinsic coordinates using the arc length $s(t)$ as an intermediate variable.

$$\vec{v}(t) = \frac{d\vec{r}}{dt}$$

We introduce the arc length $s(t)$, and apply the chain rule:

$$\vec{v}(t) = \frac{d\vec{r}}{ds} \cdot \frac{ds}{dt}$$

Since $(\frac{d\vec{r}}{ds})$ is the unit tangent vector $(\overrightarrow{U_t})$, and $(\frac{ds}{dt} = v)$ is the speed:

$$\vec{v}(t) = v\overrightarrow{U_t}$$

Acceleration vector:

$$\vec{a}(t) = \frac{d\vec{v}}{dt} = \frac{d}{dt}\left(v\overrightarrow{U_t}\right)$$

Apply the product rule:

$$\vec{a}(t) = \frac{dv}{dt}\overrightarrow{U_t} + v\frac{d\overrightarrow{U_t}}{dt}$$

We compute now $\frac{d\overrightarrow{U_t}}{dt}$. Use the chain rule again:

$$\frac{d\overrightarrow{U_t}}{dt} = \frac{d\overrightarrow{U_t}}{ds} \cdot \frac{ds}{dt} = \frac{d\overrightarrow{U_t}}{ds} \cdot v$$

We have:

$$\frac{d\overrightarrow{U_t}}{ds} = \kappa\overrightarrow{U_n}$$

$$\frac{d\overrightarrow{U_t}}{dt} = \kappa v\overrightarrow{U_n}$$

Substitute back into the expression for $(\vec{a}(t)$:

$$\vec{a}(t) = \frac{dv}{dt}\overrightarrow{U_t} + v\left(\kappa v\overrightarrow{U_n}\right)$$

So,

$$\vec{a}(t) = \frac{dv}{dt}\overrightarrow{U_t} + \kappa v^2\overrightarrow{U_n}$$

$$\vec{v}(t) = v\overrightarrow{U_t}$$

$$\vec{a}(t) = \frac{dv}{dt}\overrightarrow{U_t} + \kappa v^2\overrightarrow{U_n}$$

Relative Motion

III. Relative motion

III.1 Introduction

Relative motion refers to the motion of a particle point with respect to another moving particle point or frame of reference.

III.2 Example from real life:

Walking on a Moving Walkway (like in an airport)

- You're walking forward on a moving walkway.
- To someone standing **on the walkway**, you're just walking normally.
- To someone **standing on the ground**, you're moving faster because the walkway adds speed.

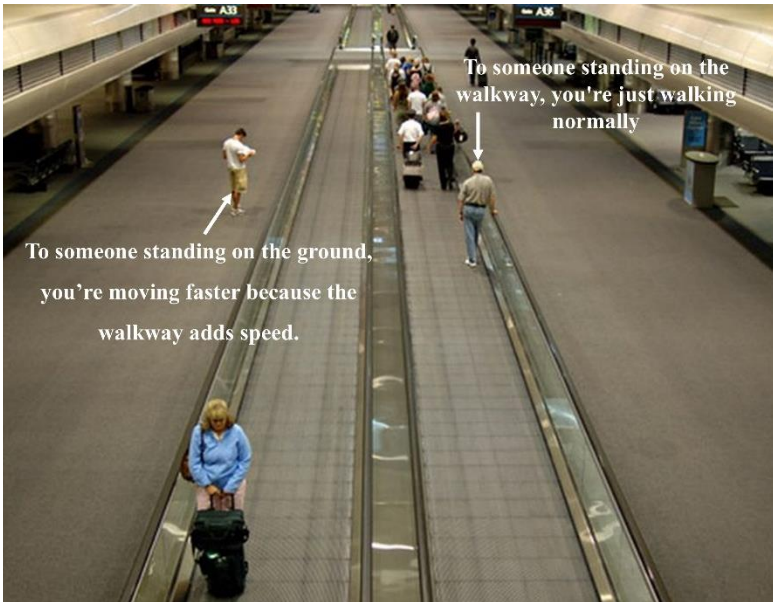

Figure 16 Walking on a moving walkway

Cars Passing Each Other

You're sitting in a car on the highway going 100 km/h.

Another car passes you going 120 km/h.

Even though both cars are moving fast, the other car looks like it's moving slowly past you — just at 20 km/h.

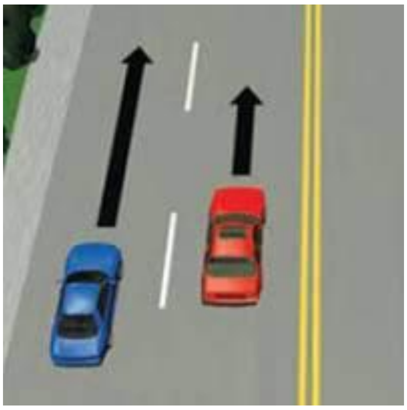

Figure 17 Cars passing each other

III.3. Definitions:

We consider two reference frames ***R*** (O, x, y, z) and ***R'*** (O', x', y', z') with the basis:

$\vec{\imath}, \quad \vec{\jmath}, \quad \vec{k}$, and $\vec{\imath'}, \quad \vec{\jmath'}, \quad \vec{k'}$ respectively

III.3.1. The absolute frame of reference:

An absolute frame of reference, also known as an inertial frame, is a coordinate system in which Newton's laws of motion apply without the need to introduce imaginary forces. It is considered stationary in space and unaccelerated.

In this reference, the unit vectors are fixed, that means the magnitude and the direction of the unit vectors do not change with time:

$$\frac{d\vec{\imath}}{dt} = \vec{0}, \qquad \frac{d\vec{\jmath}}{dt} = \vec{0}, \qquad \frac{d\vec{k}}{dt} = \vec{0}$$

III.3.2. Relative reference:

A relative reference frame is a coordinate system that is moving or rotating with respect to an absolute (inertial) frame.

III.3.3. The absolute motion:

Absolute motion refers to the motion of an object as observed from a fixed, non-moving (inertial) reference frame.

Vector Position in Absolute reference:

$$\overrightarrow{OM}(t) = \vec{r}(t) = x(t)\,\vec{\imath} + y(t)\,\vec{\jmath} + z(t)\,\vec{k}$$

Vector position in Relative reference:

$$\vec{r'}(t) = x'(t)\,\vec{\imath'} + y'(t)\,\vec{\jmath'} + z'(t)\,\vec{k'}$$

III.3.4. Motion of a Relative Reference Frame with Respect to an Absolute Reference Frame:

Absolute frame unit vectors:

$$\vec{\imath}, \quad \vec{\jmath}, \quad \vec{k}$$

Relative frame origin and unit vectors:

$$O', \quad \overrightarrow{\imath'}(t), \quad \overrightarrow{\jmath'}(t), \quad \overrightarrow{k'}(t)$$

$$\overrightarrow{OM}(t) = \vec{r}(t) = x(t)\,\vec{\imath} + y(t)\,\vec{\jmath} + z(t)\,\vec{k}$$

$$\overrightarrow{r_{O'}}(t) \quad \text{(position of } O' \text{ in absolute frame),}$$

$$\overrightarrow{r'}(t) \quad \text{(position of } M \text{ in relative frame)}$$

Case 1:

If there is only translation and no rotation, then

$$\overrightarrow{\imath'} = \vec{\imath}, \quad \overrightarrow{\jmath'} = \vec{\jmath}, \quad \overrightarrow{k'} = \vec{k}$$

$$\vec{r}(t) = \overrightarrow{r_{O'}}(t) + \overrightarrow{r'}(t) = \overrightarrow{r_{O'}}(t) + x'(t)\vec{\imath} + y'(t)\vec{\jmath} + z'(t)\vec{k}$$

Case 2

III.3.5 Rotation:

In this case:

$$\left(\frac{d\overrightarrow{\imath'}}{dt}\right)_{\mathcal{R}} \neq \vec{0},$$

$$\left(\frac{d\overrightarrow{\jmath'}}{dt}\right)_{\mathcal{R}} \neq \vec{0},$$

$$\left(\frac{d\overrightarrow{k'}}{dt}\right)_{\mathcal{R}} \neq \vec{0}$$

When the relative frame rotates with angular velocity $\vec{\omega}(t)$, the unit vectors vary as:

$$\frac{d\overrightarrow{\imath'}}{dt} = \vec{\omega} \times \overrightarrow{\imath'},$$

$$\frac{d\overrightarrow{\jmath'}}{dt} = \vec{\omega} \times \overrightarrow{\jmath'},$$

$$\frac{d\overrightarrow{k'}}{dt} = \vec{\omega} \times \overrightarrow{k'}$$

The position vector of point M in the absolute frame is:

$$\vec{r}(t) = \overrightarrow{r_{O'}}(t) + x'(t)\overrightarrow{\imath'}(t) + y'(t)\overrightarrow{\jmath'}(t) + z'(t)\overrightarrow{k'}(t)$$

The position vector of M in relative frame:

$$\overrightarrow{r'}(t) = x'(t)\,\overrightarrow{\imath'} + y'(t)\,\overrightarrow{\jmath'} + z'(t)\,\overrightarrow{k'}$$

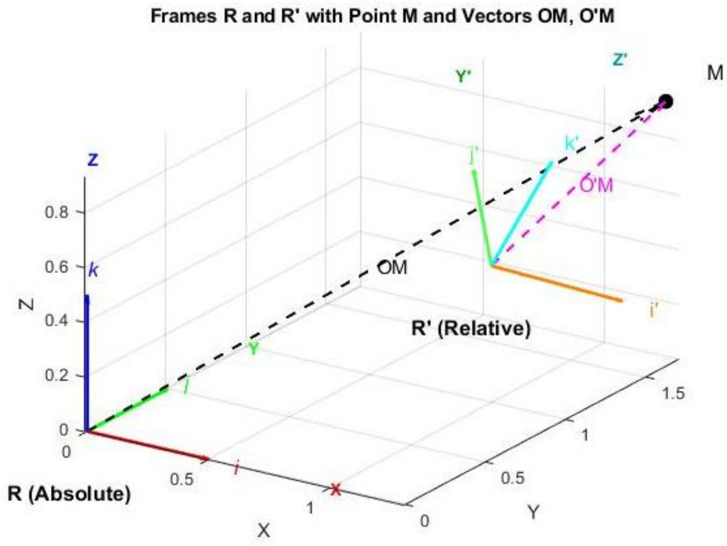

Figure 18 Absolute and relative frame

<u>Velocity in Case of Translation:</u>

In this case, we study the translation of reference R'.

$$\overrightarrow{\imath'} = \vec{\imath}, \quad \overrightarrow{\jmath'} = \vec{\jmath}, \quad \overrightarrow{k'} = \vec{k}$$

The position vector of the point M:

$$\overrightarrow{r_M} = \overrightarrow{r_{O'}} + \overrightarrow{r'_M}$$

$$\overrightarrow{V_{(M/\mathcal{R})}} = \frac{d\overrightarrow{r_M}}{dt} = \frac{d\overrightarrow{r_{O'}}}{dt} + \frac{d\overrightarrow{r'_M}}{dt} = \overrightarrow{v_{O'}} + \overrightarrow{V_{(M/\mathcal{R}')}}$$

$$\overrightarrow{v_{O'}} = \frac{d\overrightarrow{O'O}}{dt}$$

$\overrightarrow{v_{O'}} = \overrightarrow{V_e}$ the motion of reference R' compared to R

$$\overrightarrow{V_{(M/\mathcal{R}')}} = \left(\frac{d\overrightarrow{r'_M}}{dt}\right)_{\mathcal{R}'} = \frac{dx'}{dt}\overrightarrow{\imath'} + \frac{dy'}{dt}\overrightarrow{\jmath'} + \frac{dz'}{dt}\overrightarrow{k'}$$

$$\overrightarrow{V_{(M/\mathcal{R}')}} = \dot{x}'\,\overrightarrow{\imath'} + \dot{y}'\,\overrightarrow{\jmath'} + \dot{z}'\,\overrightarrow{k'}$$

$\overrightarrow{V_{(M/\mathcal{R}')}} = \overrightarrow{V_r}$ The relative velocity

$$\overrightarrow{V_{(M/\mathcal{R})}} = \frac{d\overrightarrow{O'O}}{dt} + \dot{x}'\,\overrightarrow{\imath'} + \dot{y}'\,\overrightarrow{\jmath'} + \dot{z}'\,\overrightarrow{k'}$$

$\overrightarrow{V_{(M/\mathcal{R})}} = \overrightarrow{V_a}$ The absolute velocity

In this case:

$$\frac{d\vec{\imath'}}{dt} = \vec{0},$$

$$\frac{d\vec{\jmath'}}{dt} = \vec{0},$$

$$\frac{d\vec{k'}}{dt} = \vec{0}$$

$$\overrightarrow{V_{(M/\mathcal{R})}} = \vec{V_a} = \left(\frac{d\overrightarrow{OM}}{dt}\right)_{\mathcal{R}}$$

$$= \dot{x}\vec{\imath} + \dot{y}\vec{\jmath} + \dot{z}\vec{k} + x\left(\frac{d\vec{\imath}}{dt}\right)_{\mathcal{R}} + y\left(\frac{d\vec{\jmath}}{dt}\right)_{\mathcal{R}} + z\left(\frac{d\vec{k}}{dt}\right)_{\mathcal{R}}$$

$$= \dot{x}\vec{\imath} + \dot{y}\vec{\jmath} + \dot{z}\vec{k} \quad (\text{since } \frac{d\vec{\imath}}{dt_{\mathcal{R}}} = \frac{d\vec{\jmath}}{dt_{\mathcal{R}}} = \frac{d\vec{k}}{dt_{\mathcal{R}}} = \vec{0})$$

As we mentioned before, the velocity of (M) with respect to the moving reference frame ($\mathcal{R}'$) is called the relative velocity, obtained by differentiating the position vector ($\overrightarrow{O'M}$) in frame ($\mathcal{R}'$):

$$\overrightarrow{V_{(M/\mathcal{R}')}} = \vec{V_r} = \left(\frac{d\overrightarrow{O'M}}{dt}\right)_{\mathcal{R}'}$$

$$= \dot{x'}\vec{\imath'} + \dot{y'}\vec{\jmath'} + \dot{z'}\vec{k'} + x'\left(\frac{d\vec{\imath'}}{dt}\right)_{\mathcal{R}'} + y'\left(\frac{d\vec{\jmath'}}{dt}\right)_{\mathcal{R}'} + z'\left(\frac{d\vec{k'}}{dt}\right)_{\mathcal{R}'}$$

As we mentioned before

$$\overrightarrow{OM} = \overrightarrow{OO'} + \overrightarrow{O'M}$$

$$\left(\frac{d\overrightarrow{OM}}{dt}\right)_{\mathcal{R}} = \left(\frac{d\overrightarrow{OO'}}{dt}\right)_{\mathcal{R}} + \left(\frac{d\overrightarrow{O'M}}{dt}\right)_{\mathcal{R}}$$

$$\Rightarrow \vec{V_a} = \left(\frac{d\overrightarrow{OO'}}{dt}\right)_{\mathcal{R}} + \dot{x'}\vec{\imath'} + \dot{y'}\vec{\jmath'} + \dot{z'}\vec{k'} + x'\left(\frac{d\vec{\imath'}}{dt}\right)_{\mathcal{R}} + y'\left(\frac{d\vec{\jmath'}}{dt}\right)_{\mathcal{R}} + z'\left(\frac{d\vec{k'}}{dt}\right)_{\mathcal{R}}$$

$$= \left(\frac{d\overrightarrow{OO'}}{dt}\right)_{\mathcal{R}} + \vec{V_r} + x'(\vec{\omega} \times \vec{\imath'}) + y'(\vec{\omega} \times \vec{\jmath'}) + z'(\vec{\omega} \times \vec{k'})$$

$$= \overrightarrow{V_r} + \left(\frac{d\overrightarrow{OO'}}{dt}\right)_{\mathcal{R}} + \vec{\omega} \times \left(x'\overrightarrow{\imath'} + y'\overrightarrow{\jmath'} + z'\overrightarrow{k'}\right) = \overrightarrow{V_r} + \left(\frac{d\overrightarrow{OO'}}{dt}\right)_{\mathcal{R}} + \vec{\omega} \times \overrightarrow{O'M}$$

$$\overrightarrow{V_a} = \overrightarrow{V_r} + \overrightarrow{V_e}$$

With,

$$\overrightarrow{V_e} = \left(\frac{d\overrightarrow{OO'}}{dt}\right)_{\mathcal{R}} + \vec{\omega} \times \overrightarrow{O'M}$$

Called the transport (or entrainment) velocity.

<u>The acceleration:</u>

The absolute acceleration of a point M is obtained by differentiating its absolute velocity with respect to time in the reference frame $(\mathcal{R})$:

$$\overrightarrow{a_{(M/\mathcal{R})}} = \overrightarrow{a_a} = \left(\frac{d\overrightarrow{V_a}}{dt}\right)_{\mathcal{R}} = = \ddot{x}\,\vec{\imath} + \ddot{y}\,\vec{\jmath} + \ddot{z}\,\vec{k}$$

The relative acceleration of (M) is obtained by differentiating the relative velocity in the rotating reference frame $(\mathcal{R}')$:

$$\overrightarrow{a_{(M/\mathcal{R}')}} = \overrightarrow{a_r} = \left(\frac{d\overrightarrow{V_r}}{dt}\right)_{\mathcal{R}'}$$

$$= \ddot{x}'\overrightarrow{\imath'} + \ddot{y}'\overrightarrow{\jmath'} + \ddot{z}'\overrightarrow{k'}$$

The time derivative of the velocity composition relation:

$$\overrightarrow{V_a} = \left(\frac{d\overrightarrow{OO'}}{dt}\right)_{\mathcal{R}} + \dot{x}'\overrightarrow{\imath'} + \dot{y}'\overrightarrow{\jmath'} + \dot{z}'\overrightarrow{k'} + x'\left(\frac{d\overrightarrow{\imath'}}{dt}\right)_{\mathcal{R}} + y'\left(\frac{d\overrightarrow{\jmath'}}{dt}\right)_{\mathcal{R}} + z'\left(\frac{d\overrightarrow{k'}}{dt}\right)_{\mathcal{R}}$$

$$\left(\frac{d\overrightarrow{V_a}}{dt}\right)_{\mathcal{R}} = \left(\frac{d^2\overrightarrow{OO'}}{dt^2}\right)_{\mathcal{R}} + \ddot{x}'\overrightarrow{\imath'} + \ddot{y}'\overrightarrow{\jmath'} + \ddot{z}'\overrightarrow{k'} + 2\,(\dot{x}'\left(\frac{d\overrightarrow{\imath'}}{dt}\right)_{\mathcal{R}} + \dot{y}'\left(\frac{d\overrightarrow{\jmath'}}{dt}\right)_{\mathcal{R}} + \dot{z}'\left(\frac{d\overrightarrow{k'}}{dt}\right)_{\mathcal{R}})$$
$$+ x'\left(\frac{d^2\overrightarrow{\imath'}}{dt^2}\right)_{\mathcal{R}} + y'\left(\frac{d^2\overrightarrow{\jmath'}}{dt^2}\right)_{\mathcal{R}} + z'\left(\frac{d^2\overrightarrow{k'}}{dt^2}\right)_{\mathcal{R}}$$

With:

$$\dot{x}'\left(\frac{d\overrightarrow{\imath'}}{dt}\right)_{\mathcal{R}} + \dot{y}'\left(\frac{d\overrightarrow{\jmath'}}{dt}\right)_{\mathcal{R}} + \dot{z}'\left(\frac{d\overrightarrow{k'}}{dt}\right)_{\mathcal{R}} = \dot{x}'(\vec{\omega} \times \overrightarrow{\imath'}) + \dot{y}'(\vec{\omega} \times \overrightarrow{\jmath'}) + \dot{z}'(\vec{\omega} \times \overrightarrow{k'})$$

$$= \vec{\omega} \times \left(\dot{x}'\overrightarrow{\imath'} + \dot{y}'\overrightarrow{\jmath'} + \dot{z}'\overrightarrow{k'}\right)$$

$$= \vec{\omega} \times \overrightarrow{V_r}$$

$$\left(\frac{d}{dt}\left(\frac{d\overrightarrow{\imath'}}{dt}\right)_{\mathcal{R}}\right) = \frac{d}{dt}\left(\vec{\omega} \times \overrightarrow{\imath'}\right)$$

$$= \frac{d\vec{\omega}}{dt} \times \overrightarrow{\imath'} + \vec{\omega} \times \frac{d\overrightarrow{\imath'}}{dt}$$

$$= \frac{d\vec{\omega}}{dt} \times \overrightarrow{\imath'} + \vec{\omega} \times \left(\vec{\omega} \times \overrightarrow{\imath'}\right)$$

And similarly:

$$\left(\frac{d^2\overrightarrow{\jmath'}}{dt^2}\right)_{\mathcal{R}} = \frac{d\vec{\omega}}{dt} \times \overrightarrow{\jmath'} + \vec{\omega} \times \left(\vec{\omega} \times \overrightarrow{\jmath'}\right)$$

$$\left(\frac{d^2\overrightarrow{k'}}{dt^2}\right)_{\mathcal{R}} = \frac{d\vec{\omega}}{dt} \times \overrightarrow{k'} + \vec{\omega} \times \left(\vec{\omega} \times \overrightarrow{k'}\right)$$

$$\Rightarrow \; x' \cdot \frac{d}{dt}\left(\frac{d\overrightarrow{\imath'}}{dt}\right)_{(R)} + y' \cdot \frac{d}{dt}\left(\frac{d\overrightarrow{\jmath'}}{dt}\right)_{(R)} + z' \cdot \frac{d}{dt}\left(\frac{d\overrightarrow{k'}}{dt}\right)_{(R)}$$

$$= \frac{d\vec{\omega}}{dt} \wedge \left(x' \cdot \overrightarrow{\imath'} + y' \cdot \overrightarrow{\jmath'} + z' \cdot \overrightarrow{k'}\right) \; + \; \vec{\omega} \; \wedge \; (\vec{\omega} \; \wedge \; (x'\overrightarrow{\imath'} + y' \cdot \overrightarrow{\jmath'} + z' \cdot \overrightarrow{k'})$$

$$\frac{d\vec{\omega}}{dt} \wedge \overrightarrow{O'M} + \vec{\omega} \; \wedge \; (\vec{\omega} \wedge \overrightarrow{O'M})$$

$$\overrightarrow{a_s} = \overrightarrow{a_r} + \frac{d^2\overrightarrow{OO'}}{dt^2} + \frac{d\vec{\omega}}{dt} \wedge \overrightarrow{OM} + \vec{\omega} \wedge \left(\vec{\omega} \wedge \overrightarrow{OM}\right) + 2 \cdot \vec{\omega} \wedge \overrightarrow{V_r}$$

$$\overrightarrow{a_s} = \overrightarrow{a_r} + \overrightarrow{a_e} + \overrightarrow{a_c}$$

$$\overrightarrow{a_e} = \frac{d^2\overrightarrow{OO'}}{dt^2}_{(R)} + \frac{d\vec{\omega}}{dt} \wedge \overrightarrow{OM} + \vec{\omega} \wedge \left(\vec{\omega} \wedge \overrightarrow{OM}\right)$$

$$\overrightarrow{a_c} = 2 \cdot \vec{\omega} \wedge \overrightarrow{V_r}$$

Material Point Dynamics

Material point dynamics

IV.1. Introduction:

Dynamics is a branch of classical mechanics. It studies the motion of objects and the forces that cause them. Unlike kinematics, which we studied in previous chapters, which is limited to describing how objects move (position, velocity, acceleration), dynamics seeks to explain why they move by linking motion to its causes, primarily forces, primarily through Newton's laws of motion.

There are two main types of dynamics:

Translational dynamics, which studies the linear motion of objects under the influence of forces.

Rotational dynamics, which focuses on the motion of rotating objects, taking into account torque and angular momentum.

In this chapter, we will limit ourselves to studying translational dynamics.

IV.2. Definition:

IV.2.1.The force:

it is any mechanical action exerted by one body on another, which results in one or all of the following changes:

a change in its speed (moving it or stopping it);

a change in its trajectory;

a change in its shape (deforming it).

Force is represented by a vector $\left(often\ denoted\ \vec{F}\right)$ which has the same characteristics (direction, sense, magnitude) and is linked to its point of application.

Forces can be classified according to their range of action into contact forces and distance (field) forces.

The resultant of all forces acting on a body is the vector sum of all the forces acting on it.

IV.2.2. Mass:

A scalar quantity representing the amount of matter in an object. It also measures its inertia.

IV.2.3. Acceleration:

Variation of speed with respect to time. It is a vector quantity.

IV.2.4. Velocity:

Variation of position with respect to time. It is a vector quantity.

IV.2.5. Inertia:

An object's tendency to maintain its state of motion or rest.

V.2.6. Newton's Laws:

Three fundamental laws that describe the relationship between motion and the forces acting on an object.

IV.2.7. Principle of Inertia: (also known as Newton's First Law of Motion):

A body remains at rest or in uniform straight-line motion unless acted upon by an external force.

IV.3. Inertial Reference Frame:

An inertial reference frame is a frame of reference in which Newton’s first law (the principle of inertia) holds true.

Examples of Galilean (Inertial) Reference Frames:

A car moving at constant speed in a straight line (on a smooth road, no acceleration):

Inside the car, if you toss a ball straight up, it comes back down in your hand — just like when you're standing still. This is an example of an inertial frame.

IV.4.Momentum:

Momentum (also called linear momentum) is a vector quantity defined as the product of a body’s mass and its velocity.

$$\vec{p} = m \cdot \vec{v}$$

It represents the quantity of motion of a body.

IV.4.1. Superposition Principle of Momentum:

The total momentum of a system is the vector sum of the momenta of all particles:

$$\overrightarrow{P_{\text{total}}} = \sum_{i=1}^{n} \overrightarrow{p_i} = \sum_{i=1}^{n} m_i \overrightarrow{v_i}$$

($\overrightarrow{P_{\text{total}}}$) is the total momentum of the system,

($\overrightarrow{p_i}$)is the momentum of particle (i),

(m_i)is the mass of particle (i),

($\overrightarrow{v_i}$) is the velocity of particle (i).

This principle allows us to treat the motion of each particle separately and then combine their contributions to get the overall momentum of the system.

IV.4.2. Principle of conservation of momentum:

In a closed and isolated system (no external forces), the total linear momentum remains constant:

$$\overrightarrow{P_{\text{initial}}} = \overrightarrow{P_{\text{final}}}$$

$$\sum_{i=1}^{n} m_i \overrightarrow{v_{i,\text{initial}}} = \sum_{i=1}^{n} m_i \overrightarrow{v_{i,\text{final}}}$$

$(\overrightarrow{P_{\text{initial}}})$: total momentum before the interaction,

$(\overrightarrow{P_{\text{final}}})$: total momentum after the interaction,

(m_i): mass of particle (i),

$(\overrightarrow{v_{\imath,\text{initial}}})$: velocity of particle i before,

$(\overrightarrow{v_{\imath,\text{final}}})$: velocity of particle i after.

IV.5. Newton laws:

IV.5.1. First Law:

An object at rest stays at rest, and an object in motion continues to move in a straight line at a constant speed, unless acted upon by a net external force.

Mathematically;

$$\sum \vec{F} = 0 \frac{d\vec{v}}{dt} = \vec{0}$$

$$\vec{a} = \frac{d\vec{v}}{dt} = \vec{0}$$

IV.5.2. Second Law:

The force applied to an object is equal to the mass of the object multiplied by its acceleration. Mathematically, it is written as:

$$\vec{F} = m\vec{a}$$

This law directly relates the force acting on an object to the change in its velocity.

IV.5.3. Third Law:

For every action, there is an equal and opposite reaction. This means if object A exerts a force on object B, then object B exerts a force of equal magnitude but opposite direction on object A.

$$\overrightarrow{F_{A\to B}} = -\overrightarrow{F_{B\to A}}$$

IV.6. Fundamental Principle of Translational Dynamics:

The rate of change of linear momentum of a particle is equal to the net external force applied to it:

$$\sum \vec{F} = \frac{d\vec{p}}{dt}$$

where $(\vec{p} = m\vec{v})$ is the linear momentum.

If the mass (m) is constant, then:

$$\sum \vec{F} = m\frac{d\vec{v}}{dt} = m\vec{a}$$

IV.7. Free fall

Free fall is the motion of an object under the influence of gravitational force only. When an object is dropped near the Earth's surface and air resistance is negligible, it accelerates downward due to gravity with a constant acceleration $g = 9.81$ m/s^2.

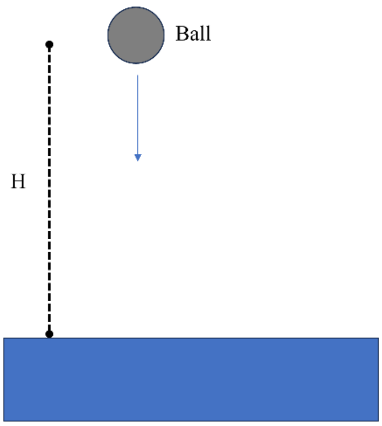

Figure 19 Free fall

The equations of motion for free fall, taking downward direction as positive, are:

$$v = gt + v_0$$

Displacement as a function of time

$$y = y_0 + v_0 t + \frac{1}{2} g t^2$$

IV.8. Normal force:

The normal force is the contact force exerted by a surface perpendicular (normal) to the object resting on it.

Key points:

Acts perpendicular to the surface.

Balances the perpendicular component of other forces like weight.

Varies depending on the situation (flat surface, incline, additional forces).

For an object of mass m resting on a horizontal surface without other vertical forces:

N = mg

For an object on an inclined plane at angle (θ):

N = mg cos θ

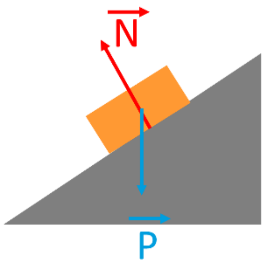

Figure 20 Inclined plane

IV.9. Friction force:

Friction is a contact force that opposes the relative motion (or tendency of motion) between two surfaces in contact. It acts parallel to the surface.

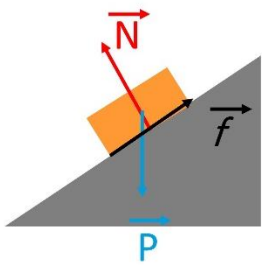

Figure 21 Inclined plane friction force

There are two main types of friction:

Static friction: Acts when the object is at rest. It prevents the object from starting to move.

$$f_s \leq \mu_s N$$

Where (f_s) is the static friction force, (μ_s) is the coefficient of static friction, and \(N\) is the normal force.

The maximum static friction is:

$$f_{s,\max} = \mu_s N$$

IV.10. Kinetic (or dynamic) friction:

Acts when the object is already moving.

$$f_k = \mu_k N$$

Where (f_k) is the kinetic friction force, and (μ_k) is the coefficient of kinetic friction.

The tension:

IV.11. Tension is the force transmitted through a string, rope, cable, or wire when it is pulled tight by forces acting from opposite ends.

Tension always acts away from the object along the direction of the rope.

In an ideal rope (massless and inextensible), the tension is the same throughout.

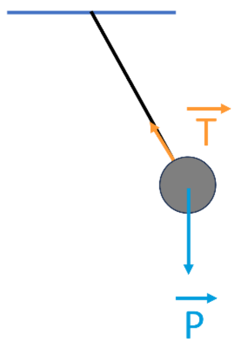

Figure 22 Tension force

IV.12. Archimedes' Principle states:

Any object fully or partially submerged in a fluid experiences an upward force (called buoyant force or Archimedes' thrust) equal to the weight of the fluid it displaces.

$$F_A = \rho_f V_d\, g$$

(F_A) *is the buoyant force,*

(ρ_f) *is the density of the fluid,*

(V_d) *is the volume of the displaced fluid,*

(g) *is the gravitational acceleration.*

IV.13. The restoring force of a spring:

The **restoring force** of a spring is the force that brings the spring back to its equilibrium (unstretched) position when it is compressed or stretched.

$$F = -k\,x$$

(F) is the restoring force (in newtons),

(k) is the spring constant (in N/m),

(x) is the displacement from the equilibrium position (in meters),

The negative sign indicates that the force acts in the direction opposite to the displacement.

If $(x > 0)$: the spring is stretched, and the force pulls back.

If ($x < 0$): the spring is compressed, and the force pushes outward.

IV.14. Hooke's Law:

The restoring force exerted by a spring is directly proportional to the displacement from its equilibrium position and acts in the opposite direction.

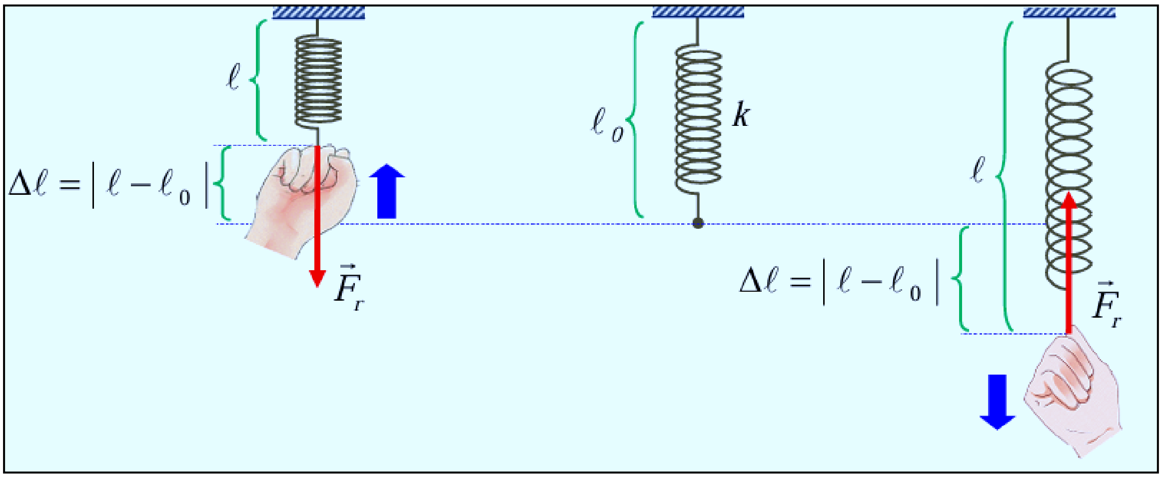

Figure 23 Restoring force

IV.15. Rotational dynamics

IV.15.1. Definition:

Rotational dynamics studies the motion of objects that rotate around an axis and the torques (rotational forces) that cause or change this motion.

IV.15.2. Moment of a force

The **moment of a force** (also called **torque**) measures the tendency of a force to cause a body to rotate around a point or an axis.

$$\vec{\tau} = \vec{r} \times \vec{F}$$

($\vec{\tau}$) is the torque (moment of the force),

($\vec{r}$) is the position vector from the axis (or point) to the point of force application,

($\vec{F}$) is the applied force,

($\times$) denotes the cross product.

The SI unit of moment is the newton-meter (N·m).

IV.15.3. Angular momentum:

Angular momentum is the rotational analog of linear momentum. It measures the amount of rotation a body has, taking into account its mass distribution and angular velocity.

$$\vec{L} = \vec{r} \times \vec{p} = \vec{r} \times (m\vec{v})$$

($\vec{L}$) is the angular momentum,

($\vec{r}$) is the position vector from the rotation point to the particle,

($\vec{v}$) is the velocity,

(m) is the mass of the particle.

IV.15.4. The Angular Momentum Theorem :

The Angular Momentum Theorem states that the rate of change of angular momentum of a system (about a point or axis) is equal to the net external torque acting on the system with respect to that point or axis.

$$\frac{d\vec{L}}{dt} = \overrightarrow{\tau_{\text{ext}}}$$

($\vec{L}$) is the angular momentum,

($\overrightarrow{\tau_{\text{ext}}}$) is the net external torque.

Work and Energy

Work and energy

V.1. Introduction:

In this chapter, we explore two fundamental concepts of classical mechanics: **work** and **energy**. These quantities are essential for understanding how forces influence motion and how physical systems store and transfer the capacity to perform tasks.

V.2. Work:

Work is a measure of the energy transferred by a force when it moves an object through a displacement $\vec{r}$.

If a constant force $(\vec{F})$is applied, and the object moves through a displacement $(\vec{r})$, the work done is:

$$W = \vec{F} \cdot \vec{r} = Fr\cos\theta$$

Where,

(W) is the work,

$(\vec{F})$ is the applied force,

$(\vec{r})$ is the displacement,

(θ) is the angle between $(\vec{F})$ and $(\vec{r})$.

If the force is variable:

$$W = \int_{x_1}^{x_2} \vec{F}(x) \cdot d\vec{x}$$

The SI unit of work is the joule (J).

<u>Example:</u>

A box of mass 10 $kilogram$is pushed along a horizontal surface by a constant force of 50 N over a distance of 5 m The force is applied in the same direction as the motion. Calculate the work done by the force.

$$W = F \cdot r \cdot \cos(\theta)$$

$$W = 250\,J$$

V.3. Power:

Power is the rate at which work is done.
Average Power:

$$P_{\text{avg}} = \frac{\Delta W}{\Delta t}$$

Instantaneous Power:

$$P = \frac{dW}{\mathrm{d}t} = \vec{F} \cdot \frac{\mathrm{d}\vec{r}}{\mathrm{dt}}$$

$$P = \vec{F} \cdot \vec{v}$$

($\vec{v}$) is the velocity at the point of application.

V.4. Kinetic Energy:

Kinetic energy is the energy possessed by an object due to its motion.

Translational:

$$E_{\text{kin}} = \frac{1}{2} m v^2$$

Rotational

$$E_{\text{rot}} = \frac{1}{2} I \omega^2$$

Where,

(m) is the mass,

(v) is the velocity,

(I) is the moment of inertia,

(ω) is the angular velocity.

In the case of a system with *n* particles, the kinetic energy of the system is the sum of the kinetic energy of each *i* particle.

$$E_{\text{kinT}} = \sum_{i=0}^{n} \frac{1}{2} m_i {v^2}_i$$

If no force is applied to the particle (or particles), its velocity and mass remain constant; therefore, the kinetic energy also remains constant. The kinetic energy changes only if the speed or the mass of the particle changes.

Rotational:

$$E_{\text{rotT}} = \sum_{i=1}^{n} \frac{1}{2} I_i {\omega^2}_i$$

V.5. Kinetic Energy Theorem

The net work done by all the forces acting on a particle is equal to the change in its kinetic energy.

$$W_{\text{net}} = \Delta E_{\text{kin}} = \frac{1}{2}mv_B^2 - \frac{1}{2}mv_A^2$$

V.6.Gravitational Potential Energy:

It is the energy stored in an object due to its position in a gravitational field.

V.6.1. Uniform Gravitational Field (Near Earth's Surface):

If the position of an object is near the Earth's surface, we define potential energy as the negative work done by gravity to move an object from a reference point h_0 to position h.

$$E_{\text{pot}}(h) = -\int_{h_0}^{h} \overrightarrow{F_g} \cdot d\vec{r}$$

$$E_{\text{pot}}(h) = mg(h - \mathrm{h}_0)$$

$$= -\int_{h_0}^{h} (-mg)\ dz$$

$$= mg(h - h_0)$$

m: mass of the particle

g: acceleration due to gravity

h: height above a reference level

Characteristics:

1- This expression is valid only near Earth's surface, where the acceleration of gravity is approximately constant.
2- E_{pot} increases with height.
3- Gravity is a conservative force.

V.6.2. General Gravitational Field (Newtonian Gravity):

When distances are large, for example, planetary motion, the gravitational field is non-uniform and follows Newton's inverse-square law.

$$\overrightarrow{F_g} = -G\frac{Mm}{r^2}\overrightarrow{u_r}$$

Potential Energy Function:

$$E_{\text{pot}}(r) = -\int_{\infty}^{r} \overrightarrow{F_g} \cdot d\vec{r}$$

$$= -\int_{\infty}^{r} \left(-G\frac{Mm}{r'^2}\right) dr'$$

$$= GMm \int_{\infty}^{r} \frac{1}{r'^2} dr'$$

$$E_{\text{pot}}(r) = -G\frac{Mm}{r}$$

V.7. Elastic (Spring) Potential Energy:

Elastic potential energy is the energy stored in a spring or elastic object when it is stretched or compressed.

For an ideal spring, the restoring force is:

$$\overrightarrow{F_{\text{spring}}} = -k(\vec{x} - \overrightarrow{x_0})$$

(k) is the spring constant,

$\vec{x}$ is the current position of the mass attached to the spring,

$\overrightarrow{x_0}$ is the equilibrium (unstretched) position,

Derivation of Potential Energy:

The potential energy is the negative work done by the spring force:

$$E_{\text{spring}}(\vec{x}) = -\int_{\overrightarrow{x_0}}^{\vec{x}} \overrightarrow{F_{\text{spring}}} \cdot d\vec{x}$$

$$= -\int_{\overrightarrow{x_0}}^{\vec{x}} \left[-k\left(\overrightarrow{x'} - \overrightarrow{x_0}\right)\right] \cdot d\overrightarrow{x'}$$

$$= k\int_{\overrightarrow{x_0}}^{\vec{x}} \left(\overrightarrow{x'} - \overrightarrow{x_0}\right) \cdot d\overrightarrow{x'}$$

$$Let(\vec{u} = \overrightarrow{x'} - \overrightarrow{x_0} \Rightarrow d\vec{u} = d\overrightarrow{x'}):$$

$$E_{\text{spring}}(\vec{x}) = k\int_{\vec{0}}^{\vec{x}-\overrightarrow{x_0}} \vec{u} \cdot d\vec{u}$$

$$E_{\text{spring}} = \frac{1}{2}kx^2$$

V.8. Mechanical energy:

Mechanical energy is the sum of a system's kinetic energy and potential energy. It represents the total energy available to perform mechanical work.

$$E_{\text{mec}} = E_{\text{kin}} + E_{\text{pot}}$$

V.9. Conservative Forces

A conservative force is a force for which the work done is independent of the path and depends only on the initial and final positions.

$$W = \int_{\vec{r_1}}^{\vec{r_2}} \vec{F} \cdot d\vec{r}$$

For a closed loop:

$$\oint \vec{F} \cdot d\vec{r} = 0$$

A conservative force can be written as the *negative gradient* of a scalar potential energy function.

$$\vec{F} = -\nabla E_{pot}(\vec{r})$$

Properties:

Work done is recoverable.

Associated with potential energy.

The mechanical energy is conserved.

Example:

Gravitational force

Spring force

Electrostatic force

V.12. Non-Conservative Forces:

A non-conservative force is a force for which the work done depends on the path taken between two points.

$$\oint \vec{F} \cdot d\vec{r} \neq 0$$

Properties:

Work depends on the path.

Converts mechanical energy into other forms (e.g., heat).

Mechanical energy is not conserved.

Often results in energy dissipation.

Examples: Kinetic friction, air resistance:

V.11. Conservation of Mechanical Energy:

In the absence of non-conservative forces, the mechanical energy of a system is conserved:

$$E_{\text{mec, initial}} = E_{\text{mec, final}} = constant$$

Example:

A block of mass m = 2 kg slides on a frictionless horizontal surface. It is moving with an initial speed v_0= 3 m/s when it strikes a massless spring of spring constant k=200 N/m fixed at a wall. What is the maximum compression $x$max of the spring?

Solution:

In this problem the force which does, work on the block is the spring force (conservative), No friction therefore mechanical energy conserved:

$$E_{\text{initial}} = \frac{1}{2} m v_0^2,$$

$$E_{\text{final}} = \frac{1}{2} k x_{\text{max}}^2$$

$$\Rightarrow \frac{1}{2} m v_0^2 = \frac{1}{2} k x_{\text{max}}^2$$

$$\Rightarrow m v_0^2 = k x_{\text{max}}^2$$

$$\Rightarrow x_{\text{max}} = \sqrt{\frac{m}{k}} v_0$$

$$\boxed{x_{\text{max}} = 0.30\,\text{m}}$$

Problems and solutions:

Problem 1:

Find the dimension of these quantities:

The velocity, acceleration, force, energy, pressure, density, electrical field, and electrical potential.

$$[V] = \frac{[l]}{[t]} = \frac{L}{T} = LT^{-1}$$

$$[a] = \frac{[dV]}{[dt]} = \frac{LT^{-1}}{T} = LT^{-2}$$

$$[F] = [m][a] = MLT^{-2}$$

$$[E] = \frac{1}{2}[m][V^2] = M\,L^2\,T^{-2}$$

$$[P] = \frac{[F]}{[S]} = \frac{MLT^{-2}}{L^2} = ML^{-1}T^{-2}$$

$$[\rho] = \frac{[m]}{[V]} = \frac{M}{L^3} = ML^{-3}$$

Problem 2:

Check if these equations are dimensionally consistent:

$E = mc^2$.

$E = mgh$.

$$[E] = M\,L^2\,T^{-2}$$

$$[\mathrm{mc}^2] = [m][\mathrm{c}^2] = M\,L^2\,T^{-2}$$

$$[mgh] = [m][g][h] = MLT^{-2}L = M\,L^2\,T^{-2}$$

Problem 3:

If $v = at^2 + bt + c$, find the dimensions of a, b, and c.

$[v] = [at^2] = [bt] = c = LT^{-1}$

So $[a] = [v]/[t]^2$

Therefore, $[a] = LT^{-3}$

$[b] = LT^{-2}$

$[c] = LT^{-1}$

Dimension of some physical quantities:

#	Physical Quantity	Dimensional Formula	SI Unit
1	Length	L	meter (m)
2	Mass	M	kilogram (kg)
3	Time	T	second (s)
4	Electric Current	I	ampere (A)
5	Temperature	Θ	kelvin (K)
6	Amount of Substance	N	mole (mol)
7	Luminous Intensity	J	candela (cd)
8	Area	L^2	square meter (m^2)
9	Volume	L^3	cubic meter (m^3)
10	Density	ML^{-3}	kg/m^3
11	Velocity	LT^{-1}	m/s
12	Acceleration	LT^{-2}	m/s^2
13	Momentum	MLT^{-1}	kg·m/s
14	Force	MLT^{-2}	newton (N)
15	Pressure	$ML^{-1}T^{-2}$	pascal (Pa)
16	Energy/Work	ML^2T^{-2}	joule (J)
17	Power	ML^2T^{-3}	watt (W)
18	Frequency	T^{-1}	hertz (Hz)
19	Charge	TI	coulomb (C)

20	Voltage	$ML^2T^{-3}I^{-1}$	volt (V)
21	Capacitance	$M^{-1}L^{-2}T^4I^2$	farad (F)
22	Resistance	$ML^2T^{-3}I^{-2}$	ohm (Ω)
23	Conductance	$M^{-1}L^{-2}T^3I^2$	siemens (S)
24	Magnetic Flux	$ML^2T^{-2}I^{-1}$	weber (Wb)
25	Magnetic Field Strength (B)	$MT^{-2}I^{-1}$	tesla (T)
26	Magnetic Field Intensity (H)	$L^{-1}I$	ampere/meter (A/m)
27	Inductance	$ML^2T^{-2}I^{-2}$	henry (H)
28	Angular Displacement	—	radian (rad)
29	Angular Velocity	T^{-1}	rad/s
30	Angular Acceleration	T^{-2}	rad/s^2
31	Moment of Inertia	ML^2	kg·m^2
32	Torque	ML^2T^{-2}	newton·meter (N·m)
33	Surface Tension	MT^{-2}	N/m
34	Heat	ML^2T^{-2}	joule (J)
35	Specific Heat Capacity	$L^2T^{-2}\Theta^{-1}$	J/(kg·K)
36	Thermal Conductivity	$MLT^{-3}\Theta^{-1}$	W/(m·K)
37	Latent Heat	L^2T^{-2}	J/kg
38	Electric Field	$MLT^{-3}I^{-1}$	V/m
39	Electric Potential Energy	ML^2T^{-2}	joule (J)
40	Electric Dipole Moment	LTI	C·m
41	Permittivity (ε)	$M^{-1}L^{-3}T^4I^2$	F/m

42	Permeability (μ)	$MLT^{-2}I^{-2}$	H/m
43	Refractive Index	—	dimensionless
44	Luminous Flux	J	lumen (lm)
45	Illuminance	JL^{-2}	lux (lx)
46	Radiant Intensity	$ML^{2}T^{-3}$	watt/steradian (W/sr)
47	Radiance	MT^{-3}	W/(sr·m²)
48	Entropy	$ML^{2}T^{-2}\Theta^{-1}$	J/K
49	Thermal Resistance	$M^{-1}L^{-2}T^{3}\Theta$	K/W
50	Strain	—	dimensionless

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