Democratic and Popular Republic of Algeria
Ministry of Higher Education and Scientific Research
Ibn Khaldoun University of Tiaret

Faculty of Applied Sciences
Department of Sciences and Technology

Handout of Analysis 03
Lessons and exercises

For Second-Year engineering Students in the Science and Technology Domain.

Presented by:

Dr. FERHAOUI Mohamed

Expertised by:

Dr. ELKHIRI Laid -Ibn Khaldoun University of Tiaret.

Dr. Tahar Mezeddek Mohamed -Mustapha Stambouli University of Mascara.

Academic Year: 2024/2025.

Contents

Introduction

This handout is intended for second-year engineering science students. It illustrates the Analysis 3 curriculum of the second year in the field of Science and Technology. It can be used by students from various fields such as Mathematics and Computer Science.

It will consist of six chapters, namely: vector analysis, infinite series, power series, Fourier series, Fourier transform, and Laplace transform. Each chapter includes fundamental definitions and results in the form of theorems or propositions. There are also illustrative examples, relevant remarks, and detailed solved exercises aimed at assimilating the course material and acquiring problem-solving techniques.

The goal of the first part is to introduce the concept of curvilinear integrals and surface integrals, to understand their properties, and especially to know how to calculate them, as well as the different theorems related to this type of integral.

The objective of the second part is to highlight the main tools used in the study of the nature of numerical series, more specifically their convergence or divergence.

The fourth chapter is dedicated to power series, their definition, domain of convergence, and their applications.

The objective of the fifth chapter is to examine Fourier series, which are a very important tool for engineers.

The sixth chapter covers Fourier transforms and their applications.

The last chapter presents the Laplace transform and its applications in solving differential equations.

Chapter 1

Vector analysis

1.1 Scalar and Vector fields

Definition 1.1.1 *Scalar fields (Scalar functions)*

A function f is called scalar function on $\mathbb{R}^3$, if it assins a number real to each point $X = (x, y, z) \in \mathbb{R}^3$.

$$\begin{array}{rcl} f : \mathbb{R}^3 & \to & \mathbb{R} \\ (x, y, z) & \longmapsto & f(x, y, z). \end{array}$$

Example 1.1.1

1) ***The temperature***

The temperature $T(x, y, z)$ as a function of special coordinates in space is scalar function.

$$\begin{array}{rcl} T : \mathbb{R}^3 & \to & \mathbb{R} \\ (x, y, z) & \longmapsto & T(x, y, z). \end{array}$$

2) ***Ecludian distance***

Let $X_0 \in \mathbb{R}^3$ and $f(X) = \left\| \overrightarrow{X_0 X} \right\|$ the distance of $X \in \mathbb{R}^3$ from the fixed point X_0.

f defines a scalar field (function) in space.

$$\begin{aligned} f:\mathbb{R}^3 &\rightarrow \mathbb{R} \\ X &\longmapsto f(X) = \left\| \overrightarrow{X_0X} \right\| = \sqrt{(x-x_0)^2 + (y-y_0)^2 + (x-z_0)^2}. \end{aligned}$$

Definition 1.1.2 *Vector fields (Vector functions)*

A function f is called Vector function on $\mathbb{R}^3$, if to each point $X = (x, y, z) \in \mathbb{R}^3$ is assined to a vector $f(X) \in \mathbb{R}^3$.

$$\begin{aligned} f:\mathbb{R}^3 &\rightarrow \mathbb{R}^3 \\ (x,y,z) &\longmapsto f(x,y,z) = (f_1(x,y,z), f_2(x,y,z), f_3(x,y,z)), \end{aligned}$$

where f_1, f_2 and f_3 are the components of f.

Example 1.1.2

The following function is a vector function.

$$\begin{aligned} f:\mathbb{R}^3 &\rightarrow \mathbb{R}^3 \\ (x,y,z) &\longmapsto f(x,y,z) = \left(x+2y, x^2-y+z, 2xy+z^2\right). \end{aligned}$$

Remark 1.1.3 *More generally, we have*

i) $g:\mathbb{R}^n \rightarrow \mathbb{R}$ is a scalar field.

ii) $f:\mathbb{R}^n \rightarrow \mathbb{R}^n$ is also a vector field, with

$$f(x,y,z) = (f_1(x,y,z), ..., f_n(x,y,z)),$$

where $f_1, ..., f_n$ are the components of f.

If $n = 2$, g (resp. f) is scalar (resp. vector) field in the plane. If $n = 3$, g (resp. f) is scalar (resp. vector) field in the space.

1.2 Circulation and gradient of a vector field

1.2.1 The gradient of a scalar field

Consider the scalar field $f : \mathbb{R}^3 \to \mathbb{R}$. The gradient of the scalar field is defined by

$$grad\,(f) = \nabla f = \frac{\partial f}{\partial x}\overrightarrow{i} + \frac{\partial f}{\partial y}\overrightarrow{j} + \frac{\partial f}{\partial z}\overrightarrow{k} = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right),$$

where $\dfrac{\partial f}{\partial x}, \dfrac{\partial f}{\partial y}$ and $\dfrac{\partial f}{\partial z}$ are the partial derivatives of f with respect to x, y and z respectively.

Remark 1.2.1

1) For $f : D \subset \mathbb{R}^3 \to \mathbb{R}$. If $grad\,(f)$ is defined at each point of D, then

$$\begin{aligned} \nabla f : \mathbb{R}^3 \quad &\to \quad \mathbb{R}^3 \\ (x, y, z) \quad &\longmapsto \quad \nabla f\,(x, y, z) = \left(\frac{\partial f\,(x, y, z)}{\partial x}, \frac{\partial f\,(x, y, z)}{\partial y}, \frac{\partial f\,(x, y, z)}{\partial z}\right), \end{aligned}$$

is a vector field.

2) In general, for $f : \mathbb{R}^n \to \mathbb{R}$, the $grad\,(f)$ is defined at each point $(x_1, ..., x_n)$ by

$$\nabla f\,(x_1, ..., x_n) = \left(\frac{\partial f\,(x_1, ..., x_n)}{\partial x_1}, \frac{\partial f\,(x_1, ..., x_n)}{\partial x_2}, ..., \frac{\partial f\,(x_1, ..., x_n)}{\partial x_n}\right).$$

Example 1.2.1

1) Let the scalar field $f : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$f\,(x, y) = x^2 + 2y,$$

we have

$$\frac{\partial f\,(x, y)}{\partial x} = \frac{\partial\,(x^2 + 2y)}{\partial x} = 2x,$$

and

$$\frac{\partial f\,(x, y)}{\partial y} = \frac{\partial\,(x^2 + 2y)}{\partial y} = 2,$$

then

$$\begin{aligned}\nabla f(x,y) &= \left(\frac{\partial f(x,y)}{\partial x}, \frac{\partial f(x,y)}{\partial y}\right) \\ &= (2x, 2).\end{aligned}$$

2) Let the scalar field $f : \mathbb{R}^3 \to \mathbb{R}$ defined by

$$f(x,y,z) = xyz + \sin x + yz,$$

we have

$$\frac{\partial f(x,y,z)}{\partial x} = yx + \cos x,\ \frac{\partial f(x,y,z)}{\partial y} = xz + z \text{ and } \frac{\partial f(x,y,z)}{\partial z} = xz + y,$$

then

$$\begin{aligned}\nabla f(x,y,z) &= \left(\frac{\partial f(x,y,z)}{\partial x}, \frac{\partial f(x,y,z)}{\partial y}, \frac{\partial f(x,y,z)}{\partial z}\right) \\ &= (yx + \cos x, xz + z, xz + y).\end{aligned}$$

1.2.2 Gradient field

A gradient field is a vector field that can be written as the gradient of another function, i.e: a vector field $g : \mathbb{R}^3 \to \mathbb{R}^3$ is a gradient field if there exists a scalar field $f : \mathbb{R}^3 \to \mathbb{R}$ such that

$$g = \nabla f.$$

Example 1.2.2
The vector field $g : \mathbb{R}^3 \to \mathbb{R}^3$ defined by

$$g(x,y,z) = (yx + \cos x, xz + z, xz + y),$$

is a gradient field, because it is a gradient of the scalar field $f : \mathbb{R}^3 \to \mathbb{R}$ defined by

$$f(x,y,z) = xyz + \sin x + yz,$$

and we have $g = \nabla f$.

1.3 Divergence and Rotation of a Vector Field

1.3.1 Divergence of a Vector Field

For $f : \mathbb{R}^3 \to \mathbb{R}^3$ a vector field, the divergence of the vector field f is a differential operator that measures the intensity with which a vector field "diverges" or "flows out" from a given point. The divergence of f is given by

$$div\,(f) = \nabla.f = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}.$$

If the divergence is positive at a point, it means there is a source generating flux at that location. If it is negative, it indicates a "sink" that absorbs the flux.

Example 1.3.1

Let $f : \mathbb{R}^3 \to \mathbb{R}^3$ a vector field defined by

$$f\,(x, y, z) = (f_1\,(x, y, z)\,, f_2\,(x, y, z)\,, f_3\,(x, y, z)) = \left(x^2y, 3y + z, z^3\right),$$

$div\,(f)$ the divergence of f is a scalar field from $\mathbb{R}^3$ to $\mathbb{R}$ defined at each point (x, y, z) by

$$div\,(f)\,(x, y, z) = \nabla.f\,(x, y, z) = \frac{\partial f_1\,(x, y, z)}{\partial x} + \frac{\partial f_2\,(x, y, z)}{\partial y} + \frac{\partial f_3\,(x, y, z)}{\partial z},$$

we have

$$\frac{\partial f_1\,(x, y, z)}{\partial x} = 2xy,\ \frac{\partial f_2\,(x, y, z)}{\partial y} = 3 \text{ and } \frac{\partial f_3\,(x, y, z)}{\partial z} = 2z^2,$$

then

$$\begin{aligned} div\,(f)\,(x, y, z) &= \frac{\partial f_1\,(x, y, z)}{\partial x} + \frac{\partial f_2\,(x, y, z)}{\partial y} + \frac{\partial f_3\,(x, y, z)}{\partial z} \\ &= 2xy + 2z^2 + 3. \end{aligned}$$

For example, for $(1, 1, 1)$ and $(1, -6, 2)$, we have

$$div\,(f)\,(1, 1, 1) = 7, \text{ and } div\,(f)\,(1, -6, 2) = -1.$$

1.3.2 Rotation and rotational field

Rotation of a vector Field

For $f : \mathbb{R}^3 \to \mathbb{R}^3$ a vector field, the curl of the vector field f is a differential operator that measures the tendency of a vector field to rotate around a point, meaning the amount of "rotation" of the field at a given point. The curl of the vector field f is given by

$$curl\,(f) = \nabla \times f = \left(\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z}, \frac{\partial f_1}{\partial z} - \frac{\partial f_3}{\partial x}, \frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y} \right).$$

$curl\,(f)$ the curl of f is a vector field from $\mathbb{R}^3$ to $\mathbb{R}^3$.

Example 1.3.2

Let $f : \mathbb{R}^3 \to \mathbb{R}^3$ a vector field defined by

$$f\,(x,y,z) = (f_1\,(x,y,z)\,, f_2\,(x,y,z)\,, f_3\,(x,y,z)) = \left(xy^2, 3yz + z, x^2 + z\right).$$

The curl of f is a vector field from $\mathbb{R}^3$ to $\mathbb{R}^3$ defined at each point (x,y,z) by

$$curl\,(f) = \nabla \times f = \left(\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z}, \frac{\partial f_1}{\partial z} - \frac{\partial f_3}{\partial x}, \frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y} \right),$$

we have

$$f_1\,(x,y,z) = xy^2,\;\; f_2\,(x,y,z) = 3yz + z \textit{ and } f_3\,(x,y,z) = x^2 + z,$$

then

$$\begin{aligned} curl\,(f)\,(x,y,z) &= \nabla \times f\,(x,y,z) \\ &= (0 - 3y - 1, 0 - 2x, 0 - 2xy) \\ &= (-3y - 1, -2x, -2xy)\,. \end{aligned}$$

For example, for $(1,0,1)$, we have

$$curl\,(f)\,(1,0,1) = \nabla \times f\,(1,0,1) = (-1,-2,0)\,.$$

Rotational fields

A rotational field is a vector field that can be described as the curl of another vector field. This means that there exists a vector field whose curl produces the given field, i.e: a vector field $g : \mathbb{R}^3 \to \mathbb{R}^3$ is curl field, if there exists a scalar field $f : \mathbb{R}^3 \to \mathbb{R}^3$ such that

$$g = \nabla \times f.$$

Example 1.3.3
Let $g : \mathbb{R}^3 \to \mathbb{R}^3$ *a vector field defined by*

$$g(x, y, z) \quad = \quad (-3y - 1, -2x, -2xy).$$

The function g *is a rotaional field, because it is the the curl of the vector field* $f : \mathbb{R}^3 \to \mathbb{R}^3$ *defined by*

$$f(x, y, z) \quad = \quad \left(xy^2, 3yz + z, x^2 + z\right),$$

and we have

$$(g_1, g_2, g_3) = \nabla \times f = \left(\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z}, \frac{\partial f_1}{\partial z} - \frac{\partial f_3}{\partial x}, \frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right).$$

1.3.3 Laplacian of a scalar field

For $f : \mathbb{R}^3 \to \mathbb{R}$ a scalar field, the Laplacian of f is denoted by $\nabla^2 f$ and it is defined as the divergence of the gradient of f

$$\nabla^2 f = \nabla . (\nabla f) = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2},$$

where $\dfrac{\partial^2 f}{\partial^2 x}, \dfrac{\partial^2 f}{\partial^2 y}$ and $\dfrac{\partial^2 f}{\partial^2 z}$ are the second partial derivatives of f with respect to x, y and z respectively.

$\nabla^2 f$ the Laplacian of f is a scalar field from $\mathbb{R}^3$ to $\mathbb{R}$.

Example 1.3.4

Let the scalar field $f : \mathbb{R}^3 \to \mathbb{R}$ *defined by*

$$f(x, y, z) = xyz^2 + \sin x + yz,$$

we have

$$\frac{\partial f(x,y,z)}{\partial x} = yz^2 + \cos x,\ \frac{\partial f(x,y,z)}{\partial y} = xz^2 + z \text{ and } \frac{\partial f(x,y,z)}{\partial z} = 2xyz + y,$$

then

$$\frac{\partial^2 f(x,y,z)}{\partial x^2} = -\sin x,\ \frac{\partial^2 f(x,y,z)}{\partial y^2} = 0 \text{ and } \frac{\partial^2 f(x,y,z)}{\partial z^2} = 2xy,$$

and

$$\begin{aligned} \nabla^2 f(x,y,z) &= \frac{\partial^2 f(x,y,z)}{\partial x^2} + \frac{\partial^2 f(x,y,z)}{\partial y^2} + \frac{\partial^2 f(x,y,z)}{\partial z^2} \\ &= 2xy - \sin x. \end{aligned}$$

1.4 Scalar potentials and vector potentials

1.4.1 Scalar potential

Let $f : \mathbb{R}^3 \to \mathbb{R}^3$ a vector field. f is called a scalar potential, if there exists $g : \mathbb{R}^3 \to \mathbb{R}$ a scalar field such us

$$f = -\nabla g.$$

Example 1.4.1

1)The electric field in an electrostatic field can be derived from a scalar potential ϕ*, called the electric potential:*

$$E = -\nabla\phi.$$

This equation indicates that the electric field E *is the negative gradient of the scalar potential* ϕ*.*

2) Let $g : \mathbb{R}^3 \to \mathbb{R}$ *the scalar field defined by*

$$g(x, y, z) = xyz + \sin x + yz,$$

we have

$$\begin{aligned}\nabla g(x,y,z) &= \left(\frac{\partial g(x,y,z)}{\partial x}, \frac{\partial g(x,y,z)}{\partial y}, \frac{\partial g(x,y,z)}{\partial z}\right) \\ &= (yx + \cos x, xz + z, xz + y).\end{aligned}$$

The vector field $f : \mathbb{R}^3 \to \mathbb{R}^3$ *defined by*

$$f(x,y,z) = (-yx - \cos x, -xz - z, -xz - y),$$

is a scalar potential, because

$$f = -\nabla g.$$

1.4.2 Vector potential

Let $f : \mathbb{R}^3 \to \mathbb{R}.^3$ a vector field. $g : \mathbb{R}^3 \to \mathbb{R}^3$ is called the vector potential associated with f if

$$f = curl(g) = \nabla \times g.$$

Example 1.4.2

1) The magnetic field B in electromagnetism can be derived from a vector potential A, called the magnetic vector potential:

$$B = curl(A).$$

This equation shows that the magnetic field B is the curl of the vector potential A.
2) Let $g : \mathbb{R}^3 \to \mathbb{R}^3$ *a vector field defined by*

$$g(x,y,z) = \left(xy^2, 3yz + z, x^2 + z\right).$$

The curl of f is defined by

$$curl(g) = \nabla \times g = \left(\frac{\partial g_3}{\partial y} - \frac{\partial g_2}{\partial z}, \frac{\partial g_1}{\partial z} - \frac{\partial g_3}{\partial x}, \frac{\partial g_2}{\partial x} - \frac{\partial g_1}{\partial y}\right),$$

then

$$
\begin{aligned}
curl\,(g)\,(x,y,z) &= \nabla \times g\,(x,y,z) \\
&= (-3y-1,-2x,-2xy)\,.
\end{aligned}
$$

Let $f:\mathbb{R}^3 \to \mathbb{R}^3$ *a vector field defined by*

$$
\begin{aligned}
g\,(x,y,z) &= \nabla \times g\,(x,y,z) \\
&= (-3y-1,-2x,-2xy)\,,
\end{aligned}
$$

then, g *is called the vector potential associated with* f.

1.5 Curvilinear integral

1.5.1 Curve in space

Definition 1.5.1

A curve can be thought of as the trajectory traced by a point moving continuously in space. It does not necessarily have to be straight and can have any form, such as a line, circle, or more complex shapes.

Example 1.5.1

1) Line segment: A straight line between two points in space, such as the path from $(0,0,0)$ *and* $(1,1,1)$.

2) A saddle-shaped curve in space: often represented by the equation $z = x^2 - y^2$.

Parametrized curve

A parametrized curve is a curve described by a vector-valued function that depends on a parameter t. This parameter typically varies over a specific interval, and the curve is defined as a function from t to a point in space. C is a parametrized curve that means it contains the points $r(t)$ such as:

$$\begin{aligned} r:[a,b] &\rightarrow \mathbb{R}^3 \\ t &\longmapsto r(t)=(x(t),y(t),z(t)). \end{aligned}$$

We can describe the curve C as follow

$$C(t)=\{r(t)=(x(t),y(t),z(t)),\quad t\in[a,b]\}.$$

Length of a curve

Let $C(t)=\{r(t)=(x(t),y(t),z(t)),\quad t\in[a,b]\}$ a parametrized curve of class C^1. The length of the curve C is given by

$$l=\int_a^b \|r'(t)\|\,dt=\int_a^b \sqrt{\left(\frac{dx(t)}{dt}\right)^2+\left(\frac{dy(t)}{dt}\right)^2+\left(\frac{dz(t)}{dt}\right)^2}dt.$$

Example 1.5.2

1) ***Helix:*** *A 3D parametrized curve that spirals along a cylinder can be represented by the parametric equations:*

$$r(t)=(r\cos t,r\sin t,ct)\ \text{for } t\in[0,\infty[.$$

Here, r is the radius of the helix, c controls the spacing between loops, and t is the parameter. As t increases, the helix moves along the z-axis while wrapping around the cylinder.

2) ***Parametrized Ellipse in Space:*** *An ellipse lying in the xy-plane can be parametrized as:*

$$r(t)=(a\cos t,b\sin t,ct)\ \text{for } t\in[0,2\pi].$$

The length of this curve for $a=b=1$ and $c=2$ is

$$\begin{aligned} l &= \int_0^{2\pi} \sqrt{\left(\frac{dx(t)}{dt}\right)^2+\left(\frac{dy(t)}{dt}\right)^2+\left(\frac{dz(t)}{dt}\right)^2}dt \\ &= \int_0^{2\pi} \sqrt{(-\sin t)^2+(\cos t)^2+4}dt=\sqrt{5}\int_0^{2\pi} dt \\ \Rightarrow &\ \ l=2\pi\sqrt{5}. \end{aligned}$$

Closed Curve:

A closed curve is a curve that returns to its starting point, forming a loop, a cuvre C parametrized by $r(t) = (x(t), y(t), z)$ for $t \in [a, b]$ is called closed if

$$r(a) = r(b).$$

Example 1.5.3

1) Circle in 3D: A circle lying in a plane, such as the xy-plane, can be represented by:

$$r(t) = (r\cos t, r\sin t, 0) \ \ for\ t \in [0, 2\pi].$$

This is a closed curve because the parameter t runs from 0 to 2π, and C returning to the starting point $(i.e\ r(0) = r(2\pi))$.

2) The parametrized curve C defined by

$$r(t) = \left(t^2, t^4 - 4, 3\right) \ \ for\ t \in [-2, 2],$$

is a closed curve because $r(-2) = r(2)$.

1.5.2 Curvilinear integral

A line integral (or curvilinear integral) is a type of integral that evaluates a function along a given curve in space. Instead of integrating a function over a flat region or within a volume, this type of integral is performed over a curve (or path). Curvilinear integrals are commonly used to compute physical quantities, such as the work done by a force along a path, or to calculate the circulation of a vector field.

Let C a curve and f a function defined at points of C, then the curvilinear integral of f along C is defined by

$$\int_C f dl,$$

where dl is the infinitesimal element of length along the curve.

There are two main types of curvilinear integrals:

Curvilinear integral of a scalar field: A scalar function is integrated along a curve.

Curvilinear integral of a vector field: A vector field is integrated along a curve.

Curvilinear integral of a scalar field

Let C a parametrized curve of class C^1 defined by

$$C(t) = \{r(t) = (x(t), y(t), z(t)), \quad t \in [a,b]\},$$

and $f : C \to \mathbb{R}$ is a scalar field. The Curvilinear integral of f along the curve C is given by

$$\int\limits_C f dl = \int\limits_a^b f(x(t), y(t), z(t)) \|r'(t)\| dt,$$

with $dl = \|r'(t)\| dt$ and

$$r'(t) = \left(\frac{dx(t)}{dt}, \frac{dy(t)}{dt}, \frac{dz(t)}{dt}\right) \text{ and } \|r'(t)\| = \sqrt{\left(\frac{dx(t)}{dt}\right)^2 + \left(\frac{dy(t)}{dt}\right)^2 + \left(\frac{dz(t)}{dt}\right)^2}.$$

Proposition 1.5.1 *Elementary properties*

From the properties of Riemann integrals, we have

i) $\int\limits_C kf dl = k\int\limits_C f dl$*, where* k *is a constant.*

ii) $\int\limits_C (f+g) dl = \int\limits_C f dl + \int\limits_C g dl,$

iii) $\int\limits_C f dl = \int\limits_{C_1} f dl + \int\limits_{C_2} f dl$*, where* $C = C_1 + C_2$.

Example 1.5.4

Let $C = \{r(t) = (\cos t, \sin t, t), \ t \in [0, 2\pi]\}$ *a parametrized curve and* $f : C \to \mathbb{R}$ *defined by*

$$f(x,y,z) = x^2 + y^2 + z^2.$$

To calculate the curvilinear integral of f *along the cuvre* C*, we need to calculate* $r'(t)$ *then* $\|r'(t)\|$

$$\begin{aligned} r'(t) &= \left(\frac{dx(t)}{dt}, \frac{dy(t)}{dt}, \frac{dz(t)}{dt}\right) \\ &= (-\sin t, \cos t, 1). \end{aligned}$$

Then

$$\begin{aligned}\|r'(t)\| &= \sqrt{\left(\frac{dx(t)}{dt}\right)^2+\left(\frac{dy(t)}{dt}\right)^2+\left(\frac{dz(t)}{dt}\right)^2}\\ &= \sqrt{(-\sin t)^2+(\cos t)^2+1}=\sqrt{2}.\end{aligned}$$

Hence

$$\begin{aligned}\int_C f dl &= \int_0^{2\pi} f(x(t),y(t),z(t))\,\|r'(t)\|\,dt,\\ &= \int_0^{2\pi}\left((\cos t)^2+(\sin t)^2+t^2\right)\sqrt{2}dt\\ &= \int_0^{2\pi}\left(1+t^2\right)\sqrt{2}dt=\sqrt{2}\left[2t+\frac{1}{3}t^3\right]_0^{2\pi}\\ &= \frac{8}{3}\sqrt{2}\pi^3+4\sqrt{2}\pi.\end{aligned}$$

Curvilinear integral of a vector field

Let C a parametrized curve of class C^1 defined by

$$C(t)=\{r(t)=(x(t),y(t),z(t)),\quad t\in[a,b]\},$$

and $f: C\to\mathbb{R}^3$ a vector field. The curvilinear integral of f along the curve C is given by

$$\int_C f dl=\int_a^b f(x(t),y(t),z(t))\,r'(t)\,dt,$$

with $r'(t)=\left(\frac{dx(t)}{dt},\frac{dy(t)}{dt},\frac{dz(t)}{dt}\right)$ and
$f(x(t),y(t),z(t))=(f_1(x(t),y(t),z(t)),f_2(x(t),y(t),z(t)),f_3(x(t),y(t),z(t)))$,

then

$$\begin{aligned}\int_C f dl &= \int_a^b (f_1(x(t),y(t),z(t)),f_2(x(t),y(t),z(t)),f_3(x(t),y(t),z(t)))\left(\frac{dx(t)}{dt},\frac{dy(t)}{dt},\frac{dz(t)}{dt}\right)dt,\\ &= \int_a^b\left[f_1(x(t),y(t),z(t))\frac{dx(t)}{dt}+f_2(x(t),y(t),z(t))\frac{dy(t)}{dt}+f_3(x(t),y(t),z(t))\frac{dz(t)}{dt}\right]dt.\end{aligned}$$

Example 1.5.5

Let $C = \{r(t) = (t, t^2, t^3), \ \ t \in [0,1]\}$ *a parametrized curve and* $f : C \to \mathbb{R}^3$ *defined by*

$$f(x,y,z) = (x, 2y, 3z).$$

To calculate the curvilinear integral of f *along the cuvre* C*, we need to calculate* $r'(t)$

$$\begin{aligned} r'(t) &= \left(\frac{dx(t)}{dt}, \frac{dy(t)}{dt}, \frac{dz(t)}{dt}\right) \\ &= \left(1, 2t, 3t^2\right). \end{aligned}$$

The curvilinear integral of f *along the cuvre* C *is given by the following relation*

$$\int_C f d\ell = \int_0^1 f(x(t), y(t), z(t))\, r'(t)\, dt.$$

First, we calculate $f(x(t), y(t), z(t))\, r'(t)$

$$\begin{aligned} f(x(t), y(t), z(t))\, r'(t) &= f\left(t, t^2, t^3\right) . \left(1, 2t, 3t^2\right) \\ &= \left(t, 2t^2, 3t^3\right) . \left(1, 2t, 3t^2\right) \\ &= t \times 1 + 2t^2 \times 2t + 3t^3 \times 3t^2 \\ &= 9t^5 + 4t^4 + t. \end{aligned}$$

Then

$$\begin{aligned} \int_C f dl &= \int_0^1 f(x(t), y(t), z(t))\, r'(t)\, dt \\ &= \int_0^1 \left(9t^5 + 4t^4 + t\right) dt \\ &= \left[3t^6 + \frac{4}{5}t^5 + \frac{1}{2}t^2\right]_0^1 = \frac{43}{10}. \end{aligned}$$

1.5.3 Green's theorem

Green's Theorem is a fundamental result in vector calculus that relates a line integral around a closed curve to a double integral over the region enclosed by the curve.

Definition 1.5.2 *Positively oriented, piecewise smooth, and simple curve*

Let $C = \{r(t) = (x(t), y(t), z(t)), \quad t \in [a, b]\}$ *a parametrized curve.*
i) C is simple curve, if it does not intersect itself, i.e

$$r(t_1) \neq r(t_2), \quad for \ \ t_1, t_2 \in \,]a, b[\ \ and \ t_1 \neq t_2.$$

ii) C is closed curve, if $r(a) = r(b)$.
iii) C is piecewise smooth curve, if C is composed of several smooth curves i.e

$$C = \{C_1, C_2, ..., C_n\},$$

where C_i, $i = 1, ..., n$ *is a smooth curve (meaning differentiable with a continuous derivative).*

iv) C is positively oriented if it is counterclockwise. This means that the area bounded by it is on the left side as you move along the curve.

Theorem 1.5.3 *Green's theorem*

Let C be a positively oriented, piecewise smooth, simple closed curve in the plane, and let D be the region enclosed by C (i.e $\partial D = C$, ∂D *is the boundry of D). If* $f = (f_1, f_2)$ *is a continuously differentiable vector field on an open region containing D and its boundary C, then Green's Theorem states:*

$$\oint_C f dl = \oint_C (f_1 dx + f_2 dy) = \iint_D \left(\frac{\partial f_2}{\partial_x} - \frac{\partial f_1}{\partial_y} \right) dxdy.$$

Example 1.5.6 *Work Done by a Force Field.*

Let C is the positively oriented circle of radius 1 centered at the origin.

$$\begin{aligned} C &= \{r(t) = (\cos t, \sin t), \quad t \in [0, 2\pi]\} \\ &= \{(x, y)(\cos t, \sin t), \quad t \in [0, 2\pi]\} \\ &\Rightarrow x = \cos t \in [-1, 1] \quad and \ \ y = \sin t \in [-1, 1]. \end{aligned}$$

Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ *a vector function defined by*

$$f(x, y) = \left(x^2 - y^2, 2xy\right).$$

According to Green's theorem, the curvilinear integral of f along the cuvre C is gievn by

$$\oint_C f dl = \oint_C (f_1 dx + f_2 dy) = \iint_D \left(\frac{\partial f_2}{\partial_x} - \frac{\partial f_1}{\partial_y}\right) dxdy.$$

We have $f_1(x,y) = x^2 - y^2$ and $f_2(x,y) = 2xy$, then

$$\frac{\partial f_2(x,y)}{\partial_x} = 2y \quad and \quad \frac{\partial f_1(x,y)}{\partial_y} = -2y,$$

then

$$\begin{aligned}\oint_C f(x,y)\, dl &= \iint_D \left(\frac{\partial f_2(x,y)}{\partial_x} - \frac{\partial f_1(x,y)}{\partial_y}\right) dxdy \\ &= \iint_D 4ydxdy \\ &= \int_{-1}^{1} dx \times \int_{-1}^{1} 4ydy \\ &= [x]_{-1}^{1} \times \left[2y^2\right]_{-1}^{1} \\ &= 2 \times 0 = 0.\end{aligned}$$

1.6 Surface integrals and Stokes theorem

In the first part, we introduce parametrized surfaces. This is the analogue of parametrized curves, when the starting domain is a subset of $\mathbb{R}^2$, and not an interval of $\mathbb{R}$. Then, we define integration on parametrized surfaces. As in the case of parametrized curves, there are two kinds of objects that can be integrated: scalar and vector fields defined on the surface. Finally, we study the Stokes' theorem.

Definition 1.6.1 *Parametrized surface*

Parametrized surface means that the coordinates of points on the surface are given as functions of two variables. A parametrized surface S is represented by a vector-valued function $r : \mathbb{R}^2 \to \mathbb{R}^3$ as follows

$$r(u,v) = (x(u,v), y(u,v), z(u,v)), \qquad (u,v) \in D \subset \mathbb{R}^2,$$

we can also define S by

$$S = \left\{ r(u,v) = (x(u,v), y(u,v), z(u,v)), \qquad (u,v) \in D \subset \mathbb{R}^2 \right\}.$$

Example 1.6.1

1) The following parametrized surface S is the sphere with radius R

$$r(u,v) = (R \sin u \cos v, R \sin u \sin v, R \cos u), \quad (u,v) \in D = [0,\pi] \times [0, 2\pi].$$

We can write

$$S = \{ r(u,v) = (R \sin u \cos v, R \sin u \sin v, R \cos u), \quad (u,v) \in D = [0,\pi] \times [0, 2\pi] \}.$$

2) The following parametrized surface S is the parameter torus a, b dans $\mathbb{R}^3$

$$r(u,v) = ((a + b \cos v) \cos u, (a + b \cos v) \sin u, b \sin v), \quad (u,v) \in D = [0, 2\pi] \times [0, 2\pi].$$

Let S a surface and f a function defined at points of S, then the surface integral of f over S is given by

$$\int_S f dS,$$

where dS is the infinitesimal surface element.

There are two main types of surface integrals: a surface integral of a scalar field and a surface integral of a vector field.

1.6.1 Surface integral of a scalar function

Let S a parametrized surface of class C^1 defined by

$$S = \{ r(u,v) = (x(u,v), y(u,v), z(u,v)), \qquad (u,v) \in D = [a,b] \times [c,d] \},$$

and $f : S \to \mathbb{R}$ a scalar field.The surface integral of f over S is given by

$$\int_S f dS = \iint_D f(r(u,v)) \left\| \frac{\partial r}{\partial u} \wedge \frac{\partial r}{\partial v} \right\| dudv,$$

with

$$dS = \left\| \frac{\partial r}{\partial u} \wedge \frac{\partial r}{\partial v} \right\| dudv,$$

where

$$\frac{\partial r}{\partial u} = \left(\frac{\partial x\left(u,v\right)}{\partial u}, \frac{\partial y\left(u,v\right)}{\partial u}, \frac{\partial z\left(u,v\right)}{\partial u} \right) \quad \text{and} \quad \frac{\partial r}{\partial v} = \left(\frac{\partial x\left(u,v\right)}{\partial v}, \frac{\partial y\left(u,v\right)}{\partial v}, \frac{\partial z\left(u,v\right)}{\partial v} \right),$$

and $\wedge$ is the cross product i.e

$$\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \wedge \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = \begin{pmatrix} a_2b_3 - a_3b_2 \\ a_3b_1 - a_1b_3 \\ a_1b_2 - a_2b_1 \end{pmatrix}.$$

Example 1.6.2

Let the following parametrized surface S which is the sphere with radius R

$$S = \left\{ r\left(u,v\right) = \left(R\sin u\cos v, R\sin u\sin v, R\cos u\right), \quad \left(u,v\right) \in D = [0,\pi] \times [0,2\pi] \right\},$$

and $f : \mathbb{R}^3 \to \mathbb{R}$ defined by

$$f\left(x,y,z\right) = x^2 + y^2 + z^2.$$

To calculate the surface integral of f above S, we need to determine $\frac{\partial r}{\partial u}, \frac{\partial r}{\partial v}$ then $\left\| \frac{\partial r}{\partial u} \wedge \frac{\partial r}{\partial v} \right\|$

$$\frac{\partial r\left(u,v\right)}{\partial u} = \left(R\cos u\cos v, -R\cos u\sin v, -\sin u\right),$$

and

$$\frac{\partial r\left(u,v\right)}{\partial v} = \left(-R\sin u\sin v, R\sin u\cos v, -R\sin u\right).$$

Then

$$\begin{aligned} \frac{\partial r}{\partial u} \wedge \frac{\partial r}{\partial v} &= R^2\sin u\left(\sin u\cos v, \sin u\sin v, \right) \\ \Rightarrow & \left\| \frac{\partial r}{\partial u} \wedge \frac{\partial r}{\partial v} \right\| = R^2\sin u. \end{aligned}$$

Now, we calculate the surface integral of f above S

$$\begin{aligned}\int_S f(x,y,z)\,dS &= \iint_D f(x,y,z)\left\|\frac{\partial r}{\partial u}\wedge\frac{\partial r}{\partial v}\right\| dudv \\ &= \iint_D R^2\left\|\frac{\partial r}{\partial u}\wedge\frac{\partial r}{\partial v}\right\| dudv,\ \ (f(x,y,z)=R^2,\ for\ (x,y,z)\in D) \\ &= \int_0^{\pi}\int_0^{2\pi} R^2R^2\sin u\,dudv \\ &= R^4\int_0^{\pi}\sin u\,du\int_0^{2\pi}dv \\ &= R^4\times 2\times 2\pi = 4R^4\pi.\end{aligned}$$

1.6.2 Surface integral of a vector function

Let S a parametrized surface of C^1 class defined by

$$S=\{r(u,v)=(x(u,v),y(u,v),z(u,v)),\qquad (u,v)\in D\},$$

and $f:S\to\mathbb{R}^3$ a vector field. The surface integral of f over S is given by

$$\int_S f dS=\iint_D f(r(u,v))\times\left(\frac{\partial r}{\partial u}\wedge\frac{\partial r}{\partial v}\right)dudv$$

with

$$dS=\frac{\partial r}{\partial u}\wedge\frac{\partial r}{\partial v}dudv,$$

where

$$\frac{\partial r}{\partial u}=\left(\frac{\partial x(u,v)}{\partial u},\frac{\partial y(u,v)}{\partial u},\frac{\partial z(u,v)}{\partial u}\right)\quad\text{and}\quad\frac{\partial r}{\partial v}=\left(\frac{\partial x(u,v)}{\partial v},\frac{\partial y(u,v)}{\partial v},\frac{\partial z(u,v)}{\partial v}\right),$$

and $\wedge$ is the cross product i.e

$$\begin{pmatrix}a_1\\a_2\\a_3\end{pmatrix}\wedge\begin{pmatrix}b_1\\b_2\\b_3\end{pmatrix}=\begin{pmatrix}a_2b_3-a_3b_2\\a_3b_1-a_1b_3\\a_1b_2-a_2b_1\end{pmatrix}.$$

Example 1.6.3

Let the following parametrized surface S which is the sphere with radius R

$$S = \{r(u,v) = (R\sin u\cos v, R\sin u\sin v, R\cos u), \quad (u,v) \in D = [0,\pi]\times[0,2\pi]\},$$

and $f : \mathbb{R}^3 \to \mathbb{R}^3$ a vector field defined by

$$f(x,y,z) = (x,y,z).$$

To calculate the surface integral of f above S, we need to calculate $\frac{\partial r}{\partial u}, \frac{\partial r}{\partial v}$ then $\frac{\partial r}{\partial u} \wedge \frac{\partial r}{\partial v}$

$$\frac{\partial r(u,v)}{\partial u} = (R\cos u\cos v, R\cos u\sin v, -R\sin u),$$

and

$$\frac{\partial r(u,v)}{\partial v} = (-R\sin u\sin v, R\sin u\cos v, -R\sin u),$$

then

$$\frac{\partial r}{\partial u} \wedge \frac{\partial r}{\partial v} = R^2\sin u(\sin u\cos v, \sin u\sin v, \cos u).$$

We have

$$f(x(t), y(t), z(t)) = (R\sin u\cos v, R\sin u\sin v, R\cos u),$$

then

$$\begin{aligned} & f(x(t), y(t), z(t)) \times \left(\frac{\partial r}{\partial u} \wedge \frac{\partial r}{\partial v}\right) \\ &= (R\sin u\cos v, R\sin u\sin v, R\cos u) \times R^2\sin u(\sin u\cos v, \sin u\sin v, \cos u) \\ &= R^3\sin u\left(\sin^2 u\cos^2 v + \sin^2 u\sin^2 v + \cos^2 u\right) \\ &= R^3\sin u\left(\sin^2 u\left(\cos^2 v + \sin^2 v\right) + \cos^2 u\right) \\ &= R^3\sin u\left(\sin^2 u + \cos^2 u\right) = R^3\sin u. \end{aligned}$$

Now, we calculate the surface integral of f above S

$$\begin{aligned}
\int_S f dS &= \iint_D f(r(u,v)) \times \left(\frac{\partial r}{\partial u} \wedge \frac{\partial r}{\partial v}\right) dudv \\
&= \iint_D R^3 \sin u du dv \\
&= R^3 \int_0^{\pi} \int_0^{2\pi} \sin u du dv \\
&= R^4 \int_0^{\pi} \sin u du \int_0^{2\pi} dv \\
&= R^4 (2)(2\pi) = 4R^4\pi.
\end{aligned}$$

Stokes' theorem

Stokes' theorem relates a surface integral of the curl of a vector field to a line integral around the boundary of that surface.

Let S be a smooth surface with boundary ∂S, and let f be a continuously differentiable vector field on an open region containing S.

$$\int_S (\nabla \times f) dS = \int_{\partial S} f dl,$$

where, $\nabla \times f$ is the curl of f and dl is the differential line element around the curve ∂S.

Example 1.6.4

Let a circular disk S of radius R, centered at the origin in the xy-plane (at $z = 0$). The boundary ∂S of this disk is the circle of radius R, also centered at the origin.

$$\begin{aligned}
S &= \left\{(x,y,z), \quad x^2 + y^2 \leq R \text{ and } z = 0\right\} \\
&= \left\{r(u,v) = (u\cos v, u \sin v, 0), \quad (u,v) \in D = [0,R] \times [0,2\pi]\right\},
\end{aligned}$$

then

$$\begin{aligned}
\partial S &= \left\{(x,y,z), \quad x^2 + y^2 = R \text{ and } z = 0\right\} \\
&= \left\{r(t) = (R\cos t, R\sin t, 0), \quad t \in [0,2\pi]\right\}.
\end{aligned}$$

Let $f : \mathbb{R}^3 \to \mathbb{R}^3$ *a vector function of class* C^1 *defined by*

$$f(x, y, z) = (-y, x, 0).$$

According to Stokes's theorem, we have

$$\int_S (\nabla \times f)\, dS = \int_{\partial S} f dl.$$

1) First, we calculate the left-hand side which is the surface integral of the curl of a vector field f *over the surface* S

$$\int_S (\nabla \times f)\, dS,$$

we start whith $\nabla \times f$ *and* dS

$$\begin{aligned} \nabla \times f &= \left(\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z}, \frac{\partial f_1}{\partial z} - \frac{\partial f_3}{\partial x}, \frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y} \right) \\ &\Rightarrow \nabla \times f(x, y, z) = (0, 0, 2), \end{aligned}$$

and

$$dS = \frac{\partial r}{\partial u} \wedge \frac{\partial r}{\partial v} du dv,$$

with

$$\frac{\partial r(u, v)}{\partial u} = (\cos v, \sin v, 0) \quad \textit{and} \quad \frac{\partial r(u, v)}{\partial v} = (-u \sin v, u \cos v, 0),$$

then

$$\frac{\partial r}{\partial u} \wedge \frac{\partial r}{\partial v} = u(0, 0, 1).$$

According to the definition of a surface integral of a vector field, we have

$$\begin{aligned}\int\limits_S (\nabla \times f)\, dS &= \iint\limits_D (\nabla \times f)\,(r\,(u,v)) \times \left(\frac{\partial r}{\partial u} \wedge \frac{\partial r}{\partial v}\right) dudv \\ &= \int\limits_0^R \int\limits_0^{2\pi} (0,0,2) \times u\,(0,0,1)\, dudv \\ &= \int\limits_0^R \int\limits_0^{2\pi} 2ududv \\ &= 2\int\limits_0^R udu \times \int\limits_0^{2\pi} dv = 2\left(\frac{R^2}{2}\right)(2\pi) \\ &= 2\pi R^2,\end{aligned}$$

then

$$\int\limits_S (\nabla \times f)\, dS = 2\pi R^2.$$

2) Second, we calculate the right-hand side which is an integral of the vector field f along the boundary of the surface ∂S.

$$\int\limits_{\partial S} f dl = \int\limits_0^{2\pi} f\,(x\,(t)\,, y\,(t)\,, z\,(t))\, r'\,(t)\, dt,$$

we have

$$r'\,(t) = (-R\sin t, R\cot t, 0)\,,$$

and

$$f\,(x\,(t)\,, y\,(t)\,, z\,(t)) = (-R\sin t, R\cot t, 0)\,,$$

then

$$f\,(x\,(t)\,, y\,(t)\,, z\,(t))\, r'\,(t) = R^2.$$

Now we calculate the integral of f along ∂S.

$$\begin{aligned}\int_{\partial S} f dl &= \int_0^{2\pi} f\left(x\left(t\right), y\left(t\right), z\left(t\right)\right) r'\left(t\right) dt \\ &= \int_0^{2\pi} R^2 dt = 2\pi R^2,\end{aligned}$$

then

$$\oint_{\partial S} f dl = 2\pi R^2.$$

Ostrogradsky's theorem

Ostrogradsky's theorem relates a surface integral of a vector field over a closed surface to a volume integral of the divergence of the vector field inside the surface.

Let V be a volume enclosed by a smooth closed surface S, and let f be a continuously differentiable vector field. The Divergence Theorem (Ostrogradsky's Theorem) states:

$$\int_S f dS = \int_V \nabla . f dV,$$

where, $\nabla . f$ is the divergence of f and dV is the volume element inside the volume V.

Example 1.6.5

Let V be the volume of a sphere of radius R centered at the origin. The surface S is the boundary of the sphere (i.e., the surface of the sphere) defined by

$$S = \left\{r\left(u, v\right) = \left(R \sin u \cos v, R \sin u \sin v, R \cos u\right), \quad \left(u, v\right) \in D = [0, \pi] \times [0, 2\pi]\right\}.$$

Let $f : \mathbb{R}^3 \to \mathbb{R}^3$ be a continuously differentiable vector field defined by

$$f\left(x, y, z\right) = \left(x, y, z\right).$$

According to Ostrogradsky's theorem (Divergence Theorem), we have

$$\int_S f dS = \int_V \nabla . f dV.$$

1) First, we calculate the left-hand side which is the surface integral of f above S

$$\int_S f dS = \iint_D f(r(u,v)) \times \left(\frac{\partial r}{\partial u} \wedge \frac{\partial r}{\partial v}\right) dudv,$$

we need to determine $\frac{\partial r}{\partial u}, \frac{\partial r}{\partial v}$ *then* $\frac{\partial r}{\partial u} \wedge \frac{\partial r}{\partial v}$

$$\frac{\partial r(u,v)}{\partial u} = (R\cos u \cos v, R\cos u \sin v, -R\sin u),$$

and

$$\frac{\partial r(u,v)}{\partial v} = (-R\sin u \sin v, R\sin u \cos v, -R\sin u),$$

then

$$\frac{\partial r}{\partial u} \wedge \frac{\partial r}{\partial v} = R^2 \sin u (\sin u \cos v, \sin u \sin v, \cos u).$$

We have

$$f(x(t), y(t), z(t)) = (R\sin u \cos v, R\sin u \sin v, R\cos u),$$

then

$$f(x(t), y(t), z(t)) \times \left(\frac{\partial r}{\partial u} \wedge \frac{\partial r}{\partial v}\right)$$

$$\begin{aligned}
&= (R\sin u \cos v, R\sin u \sin v, R\cos u) \times R^2 \sin u (\sin u \cos v, \sin u \sin v, \cos u) \\
&= R^3 \sin u \left(\sin^2 u \cos^2 v + \sin^2 u \sin^2 v + \cos^2 u\right) \\
&= R^3 \sin u \left(\sin^2 u \left(\cos^2 v + \sin^2 v\right) + \cos^2 u\right) \\
&= R^3 \sin u \left(\sin^2 u + \cos^2 u\right) = R^3 \sin u.
\end{aligned}$$

Now, we calculate the surface integral of f above S

$$\begin{aligned}\int_S f dS &= \iint_D f(r(u,v)) \times \left(\frac{\partial r}{\partial u} \wedge \frac{\partial r}{\partial v}\right) dudv \\ &= \iint_D R^3 \sin u du dv \\ &= R^3 \int_0^{\pi}\int_0^{2\pi} \sin u du dv \\ &= R^3 \int_0^{\pi} \sin u du \int_0^{2\pi} dv \\ &= R^3 \times 2 \times 2\pi = 4\pi R^3.\end{aligned}$$

2) Second, we calculate the right-hand side which is the total divergence of the vector field inside the volume V.

$$\int_V \nabla . f dV.$$

The vector field f is continuously differentiable vector, we cancalculate its divergence $\nabla . f$:

$$\begin{aligned}\nabla . f(x,y,z) &= \frac{\partial f_1(x,y,z)}{\partial x} + \frac{\partial f_2(x,y,z)}{\partial y} + \frac{\partial f_3(x,y,z)}{\partial z}. \\ &= \frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z} = 3,\end{aligned}$$

the element dV for this surface S (the sphere of radius R centered at the origin) is given by

$$dV = dxdydz = R^2 \sin u dR du dv.$$

Now, we calculate the integral of $\nabla . f$ over the volume V

$$\begin{aligned}\int_v \nabla . f dV &= \int_0^{R}\int_0^{\pi}\int_0^{2\pi} 3R^2 \sin u dR du dv \\ &= 3\int_0^{R} R^2 dR \times \int_0^{\pi} \sin u du \times \int_0^{2\pi} dv \\ &= 3\left(\frac{R^3}{3}\right)(2)(2\pi) = 4\pi R^3.\end{aligned}$$

Chapter 2

Infinite series

2.1 Sequences

A sequence is an ordered list of numbers, the numbers are called terms (or elements) of the sequence. It is a function with domain the natural numbers $\mathbb{N} = \{0, 1, 2, 3, ...\}$ or the non-negative integers $\mathbb{Z}^+ = \{0, 1, 2, 3, ...\}$. Usually, we denote a sequence by $(u_n)_{n\geq 0}$ or $(v_n)_{n\geq 0}$ where

$$\begin{aligned} U : \mathbb{N} &\to \mathbb{R} \\ n &\longmapsto U_n. \end{aligned}$$

Example 2.1.1

1) The sequence $(u_n)_{n\geq 0}$ *defined by*

$$u_n = \ln(n+1), \quad n \in \mathbb{N}.$$

2) The sequence $(u_n)_{n\geq 0}$ *defined by*

$$u_n = 3n + 2, \quad n \in \mathbb{N}.$$

$(u_n)_{n\geq 0}$ *is called the arithmetic sequence of initial (or first) term* $u_0 = 2$ *and the common difference* $d = 3$. *(*$u_n = u_0 + nd$ *or* $u_n = u_1 + (n-1)d$ *are the general term of the* *arithmetic sequence).*

3) The sequence $(v_n)_{n\geq 0}$ *defined by*

$$v_n = 4 \times 3^n, \quad n \geq 0.$$

$(v_n)_{n\geq 0}$ *is called the geometic sequence of initial (or first) term* $v_o = 4$ *and the common ratio* $q = 3$. *(*$v_n = v_0 \times q^n$ *or* $v_n = v_1 \times q^{(n-1)}$ *are the general term of the geometric seauence).*

2.2 Infinite series

A series is the sum of an infinite number of terms of the sequence $(u_n)_{n\geq 0}$, i.e

$$\begin{aligned} S &= u_0 + u_1 + u_2 + ... \\ &= \sum_{n=0}^{+\infty} u_n. \end{aligned}$$

Proposition 2.2.1

A series has the following properties

i) $\sum_{n=0}^{+\infty} (u_n + v_n) = \sum_{n=0}^{+\infty} u_n + \sum_{n=0}^{+\infty} v_n$

ii) $\sum_{n=0}^{+\infty} a u_n = a \sum_{n=0}^{+\infty} u_n.$

Definition 2.2.1 *Partial sum of a sequence*

Let $(u_n)_{n\geq 0}$ *be a sequence. the partial sum of this sequence is given by*

$$\begin{aligned} S_n &= u_0 + u_1 + u_2 + ... + u_n \\ &= \sum_{n=0}^{n} u_k. \end{aligned}$$

Example 2.2.1

1) For a sequence $(u_n)_{n\geq 0}$, *we have for example*

$$S_0 = u_0, \; S_1 = u_0 + u_1, \; \text{and } S_5 = u_0 + u_1 + u_2 + u_3 + u_4 + u_5.$$

2) For $(u_n)_{n\geq 0}$ *an arithmetic sequence of the initial (or first) term* u_o *and the common difference* d*, the partial sum is given by the following relation*

$$S_n = \sum_{n=0}^{n} u_k = \frac{n+1}{2}\left(u_0 + u_n\right).$$

If the initial term of this sequence is u_1*, then*

$$S_n = \sum_{n=1}^{n} u_k = \frac{n}{2}\left(u_1 + u_n\right).$$

3) For $(u_n)_{n\geq 0}$ *a geometic sequence of the initial term* v_o *and the common ratio* q*, the partial sum is given by the following relation*

$$S_n = \sum_{n=0}^{n} u_k = \frac{1-q^{n+1}}{1-q} u_0.$$

If the initial term of this sequence is u_1*, then*

$$S_n = \sum_{n=1}^{n} u_k = \frac{1-q^{n}}{1-q} u_1.$$

Proposition 2.2.2 *Sum of a sequence whose partial sum is known*

Let $(u_n)_{n\geq 0}$ *be a sequence and* S_n *is its partial sum, then* S *the sum of this sequence is given by*

$$S = \lim_{n\to+\infty} S_n.$$

Example 2.2.2

1) Let $(u_n)_{n\geq 0}$ *be a geometric sequence of the general term* $u_n = 4 \times \left(\frac{1}{2}\right)^n$*, we have*

$$\begin{aligned} S_n &= \frac{1-q^{n+1}}{1-q} u_0 = \frac{1-\left(\frac{1}{2}\right)^{n+1}}{1-\frac{1}{2}} \times 4 \\ &= 8 - 8\left(\frac{1}{2}\right)^{n+1}. \end{aligned}$$

By using the previous proposition, we can calculate the sum S *of the sequence* $(u_n)_{n\geq 0}$

$$\begin{aligned} S &= \lim_{n\to+\infty} S_n \\ &= \lim_{n\to+\infty}\left(8 - 8\left(\frac{1}{2}\right)^{n+1}\right) = 8. \end{aligned}$$

2) Let $(u_n)_{n\geq 0}$ *be a sequence of a general term*

$$u_n = \frac{1}{n+1} - \frac{1}{n+2},$$

we have

$$\begin{aligned} S_n &= u_0 + u_1 + u_2 + ... + u_{n-1} + u_n \\ &= 1 - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \frac{1}{3} - \frac{1}{4} + ... + \frac{1}{n} - \frac{1}{n+1} + \frac{1}{n+1} - \frac{1}{n+2} \\ &= 1 - \frac{1}{n+2}. \end{aligned}$$

By using the previous proposition, we have

$$\begin{aligned} S &= \lim_{n\to+\infty} S_n \\ &= \lim_{n\to+\infty} \left(1 - \frac{1}{n+2}\right) = 1. \end{aligned}$$

2.3 Nature of a series

The nature of a series means that the series is either convergent or divergent. The nature of a series can be determined from its sum or by using other techniques called convergence criteria.

Proposition 2.3.1

Let $(u_n)_{n\geq 0}$ *be a sequence of a sum* S*, then*

i) If S *exists and finite, then the series* $\sum_{n=0}^{+\infty} u_n$ *is convergent.*

ii) If S *does not exist or is not finite, then the series* $\sum_{n=0}^{+\infty} u_n$ *is divergent.*

Example 2.3.1

1) Let $\sum_{n=0}^{+\infty} u_n$ *be a geometric series of general term*

$$u_n = 4 \times \left(\frac{1}{2}\right)^n.$$

In the previous example, we proved that

$$\begin{aligned} S &= \lim_{n\to+\infty} S_n \\ &= \lim_{n\to+\infty}\left(8-8\left(\frac{1}{2}\right)^{n+1}\right)=8, \end{aligned}$$

then, the series $\sum_{n=0}^{+\infty} 4\times\left(\frac{1}{2}\right)^n$ *is convergent.*

2) Let $\sum_{n=0}^{+\infty} u_n$ *be a series of term general*

$$u_n = a, \quad a \neq 0.$$

The partial sum of this series is

$$\begin{aligned} S_n &= u_0+u_1+u_2+...+u_{n-1}+u_n \\ &= a+a+a+...+a \\ &= an. \end{aligned}$$

then

$$\begin{aligned} S &= \lim_{n\to+\infty} S_n \\ &= \lim_{n\to+\infty} an = +\infty, \end{aligned}$$

the series $\sum_{n=0}^{+\infty} a$ *is divergent.*

Lemma 2.3.1 *Let* $\sum_{n=0}^{+\infty} u_n$ *and* $\sum_{n=0}^{+\infty} v_n$ *be two infinite series,*

i) $\sum_{n=0}^{+\infty} u_n$ *and* $\sum_{n=0}^{+\infty} \alpha\times u_n$ *have the same nature,* $(\alpha\neq 0)$.

ii) $\sum_{n=0}^{+\infty} u_n$ *converges and* $\sum_{n=0}^{+\infty} v_n$ *converges, then* $\sum_{n=0}^{+\infty}(u_n+v_n)$ *converges.*

iii) $\sum_{n=0}^{+\infty} u_n$ *converges and* $\sum_{n=0}^{+\infty} v_n$ *diverges, then* $\sum_{n=0}^{+\infty}(u_n+v_n)$ *diverges.*

iii) $\sum_{n=0}^{+\infty} u_n$ *diverges and* $\sum_{n=0}^{+\infty} v_n$ *diverges, then we cannot conclude anything about* $\sum_{n=0}^{+\infty}(u_n+v_n)$.

Proposition 2.3.2 *Convergence of geometric series*

$\sum\limits_{n=0}^{+\infty} u_0 \times q^n$ *a geometic series of an initial term* u_0 *and a common ratio* q*, then*

$$\sum_{n=0}^{+\infty} u_0 \times q^n \begin{cases} \textit{is comvergent} & \textit{if} \quad -1 < q < 1 \\ \textit{is divergent} & \textit{if not} \end{cases}$$

Proof.

i) For $-1 < q < 1$, we have

$$S = \lim_{n \to +\infty} S_n = \lim_{n \to +\infty} \frac{1 - q^{n+1}}{1 - q} u_0 = \frac{1}{1 - q} u_0,$$

then $\sum\limits_{n=0}^{+\infty} u_0 \times q^n$ is convergent.

ii) For $q \geq 1$, we have

$$S = \lim_{n \to +\infty} S_n = \lim_{n \to +\infty} \frac{1 - q^{n+1}}{1 - q} u_0 = +\infty, \quad (= -\infty \;\; if \;\; u_0 < 0),$$

then $\sum\limits_{n=0}^{+\infty} u_0 \times q^n$ is divergent.

iii) For $q = 1$, we have

$$S_n = (n + 1) u_0 \Rightarrow S = \lim_{n \to +\infty} S_n = +\infty, \quad (= -\infty \;\; if \;\; u_0 < 0),$$

then $\sum\limits_{n=0}^{+\infty} u_0 \times q^n$ is divergent.

iv) For $q \leq -1$, we have

$$S = \lim_{n \to +\infty} \frac{1 - q^{n+1}}{1 - q} u_0 \text{ does not exist},$$

then $\sum\limits_{n=0}^{+\infty} u_0 \times q^n$ is divergent. ■

Theorem 2.3.1 *Necessary condition for the convergent series*

Let $\sum\limits_{n=0}^{+\infty} u_n$ *be a convergent series, then*

$$\lim_{n \to +\infty} u_n = 0.$$

Corollary 2.3.2

Let $\sum_{n=0}^{+\infty} u_n$ *be a series, then*

$$\lim_{n\to+\infty} u_n \neq 0 \Rightarrow \sum_{n=0}^{+\infty} u_n \text{ is divergent.}$$

Example 2.3.2 *Let the following series*

$$1)\sum_{n=0}^{+\infty} \frac{5n^2+2n}{4n^2-9}, \qquad 2)\ \sum_{n=2}^{+\infty} \frac{n^2+2}{\ln n}.$$

1) For $\sum_{n=0}^{+\infty} \frac{5n^2+2n}{4n^2-9}$*,with* $u_n = \frac{5n^2+2n}{4n^2-9}$*, we have*

$$\lim_{n\to+\infty} u_n = \lim_{n\to+\infty} \frac{5n^2+2n}{4n^2-9} = \frac{5}{4} \neq 0,$$

then the series $\sum_{n=0}^{+\infty} u_n$ *is divergent.*

2)For $\sum_{n=2}^{+\infty} \frac{n^2+2}{\ln n}$*, with* $u_n = \frac{n^2+2}{\ln n}$*, we have*

$$\lim_{n\to+\infty} u_n = \lim_{n\to+\infty} \frac{n^2+2}{\ln n} = +\infty,$$

then the series $\sum_{n=0}^{+\infty} u_n$ *is divergent.*

Definition 2.3.3 *Reimann's series*

The series $\sum_{n=1}^{+\infty} \frac{1}{n^\alpha}$ *is called Reimann's series. Its convergence is related to the value of* α

$$\sum_{n=1}^{+\infty} \frac{1}{n^\alpha} : \begin{cases} \text{is convergent} & \text{if } \alpha > 1 \\ \text{is divergent} & \text{if } \alpha \leq 1 \end{cases},$$

for $\alpha = 1$*, the series* $\sum_{n=1}^{+\infty} \frac{1}{n}$ *is called harmonic series.*

2.3.1 Convergence's criteria for series with positive terms

Comparison criteria

Let $\sum_{n=0}^{+\infty} u_n$ and $\sum_{n=0}^{+\infty} v_n$ be two series with positive terms. If

$$u_n \leq v_n, \quad \forall n \in \mathbb{N}.$$

Then

$$\sum_{n=0}^{+\infty} v_n \text{ is convergent} \Rightarrow \sum_{n=0}^{+\infty} u_n \text{ is convergent}$$
$$\sum_{n=0}^{+\infty} u_n \text{ is divergent} \Rightarrow \sum_{n=0}^{+\infty} v_n \text{ is divergent.}$$

Example 2.3.3

1) $\sum_{n=1}^{+\infty} \frac{1}{n^3+3n}$, *with* $u_n = \frac{1}{n^3+3n}$.
We have $u_n \geq 0$ *for all* $n \geq 1$, *and*

$$\frac{1}{n^3+3n} \leq \frac{1}{n^3} \qquad \textit{(car } n^3 \leq n^3+3n\textit{)}.$$

The series $\sum_{n=1}^{+\infty} \frac{1}{n^3}$ *is Reimann series with* $\alpha = 3 > 1$, *then it is convergent.*

By using the comparison's criteria, we conclude that $\sum_{n=1}^{+\infty} \frac{1}{n^3+3n}$ *is convergent.*

2) $\sum_{n=1}^{+\infty} \frac{e^n}{n}$, *with* $u_n = \frac{e^n}{n}$.
We have $u_n \geq 0$ *for all* $n \geq 1$, *and*

$$\frac{1}{n} \leq \frac{e^n}{n} \qquad \textit{(because } e_n \geq 1\textit{)}.$$

The series $\sum_{n=1}^{+\infty} \frac{1}{n}$ *is harmonic series, then it is divergent.*

By using the comparison's criteria, we conclude that $\sum_{n=1}^{+\infty} \frac{e^n}{n}$ *is divergent.*

Equivalence's criteria

Let $\sum_{n=0}^{+\infty} u_n$ and $\sum_{n=0}^{+\infty} v_n$ be two series with strictly positive terms. If u_n is equivalent to v_n (we denote $u_n \sim v_n$):

$$u_n \underset{+\infty}{\sim} v_n \Leftrightarrow \lim_{n\to+\infty} \frac{u_n}{v_n} = 1.$$

Then, the series $\sum_{n=0}^{+\infty} u_n$ and $\sum_{n=0}^{+\infty} v_n$ are of the same nature.

Example 2.3.4

1) $\sum_{n=1}^{+\infty} \frac{5n^2+4}{3n^3+2n}$, *with* $u_n = \frac{5n^2+4}{3n^3+2n}$.
We have $u_n > 0$ *for all* $n \geq 1$, *and we know that*

$$u_n = \frac{5n^2+4}{3n^3+2n} \underset{+\infty}{\sim} \frac{5n^2}{3n^3} = \frac{5}{3}\frac{1}{n} = v_n \qquad \text{(because } \lim_{n\to+\infty} \frac{u_n}{v_n} = 1\text{)}.$$

The series $\frac{5}{3}\sum_{n=1}^{+\infty} \frac{1}{n}$ *is a harmonic series which is divergent.*

By using the equivalence's creteria, we conclude that $\sum_{n=1}^{+\infty} \frac{5n^2+4}{3n^3+2n}$ *is divergent.*

2) $\sum_{n=1}^{+\infty} \ln\left(1+\left(\frac{2}{3}\right)^n\right)$, *with* $u_n = \ln\left(1+\left(\frac{2}{3}\right)^n\right)$
We have $u_n > 0$ *for all* $n \geq 1$, *and we know that*

$$u_n = \ln\left(1+\left(\frac{2}{3}\right)^n\right) \underset{+\infty}{\sim} \left(\frac{2}{3}\right)^n = v_n \qquad \text{(we know that } \ln(1+y) \sim y \text{ when } y \to 0\text{)}.$$

The series $\sum_{n=1}^{+\infty} \left(\frac{2}{3}\right)^n$ *is a geometric series with ratio* $q = \frac{2}{3}$ $(-1 < q < 1)$, *then it is convergent*

By using the equivalence's creteria, we conclude that $\sum_{n=1}^{+\infty} \frac{5n^2+4}{3n^3+2n}$ *is convergent.*

Cauchy's criteria

Let $\sum_{n=0}^{+\infty} u_n$ be a series with positive terms. Suppose that

$$\lim_{n\to+\infty} \sqrt[n]{u_n} = l,$$

then

1) If $l < 1$, the series $\sum_{n=0}^{+\infty} u_n$ is convergent.

2) If $l > 1$, the series $\sum_{n=0}^{+\infty} u_n$ is divergent.

3) If $l = 1$, we cannot say anything about the series $\sum_{n=0}^{+\infty} u_n$.

Example 2.3.5

1) $\sum_{n=1}^{+\infty} \left(\frac{5n+4}{n^2+2n}\right)^n$ *with* $u_n = \left(\frac{5n+4}{n^2+2n}\right)^n$

We have $u_n \geq 0$ *for all* $n \geq 1$, *and*

$$\lim_{n\to+\infty} \sqrt[n]{u_n} = \lim_{n\to+\infty} \sqrt[n]{\left(\frac{5n+4}{n^2+2n}\right)^n} = \lim_{n\to+\infty} \left(\frac{5n+4}{n^2+2n}\right) = 0 = l.$$

By using Cauchy's criteria, and since $l = 0 < 1$, *the series* $\sum_{n=1}^{+\infty} \left(\frac{5n+4}{n^2+2n}\right)^n$ *is convergent.*

2) $\sum_{n=2}^{+\infty} \left(\frac{n}{\ln n}\right)^n$ *with* $u_n = \left(\frac{n}{\ln n}\right)^n$

We have $u_n \geq 0$ *for all* $n \geq 2$, *and*

$$\lim_{n\to\ +\infty} \sqrt[n]{u_n} = \lim_{n\to+\infty} \sqrt[n]{\left(\frac{n}{\ln n}\right)^n} = \lim_{n\to+\infty} \left(\frac{n}{\ln n}\right) = +\infty = l.$$

By using Cauchy's criteria, and since $l > 1$, *the series* $\sum_{n=1}^{+\infty} \left(\frac{5n+4}{n^2+2n}\right)^n$ *is divergent.*

D'Alembert's criteria

Let $\sum_{n=0}^{+\infty} u_n$ be a series with positive terms. Suppose that

$$\lim_{n\to+\infty} \frac{u_{n+1}}{u_n} = l,$$

then

1) If $l < 1$, the series $\sum_{n=0}^{+\infty} u_n$ is convergent.

2) If $l > 1$, the series $\sum_{n=0}^{+\infty} u_n$ is divergent.

3) If $l = 1$, we cannot say anything about the series $\sum_{n=0}^{+\infty} u_n$.

Example 2.3.6

1) $\sum_{n=0}^{+\infty} \frac{3^n}{n!}$ *with* $u_n = \frac{3^n}{n!}$

We have $u_n > 0$ *for all* $n \geq 0$*, and*

$$u_{n+1} = \frac{3^{n+1}}{(n+1)!} = \frac{3 \times 3^n}{(n+1)\, n!},$$

then

$$\frac{u_{n+1}}{u_n} = \frac{3 \times 3^n}{(n+1)\, n!} \times \frac{n!}{3^n} = \frac{3}{(n+1)},$$

so

$$\lim_{n \to +\infty} \frac{u_{n+1}}{u_n} = \lim_{n \to +\infty} \frac{3}{(n+1)} = 0.$$

By using D'Alembert's criteria, and since $l = 0 < 1$*, the series* $\sum_{n=1}^{+\infty} \frac{3^n}{n!}$ *is convergent.*

2) $\sum_{n=0}^{+\infty} \frac{1 \times 3 \times 5 \times ... \times (2n+3)}{(n+1)!}$ *with* $u_n = \frac{1 \times 3 \times 5 \times ... \times (2n+3)}{(n+1)!}$

We have $u_n > 0$ *for all* $n \geq 0$*, and*

$$u_{n+1} = \frac{1 \times 3 \times 5 \times ... \times (2n+5)}{(n+2)!} = \frac{1 \times 3 \times 5 \times ... \times (2n+3) \times (2n+5)}{(n+2)\,(n+1)!},$$

then

$$\begin{aligned} \frac{u_{n+1}}{u_n} &= \frac{1 \times 3 \times 5 \times ... \times (2n+3) \times (2n+5)}{(n+2)\,(n+1)!} \times \frac{(n+1)!}{1 \times 3 \times 5 \times ... \times (2n+3)} \\ &= \frac{2n+5}{n+2}, \end{aligned}$$

so

$$\lim_{n \to +\infty} \frac{u_{n+1}}{u_n} = \lim_{n \to +\infty} \frac{2n+5}{n+2} = 2.$$

By using D'Alembert's criteria, and since $l = 2 > 1$*, then the series* $\sum_{n=1}^{+\infty} \frac{1 \times 3 \times 5 \times ... \times (2n+3)}{(n+1)!}$ *is divergent.*

2.3.2 Convergence's creteria for series with real terms

Absolute convergence

Definition 2.3.4

Let $\sum_{n=0}^{+\infty} u_n$ *be a series with real term. We say that* $\sum_{n=0}^{+\infty} u_n$ *is absolutely convergent if and only if*

$$\sum_{n=0}^{+\infty} |u_n|,$$

is convergent.

1) **Criteria of absolute convergence**

Let $\sum_{n=0}^{+\infty} u_n$ is a series with real term. if $\sum_{n=0}^{+\infty} u_n$ is absolutely convergent then it is convergent.

Remark 2.3.5

If $\sum_{n=0}^{+\infty} u_n$ *is not absolutely convergent, we cannot directly conclude the nature of* $\sum_{n=0}^{+\infty} u_n$.

Example 2.3.7

1) $\sum_{n=1}^{+\infty} \frac{(-1)^n}{n^3+n}$ *with* $u_n = \frac{(-1)^n}{n^3+n}$ *We have* u_n *is a real term, so we try to know the nature of* $\sum_{n=1}^{+\infty} |u_n|$

$$|u_n| = \left|\frac{(-1)^n}{n^3+n}\right| = \frac{1}{n^3+n} \leq \frac{1}{n^3}.$$

The series $\sum_{n=1}^{+\infty} \frac{1}{n^3}$ *is Reimann series with* $\alpha = 3 > 1$*, then it is convergent.*

By using the comparison criteria, $\sum_{n=1}^{+\infty} |u_n|$ *is convergent. So,* $\sum_{n=1}^{+\infty} \frac{(-1)^n}{n^3+n}$ *is absolutely convergent.*

From Criteria of absolute convergence, we conclude that $\sum_{n=1}^{+\infty} \frac{(-1)^n}{n^3+n}$ *is convergent.*

2) $\displaystyle\sum_{n=1}^{+\infty}\frac{\sin n}{2^n}$ *with* $u_n = \dfrac{\sin n}{2^n}$

We have u_n *is a real term, so we try to know the nature of* $\displaystyle\sum_{n=1}^{+\infty}|u_n|$

$$|u_n| = \left|\frac{\sin n}{2^n}\right| = \frac{|\sin n|}{2^n} \leq \frac{1}{2^n} = \left(\frac{1}{2}\right)^n.$$

The series $\displaystyle\sum_{n=1}^{+\infty}\left(\frac{1}{2}\right)^n$ *is a geometric series with ratio* $q = \frac{1}{2}$ $(-1 < q < 1)$*, then it is convergent.*

By using the comparison creteria, $\displaystyle\sum_{n=1}^{+\infty}|u_n| = \sum_{n=1}^{+\infty}\frac{|\sin n|}{2^n}$ *is convergent. So,* $\displaystyle\sum_{n=1}^{+\infty}\frac{\sin n}{2^n}$ *is absolutly convergent.*

From Criteria of absolute convergence, we conclude $\displaystyle\sum_{n=1}^{+\infty}\frac{\sin n}{2^n}$ *is convergent.*

Leibniz's creteria for alternating series

Definition 2.3.6 *Alternating series*

$\displaystyle\sum_{n=0}^{+\infty}u_n$ *is said alternating series if*

$$u_n = (-1)^n b_n \text{ and } b_n \geq 0, \quad \forall n \geq 0.$$

Leibniz's creteria

Let $\displaystyle\sum_{n=0}^{+\infty}(-1)^n b_n$ be an alternating series. This $\displaystyle\sum_{n=0}^{+\infty}(-1)^n b_n$ is convergent if

i) $\lim\limits_{n\to+\infty} b_n = 0.$

ii) $(b_n)_n$ is decreasing.

Example 2.3.8

1) $\displaystyle\sum_{n=1}^{+\infty}\frac{(-1)^n}{\sqrt{n}}$ *with* $u_n = \dfrac{(-1)^n}{\sqrt{n}} = (-1)^n b_n$

We have u_n *is an alternating term, and* $b_n = \dfrac{1}{\sqrt{n}}$

$$i) \lim_{n\to+\infty} b_n = \lim_{n\to+\infty}\frac{1}{\sqrt{n}} = 0.$$

ii) $(b_n)_n$ is decreasing ($b_{n+1} \leq b_n, \ \ \forall n \geq 1$).

By using Leibniz's creteria, the series $\sum_{n=1}^{+\infty} \frac{(-1)^n}{\sqrt{n}}$ *is convergent.*

2) $\sum_{n=2}^{+\infty} \frac{(-1)^n}{\ln n}$ *with* $u_n = \frac{(-1)^n}{\ln n} = (-1)^n b_n$

We have u_n is an alternating term, and $b_n = \frac{1}{\ln n}$

i) $\lim_{n\to+\infty} b_n = \lim_{n\to+\infty} \frac{1}{\ln n} = 0.$

ii) $(b_n)_n$ is decreasing ($b_{n+1} \leq b_n, \ \ \forall n \geq 2$).

By using Leibniz's creteria, the $\sum_{n=2}^{+\infty} \frac{(-1)^n}{\ln n}$ *is convergent.*

2.3.3 Semi-convergent series

Definition 2.3.7 *Semi-convergent series*

Let $\sum_{n=0}^{+\infty} u_n$ *is a series with real term.* $\sum_{n=0}^{+\infty} u_n$ *is said semi-convergent (or conditionally convergent) if*

i) $\sum_{n=0}^{+\infty} |u_n|$ *is divergent (*$\sum_{n=0}^{+\infty} u_n$ *does not converge absolutely).*

ii) $\sum_{n=0}^{+\infty} u_n$ *is convergent.*

Example 2.3.9

1) $\sum_{n=1}^{+\infty} \frac{(-1)^n}{\sqrt{n}}$ *with* $u_n = \frac{(-1)^n}{\sqrt{n}}$.

i) Study the absolute convergence of $\sum_{n=1}^{+\infty} \frac{(-1)^n}{\sqrt{n}}$.

$$|u_n| = \left|\frac{(-1)^n}{\sqrt{n}}\right| = \frac{1}{\sqrt{n}} = \frac{1}{n^{\frac{1}{2}}}.$$

The series $\sum_{n=1}^{+\infty} \frac{1}{n^{\frac{1}{2}}}$ *is Reimann series with* $\alpha = \frac{1}{2} < 1$*, then it is divergent.*

Then, $\sum_{n=1}^{+\infty} \frac{(-1)^n}{\sqrt{n}}$ *does not converge absolutely.*

ii) Study the convergence of $\sum_{n=1}^{+\infty} \frac{(-1)^n}{\sqrt{n}}$.

We proved in the previous example that $\sum_{n=1}^{+\infty} \frac{(-1)^n}{\sqrt{n}}$ *is convergent (Leibniz's creteria).*

Then, the series $\sum_{n=1}^{+\infty} \frac{(-1)^n}{\sqrt{n}}$ *is semi-convergent (or conditionally convergent).*

2) $\sum_{n=2}^{+\infty} \frac{(-1)^n}{\ln n}$ *with* $u_n = \frac{(-1)^n}{\ln n}$.

i) Study the absolute convergence of $\sum_{n=2}^{+\infty} \frac{(-1)^n}{\ln n}$.

$$|u_n| = \left| \frac{(-1)^n}{\ln n} \right| = \frac{1}{\ln n},$$

we have

$$\frac{1}{n} \leq \frac{1}{\ln n}.$$

The series $\sum_{n=2}^{+\infty} \frac{1}{n}$ *is divergent (harmonic series).*

By using the comparison creteria, $\sum_{n=2}^{+\infty} |u_n| = \sum_{n=2}^{+\infty} \frac{1}{n}$ *is divergent. So* $\sum_{n=2}^{+\infty} \frac{(-1)^n}{\ln n}$ *does not converge absolutely.*

ii) Study the convergence of $\sum_{n=1}^{+\infty} \frac{(-1)^n}{\ln n}$.

We proved in the previous example that $\sum_{n=2}^{+\infty} \frac{(-1)^n}{\ln n}$ *is convergent (Leibniz's creteria).*

Then, the series $\sum_{n=2}^{+\infty} \frac{(-1)^n}{\sqrt{n}}$ *is semi-convergent (or conditionally convergent).*

Chapter 3

Power series

Definition 3.0.8 *Power series*

We say a power series, the series of the form

$$\sum_{n=0}^{+\infty} a_n x^n,$$

where $x \in \mathbb{R}$ *and* $(a_n)_{n\geq 0}$ *is a sequence (also we say that* $a_0, ..., a_n$ *are the coefficients of the series).*

More general, for $x_0 \in \mathbb{R}$*, the power series associate to* x_0 *is given by*

$$\sum_{n=0}^{+\infty} a_n \left(x - x_0\right)^n.$$

Example 3.0.10

1) Polynomials of degree p *are a particular type of power series where* $a_n = 0$ *for all* $n > p$*.*

$$3x^2 - 2x + 7 = \sum_{n=0}^{+\infty} a_n x^n \text{ where } \begin{cases} a_0 = 7,\ a_1 = -2,\ a_2 = 3 \\ \text{and } a_n = 0 \text{ for all } n > p \end{cases}$$

2) $\displaystyle\sum_{n=0}^{+\infty} \frac{x^n}{n^2+1}$ *with* $a_n = \dfrac{1}{n^2+1}$.

3.1 Domain of convergence of a power series

Definition 3.1.1

The domain of convergence of a power series is

$$D = \left\{ x \in \mathbb{R},\ \textit{where } \sum_{n=0}^{+\infty} a_n x^n \textit{ is convergent} \right\}.$$

Remark 3.1.2

Si $x = 0$*, then* $a_n x^n = 0$*, we conclude that the power series* $\sum_{n=0}^{+\infty} a_n x^n$ *is convergent. So,* $0 \in D$*, then*

$$D \neq \emptyset.$$

Lemma 3.1.1 *Abel's lemma*

If a power series $\sum_{n=0}^{+\infty} a_n x^n$ *is convergent for* $x = x_0 \neq 0$*, then it is convergent for all* x *such as* $-x_0 < x < x_0$

3.2 Radius of convergence of a power series

Definition 3.2.1 *Radius of convergence of a power series*

Let D *be the domain of convergence of a power series* $\sum_{n=0}^{+\infty} a_n x^n$*. The numder* $R = \sup_{x \in D} |x|$ *is called the radius of convergence of a the series.*

Remark 3.2.2

R *varies from* 0 *to* $+\infty$*.*

Using Abel's lemma and the definition of radius of convergence, we get the following proposition.

Proposition 3.2.1

Let $\sum_{n=0}^{+\infty} a_n x^n$ *a power series with a radius* R*. Then*

i) For $|x| < R$*, the series* $\sum_{n=0}^{+\infty} a_n x^n$ *is convergent.*

ii) For $|x| > R$*, the series* $\sum_{n=0}^{+\infty} a_n x^n$ *is divergent.*

iii) For $|x| = R$*, we can not say anything about the series* $\sum_{n=0}^{+\infty} a_n x^n$*.*

Remark 3.2.3

For the third case ($|x| = R$*), we have to study the nature of series* $\sum_{n=0}^{+\infty} a_n x^n$ *for* $x = R$ *and* $x = -R$*.*

Techniques to calculate the radius of convergence R

The radius of convergence R can be calculated using the following Hadamard's lemma.

Lemma 3.2.1 *(Hadamard's Lemma)*

Let $\sum_{n=0}^{+\infty} a_n x^n$ *be a power series, then*

$$R = \frac{1}{l},$$

where

$$l = \lim_{n \to +\infty} \left| \frac{a_{n+1}}{a_n} \right| \text{ or } l = \lim_{n \to +\infty} \sqrt[n]{|a_n|}.$$

Example 3.2.1

For each series of the following series, we will calculate its radius of convergence and we determine its domain of convergence.

1)$\sum_{n=0}^{+\infty} (-1)^n (n+2)! x^n$ *with* $a_n = (-1)^n (n+2)!$

i) Rdius of convergence R: $R = \dfrac{1}{l}$

We have $l = \lim_{n \to +\infty} \left| \frac{a_{n+1}}{a_n} \right|$

$\cdot\ a_{n+1} = (-1)^{n+1} (n+3)! = (-1)^{n+1} (n+3)(n+2)!.$

$\cdot\ \left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(-1)^{n+1}(n+3)(n+2)!}{(-1)^n (n+2)!}\right| = n+3.$

then

$$\begin{aligned} l &= \lim_{n\to+\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to+\infty} n+3 \\ &= +\infty, \end{aligned}$$

since $R = \frac{1}{l}$, *we get*

$$R = 0.$$

ii) Domain of convergence D

Since $R = 0$, *then the series* $\sum_{n=0}^{+\infty} a_n x^n = \sum_{n=0}^{+\infty} (-1)^n (n+2)! x^n$ *is convergent on its domain* D *which is*

$$D = \{0\}.$$

2) $\sum_{n=1}^{+\infty} \left(\frac{n+2}{n^2}\right)^n x^n$ *with* $a_n = \left(\frac{n+2}{n^2}\right)^n$

i) Radius of convergence R: $R = \dfrac{1}{l}$

We have $l = \lim_{n\to+\infty} \sqrt[n]{u_n}$

$$\sqrt[n]{u_n} = (u_n)^{\frac{1}{n}} = \left(\left(\frac{n+2}{n^2}\right)^n\right)^{\frac{1}{n}} = \frac{n+2}{n^2},$$

then

$$\begin{aligned} l &= \lim_{n\to+\infty} \sqrt[n]{a_n} = \lim_{n\to+\infty} \frac{n+2}{n^2} \\ &= 0, \end{aligned}$$

since $R = \dfrac{1}{l}$, *we get*

$$R = +\infty.$$

ii) Domain of convergence D

For $|x| < R = +\infty$, *the series* $\sum_{n=0}^{+\infty} a_n x^n = \sum_{n=1}^{+\infty} \left(\frac{n+2}{n^2}\right)^n x^n$ *is convergent. Then*

$$D = \mathbb{R} = \left]-\infty, +\infty\right[.$$

3) $\sum\limits_{n=1}^{+\infty} \frac{3^n}{(n+1)^2} x^n$ *with* $a_n = \frac{3^n}{(n+1)^2}$

i) Radius of convergence R: $R = \frac{1}{l}$

We have $l = \lim\limits_{n\to+\infty} \left| \frac{a_{n+1}}{a_n} \right|$

$$a_{n+1} = \frac{3^{n+1}}{(n+2)^2} = \frac{3 \times 3^n}{(n+2)^2},$$

and

$$\left| \frac{a_{n+1}}{a_n} \right| = \left| \frac{3 \times 3^n}{(n+2)^2} \times \frac{(n+1)^2}{3^n} \right| = \frac{3(n+1)^2}{(n+2)^2},$$

then

$$\begin{aligned} l &= \lim_{n\to+\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to+\infty} \frac{3(n+1)^2}{(n+2)^2} \\ &= 3, \end{aligned}$$

since $R = \frac{1}{l}$*, we get*

$$R = \frac{1}{3}.$$

ii) Domain of convergence D

a) For $|x| < R = \frac{1}{3}$, $\left(-\frac{1}{3} < x < \frac{1}{3} \right)$

The series $\sum\limits_{n=1}^{+\infty} \frac{3^n}{(n+1)^2} x^n$ *is convergent.*

b) For $|x| > R = \frac{1}{3}$, $\left(x < -\frac{1}{3} \text{ or } x > \frac{1}{3} \right)$

The series $\sum\limits_{n=0}^{+\infty} \frac{3^n}{(n+1)^2} x^n$ *is divergent.*

c) For $|x| = R = \frac{1}{3}$, $\left(x = -\frac{1}{3} \text{ or } x = \frac{1}{3} \right)$

Nothing can be concluded about the series $\sum\limits_{n=1}^{+\infty} \frac{3^n}{(n+1)^2} x^n$.

$\cdot$ *For* $x = \frac{1}{3}$*, we have*

$$\sum_{n=1}^{+\infty} \frac{3^n}{(n+1)^2} x^n = \sum_{n=1}^{+\infty} \frac{3^n}{(n+1)^2} \left(\frac{1}{3}\right)^n = \sum_{n=1}^{+\infty} \frac{1}{(n+1)^2}.$$

Studying the nature of $\sum_{n=1}^{+\infty}\frac{1}{(n+1)^2}$ *with* $u_n = \frac{1}{(n+1)^2}$
we have $u_n \geq 0$ *and*

$$u_n = \frac{1}{(n+1)^2} \underset{\infty}{\sim} \frac{1}{n^2} = v_n \quad \textit{(because } \lim_{n\to+\infty}\frac{u_n}{v_n} = 1\textit{).}$$

$\sum_{n=1}^{+\infty} v_n = \sum_{n=1}^{+\infty}\frac{1}{n^2}$ *is convergent (Reimann's series with* $\alpha = 2 > 1$*)*
then, $\sum_{n=1}^{+\infty}\frac{1}{(n+1)^2}$ *is convergent (from the equivalence's criteria).*
So, the series $\sum_{n=1}^{+\infty}\frac{3^n}{(n+1)^2}x^n$ *is convergent for* $x = \frac{1}{3}$.
· *For* $x = -\frac{1}{3}$, *we have*

$$\sum_{n=1}^{+\infty}\frac{3^n}{(n+1)^2}x^n = \sum_{n=1}^{+\infty}\frac{3^n}{(n+1)^2}\left(\frac{-1}{3}\right)^n = \sum_{n=1}^{+\infty}\frac{(-1)^n}{(n+1)^2}.$$

Studying the nature of the series $\sum_{n=1}^{+\infty}\frac{(-1)^n}{(n+1)^2}$ *with* $u_n = \frac{(-1)^n}{(n+1)^2}$
We have $(u_n)_n$ *is sequence with real term, then we use the criteria of absolute convergence to know the nature of the series* $\sum_{n=1}^{+\infty}\frac{(-1)^n}{(n+1)^2}$.

$$\sum_{n=1}^{+\infty}|u_n| = \sum_{n=1}^{+\infty}\frac{1}{(n+1)^2},$$

we proved in the previous part that $\sum_{n=1}^{+\infty}|u_n| = \sum_{n=1}^{+\infty}\frac{1}{(n+1)^2}$ *is convergent, then* $\sum_{n=1}^{+\infty}u_n = \sum_{n=1}^{+\infty}\frac{(-1)^n}{(n+1)^2}$ *is absolutely convergent, so* $\sum_{n=1}^{+\infty}\frac{(-1)^n}{(n+1)^2}$ *is convergent.*
We conclude that the series $\sum_{n=1}^{+\infty}\frac{3^n}{(n+1)^2}x^n$ *is convergent for* $x = -\frac{1}{3}$.

Conclusion

The domain of convergence D *of the series* $\sum_{n=1}^{+\infty}\frac{3^n}{(n+1)^2}x^n$ *is*

$$D = \left[-\frac{1}{3}, \frac{1}{3}\right].$$

3.3 Properties of power series

Let $S(x)=\sum\limits_{n=0}^{+\infty}a_nx^n$ a power series with a radius R and domain $D=\left]-R,R\right[$.

i) $S(x)$ is continuous in D.

ii) For all $x\in D$, we have

$$S'(x)=\sum_{n=1}^{+\infty}na_nx^{n-1},$$

and

$$\int_0^{+\infty}S(x)\,dx=\sum_{n=0}^{+\infty}\frac{a_n}{n+1}x^{n+1}.$$

iii) The series $\sum\limits_{n=1}^{+\infty}na_nx^{n-1}$ and $\sum\limits_{n=0}^{+\infty}\frac{a_n}{n+1}x^{n+1}$ have the same radius R.

3.4 Power series expansion near zero of a function of a real variable

3.4.1 Function expandable in a power series over the open interval of convergence

Proposition 3.4.1 *Function expandable in a power series*

A function f defined in a neighborhood of 0 is said to be expandable in a power series around 0 if there exists $R>0$ such that

$$\forall x\in\left]-R,R\right[,\quad f(x)=\sum_{n=0}^{+\infty}a_nx^n.$$

More general, a function f defined in a neighborhood of x_0 is said to be expandable in a power series around x_0 if there exists $R>0$ such that

$$\forall x\in\left]x_0-R,x_0+R\right[,\quad f(x)=\sum_{n=0}^{+\infty}a_n(x-x_0)^n.$$

3.4.2 Taylor-Maclaurin series of a function of class C^{∞}

Definition 3.4.1 *Taylor-Maclaurin series*

Let f be a function of class C^{∞} in a neighborhood of the point $x = 0$. The Taylor-Maclaurin series of the function f in a neighborhood of $x = 0$ is ginev by

$$f(x) = \sum_{n=0}^{+\infty} \frac{f^{(n)}(0)}{n!} x^n.$$

Example 3.4.1

1) $f(x) = e^x$, we use the relation

$$f(x) = \sum_{n=0}^{+\infty} \frac{f^{(n)}(0)}{n!} x^n.$$

we need to calculate $f^{(n)}(0)$

$$f(x) = e^x,\ f'(x) = e^x, ..., f^{(n)}(x) = e^x,$$

then

$$f(0) = 1,\ f'(0) = 1, ..., f^{(n)}(0) = 1,$$

we conclude

$$\begin{aligned} f(x) &= e^x = \sum_{n=0}^{+\infty} \frac{f^{(n)}(0)}{n!} x^n. \\ &\Rightarrow e^x = \sum_{n=0}^{+\infty} \frac{1}{n!} x^n. \end{aligned}$$

2) $f(x) = \dfrac{1}{1-x}$, we use the relation

$$f(x) = \sum_{n=0}^{+\infty} \frac{f^{(n)}(0)}{n!} x^n,$$

we need to calculate $f^{(n)}(0)$

$$f(x) = \frac{1}{1-x},\ f'(x) = \frac{1}{(1-x)^2},\ f''(x) = \frac{2}{(1-x)^3},$$

$$f^{(3)}(x) = \frac{2 \times 3}{(1-x)^4}, ..., f^{(n)}(x) = \frac{n!}{(1-x)^n},$$

then

$$f^{(n)}(0) = \frac{n!}{(1-0)^n} = n!,$$

we conclude that

$$\begin{aligned} f(x) &= \frac{1}{1-x} = \sum_{n=0}^{+\infty} \frac{f^{(n)}(0)}{n!} x^n. \\ &\Rightarrow \frac{1}{1-x} = \sum_{n=0}^{+\infty} x^n. \end{aligned}$$

3) $f(x) = \dfrac{1}{1+x}$, *by following the same steps as the previous example we get*

$$f^{(n)}(x) = \frac{(-1)^n n!}{(1+x)^n},$$

then

$$\frac{1}{1+x} = \sum_{n=0}^{+\infty} (-1)^n x^n.$$

4) $f(x) = \sin x$, *we use the relation*

$$f(x) = \sum_{n=0}^{+\infty} \frac{f^{(n)}(0)}{n!} x^n,$$

we need to calculate $f^{(n)}(0)$

$$f(x) = \sin x, \;\; f'(x) = \cos x, \;\; f''(x) = -\sin x,$$

$$f^{(3)}(x) = -\cos x, , f^{(4)}(x) = \sin x, ...,$$

then

$$f^{(n)}(x) = \begin{cases} (-1)^p \sin x & si\ n = 2p \\ (-1)^p \cos x & si\ n = 2p+1 \end{cases}$$

for $x = 0$, *we get*

$$f^{(n)}(0) = \begin{cases} 0 & si\ n = 2p \\ (-1)^p & si\ n = 2p+1 \end{cases},$$

we conclude that

$$\begin{aligned} f(x) &= \sin x = \sum_{n=0}^{+\infty} \frac{f^{(n)}(0)}{n!} x^n. \\ &\Rightarrow \sin x = \sum_{n=0}^{+\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}. \end{aligned}$$

5) $f(x) = cosx$, by following the same steps as the previous example we get

$$\cos x = \sum_{n=0}^{+\infty} \frac{(-1)^n}{(2n)!} x^{2n}.$$

3.5 Applications

3.5.1 Solving Differential Equations

We seek a solution in the form of a power series with undetermined coefficients of a differential equation. By identification, we obtain these coefficients . It is then sufficient to study the convergence of this series to determine the solution of the equation in the convergence interval.

Example 3.5.1

1) Finding the power series solution to the equation

$$y' - y = 0, \quad y(0) = 1.$$

We pose $y(x) = \sum_{n=0}^{+\infty} a_n x^n$ then $y'(x) = \sum_{n=1}^{+\infty} n a_n x^{n-1}$
By a change of index in $y'(x)$ we obtain

$$y'(x) = \sum_{n=1}^{+\infty} n a_n x^{n-1} = \sum_{k=0}^{+\infty} (k+1) a_{k+1} x^k = \sum_{n=0}^{+\infty} (n+1) a_{n+1} x^n,$$

we get

$$\begin{aligned}
(1-x)\,y'-y &= 0 \Leftrightarrow (1-x)\sum_{n=1}^{+\infty} n a_n x^{n-1}-\sum_{n=0}^{+\infty} a_n x^n=0\\
&\Leftrightarrow \sum_{n=1}^{+\infty} n a_n x^{n-1}-\sum_{n=1}^{+\infty} n a_n x^n-\sum_{n=0}^{+\infty} a_n x^n=0\\
&\Leftrightarrow \sum_{k=0}^{+\infty}(n+1)\,a_{n+1}x^n-\sum_{n=1}^{+\infty} n a_n x^n-\sum_{n=0}^{+\infty} a_n x^n=0\\
&\Leftrightarrow \sum_{k=0}^{+\infty}\left((n+1)\,a_{n+1}-n a_n-a_n\right)x^n=0\\
&\Leftrightarrow \sum_{k=0}^{+\infty}\left((n+1)\,a_{n+1}-(n+1)\,a_n\right)x^n=0\\
&\Leftrightarrow (n+1)\,a_{n+1}-(n+1)\,a_n=0\\
&\Leftrightarrow a_{n+1}=a_n,\qquad \forall n\in\mathbb{N}.
\end{aligned}$$

From $y(0)=1$, *we get* $a_0=1$, *then*

$$a_n=1,\qquad \forall n\in\mathbb{N}.$$

This implies that

$$y(x)=\sum_{n=0}^{+\infty} x^n.$$

From the previous example, we have

$$\frac{1}{1-x}=\sum_{n=0}^{+\infty} x^n,\quad |x|<1,$$

then

$$y(x)=\frac{1}{1-x},\quad |x|<1.$$

2) Finding the power series solution of the equation

$$(E)\begin{cases} y''+2xy'+2y=0\\ y(0)=1 \text{ et } y'(0)=1\end{cases}.$$

We pose $y(x)=\sum\limits_{n=0}^{+\infty}a_nx^n$ *then*

$$y'(x)=\sum_{n=1}^{+\infty}na_nx^{n-1} \text{ and } y''(x)=\sum_{n=2}^{+\infty}n(n-1)a_nx^{n-2}.$$

By a change of index in $y'(x)$ *and* $y''(x)$ *we obtain*

$$y'(x)=\sum_{n=1}^{+\infty}na_nx^{n-1}=\sum_{k=0}^{+\infty}(k+1)a_{k+1}x^k=\sum_{n=0}^{+\infty}(n+1)a_{n+1}x^n,$$

and

$$y''(x)=\sum_{n=2}^{+\infty}n(n-1)a_nx^{n-2}=\sum_{k=0}^{+\infty}(k+2)(k+1)a_{k+2}x^k=\sum_{n=0}^{+\infty}(n+2)(n+1)a_{n+2}x^n,$$

we get

$$\begin{aligned}
y''+2xy'+2y &= 0 \Leftrightarrow \sum_{n=2}^{+\infty}n(n-1)a_nx^{n-2}+2x\sum_{n=1}^{+\infty}na_nx^{n-1}+2\sum_{n=0}^{+\infty}a_nx^n=0\\
&\Leftrightarrow \sum_{n=2}^{+\infty}n(n-1)a_nx^{n-2}+2\sum_{n=1}^{+\infty}na_nx^n+2\sum_{n=0}^{+\infty}a_nx^n=0\\
&\Leftrightarrow \sum_{n=0}^{+\infty}(n+2)(n+1)a_{n+2}x^n+2\sum_{n=1}^{+\infty}na_nx^n+2\sum_{n=0}^{+\infty}a_nx^n=0\\
&\Leftrightarrow \sum_{n=0}^{+\infty}((n+2)(n+1)a_{n+2}+2na_n+2a_n)x^n=0\\
&\Leftrightarrow (n+2)(n+1)a_{n+2}+2(n+1)a_n=0\\
&\Leftrightarrow a_{n+2}=\frac{2}{(n+2)}a_n, \quad \forall n\in\mathbb{N}.....(*).
\end{aligned}$$

From $y(0)=0$, *we get* $a_0=1$ *and from* $y'(0)=0$, *we get* $a_1=0$.

From the relation $(*)$, *we find*

$$a_2=-1,\ a_3=0,\ a_4=\frac{1}{2},\ a_5=0,\ a_6=-\frac{1}{2\times 3}.$$

We note that the coefficients of odd indices are 0. So:

$$a_n=\begin{cases}\dfrac{(-1)^p}{p!}, & si \quad n=2p\\ 0, & si \quad n=2p+1\end{cases}.$$

Then

$$\begin{aligned} y(x) &= \sum_{n=0}^{+\infty} a_n x^n = \sum_{n=0}^{+\infty} a_{2n} x^{2n} + \sum_{n=0}^{+\infty} a_{2n+1} x^{2n+1} \\ &= \sum_{n=0}^{+\infty} a_{2n} x^{2n} = \sum_{n=0}^{+\infty} \frac{(-1)^n}{n!} x^{2n}. \end{aligned}$$

From the previous example, we have

$$e^x = \sum_{n=0}^{+\infty} \frac{1}{n!} x^n$$

then

$$\begin{aligned} y(x) &= \sum_{n=0}^{+\infty} \frac{(-1)^n}{n!} x^{2n} = \sum_{n=0}^{+\infty} \frac{(-x^2)^n}{n!} \\ &= e^{-x^2}. \end{aligned}$$

Chapter 4

Fourier series

In this chapter, we study Fourier series, which are a fundamental tool in the analysis of periodic functions. Their applications are quite numerous in other areas of mathematics, notably in differential equations and partial differential equations.

4.1 Trigonometric series

Definition 4.1.1 *A real trigonometric series is a series of functions of the form:*

$$\frac{a_0}{2} + \sum_{n=1}^{+\infty} [a_n \cos(nx) + b_n \sin(nx)], \tag{4.1.1}$$

with $x \in \mathbb{R}$ *et* a_0 a_n *and* $b_n \in \mathbb{R}$, $\forall n \geq 1$.

Definition 4.1.2 (*Periodic fimction*)*:*
A function f *from* $\mathbb{R}$ *to* $\mathbb{C}$ *is called periodic, if there exists a number* T *such that, for all* $x \in \mathbb{R}$

$$f(x) = f(x+T) \quad \forall x \in \mathbb{R}.$$

The smallest such positive number T *is called the fundamental period of* f*, and we say that* f *is* T*-periodic*

Example 4.1.1

$f_1(x) = \cos x$ and $f_2(x) = \sin x$ are 2π-periodic functions:

$$\cos(x + 2\pi) = \cos x \text{ and } \sin(x + 2\pi) = \sin x.$$

Proposition 4.1.1

If the infinite series $\sum a_n$ et $\sum b_n$ are absolutely convergent then the series defined by (4.1.1) is normally convergent.

Remark 4.1.3

Suppose that the series defined by (4.1.1) is convergent. Then the function defined by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{+\infty} [a_n \cos(nx) + b_n \sin(nx)],$$

is 2π-periodic function.

4.1.1 Calculation of the coefficients of the trigonometric series

We suppose that the series defined by (4.1.1) is uniformly convergent. Then

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{+\infty} [a_n \cos(nx) + b_n \sin(nx)], \tag{4.1.2}$$

is 2π-periodic function. In this case, the coefficients a_0, a_n and b_n are given by the following relations

$$a_0 = \frac{1}{\pi} \int_0^{2\pi} f(x)dx,$$

$$a_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \cos(nx)\, dx \text{ and } b_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \sin(nx)\, dx \quad \forall n \geq 1.$$

Remark 4.1.4

If f is T-periodic function and continuous sur $[0, T]$, then:

$$\int_0^T f(x)\, dx = \int_\alpha^{\alpha+T} f(x)\, dx \quad \forall \alpha \in \mathbb{R}.$$

Using this remark, and since the function f defined by (4.1.2) is 2π-periodic, we have

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{+\pi} f(x)\,dx.$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{+\pi} f(x)\cos(nx)\,dx, \text{ and } b_n = \frac{1}{\pi}\int_{-\pi}^{+\pi} f(x)\sin(nx)\,dx \ \ \forall n \geq 1.$$

4.2 Fourier series

Definition 4.2.1 *Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be a 2π-periodic function. The Fourier series associated with the function f is the trigonometric series:*

$$\frac{a_0}{2} + \sum_{n=1}^{+\infty}[a_n \cos(nx) + b_n \sin(nx)],$$

where

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{+\pi} f(x)\,dx,$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{+\pi} f(x)\cos(nx)\,dx, \text{ and } b_n = \frac{1}{\pi}\int_{-\pi}^{+\pi} f(x)\sin(nx)\,dx \ \ \forall n \geq 1.$$

Theorem 4.2.2

Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be a 2π-periodic function satisfying the following conditions:

1) *There exists $M > 0$ such as $|f(x)| \leq M$, $\forall x \in \mathbb{R}$.*

2) *f is slice-monotone on the interval $[a, b]$ (i.e we can divide $[a, b]$ into subintervals such that the function f is monotonic on each subinterval).*

Then, the Fourier series associated with the function f is convergent, and we have:

$$S(x) = \frac{a_0}{2} + \sum_{n=1}^{+\infty}[a_n \cos(nx) + b_n \sin(nx)],$$

with

$$S(y) = \begin{cases} f(y) & \text{if } f \text{ is continuous at } y \\ \frac{f^+(y)+f^-(y)}{2} & \text{if } f \text{ is not continuous at } y \end{cases}$$

where

$$f^+(y) = \lim_{\substack{x \to y \\ >}} f(x) \quad \text{and} \quad f^-(y) = \lim_{\substack{x \to y \\ <}} f(x).$$

To give more simpler format for the coefficients a_0, a_n and b_n for a particular type of function, we need the following proposition.

Proposition 4.2.1

Let $g : [-k, +k] \to \mathbb{R}$ be a continuous function. then:

i) If f is an even function, then $\int_{-k}^{+k} g(x)\, dx = 2 \int_{0}^{+k} g(x)\, dx.$

ii) If f is an odd function, then $\int_{-k}^{+k} g(x)\, dx = 0.$

Using this proposition, the coefficients a_0, a_n and b_n are given as follows

First case: If f is an even function, then

$$f(\cdot)\cos(\cdot) \text{ is an even function,}$$

and

$$f(\cdot)\sin(\cdot) \text{ is an odd function.}$$

So,

$$a_0 = \frac{2}{\pi} \int_{0}^{\pi} f(x)\, dx$$

and

$$a_n = \frac{2}{\pi} \int_{0}^{\pi} f(x)\cos(nx)\, dx \quad \text{and} \quad b_n = 0 \ \ \forall n \geq 1.$$

Second case: If f is an odd function, then

$$f(\cdot)\cos(\cdot) \text{ is an odd function,}$$

and

$$f(\cdot)\sin(\cdot) \text{ is an even function.}$$

So, $a_0 = a_n = 0 \ \ \forall n \geq 1$, and

$$b_n = \frac{2}{\pi} \int_0^{\pi} f(x)\sin(nx)\, dx \ \ \forall n \geq 1.$$

4.3 Parseval's equality

Theorem 4.3.1

Let f be a 2π-periodic function and developable in Fourier series, then we have

$$\frac{1}{\pi} \int_{-\pi}^{+\pi} |f(x)|^2 \, dx = \frac{|a_0|^2}{2} + \sum_{n=1}^{+\infty} \left(|a_n|^2 + |b_n|^2\right).$$

1) If f is an even function, f^2 is also an even function and $b_n = 0$, then:

$$\frac{2}{\pi} \int_0^{\pi} |f(x)|^2 \, dx = \frac{|a_0|^2}{2} + \sum_{n=1}^{+\infty} |a_n|^2.$$

2) If f is an odd function, f^2 is an even function and $a_n = 0$, then:

$$\frac{2}{\pi} \int_0^{\pi} |f(x)|^2 \, dx = \sum_{n=1}^{+\infty} |b_n|^2.$$

Example 4.3.1

Let f be a 2π-periodic function defined by

$$f(x) = \pi - |x| \quad et \quad -\pi \leq x \leq +\pi.$$

1) *Plot the graph of f over the interval $[-3\pi, 3\pi]$.*

2) *Calculate the Fourier coefficients of f.*

3) *Obtain a Fourier series expansion of the function f.*

4) *Deduce the sums of the following infinite convergent series:*

$$A = \sum_{n=0}^{+\infty} \frac{1}{(2n+1)^2} \ \ \text{and} \ B = \sum_{n=0}^{+\infty} \frac{1}{(2n+1)^4}.$$

Solution:

1) The graph of f

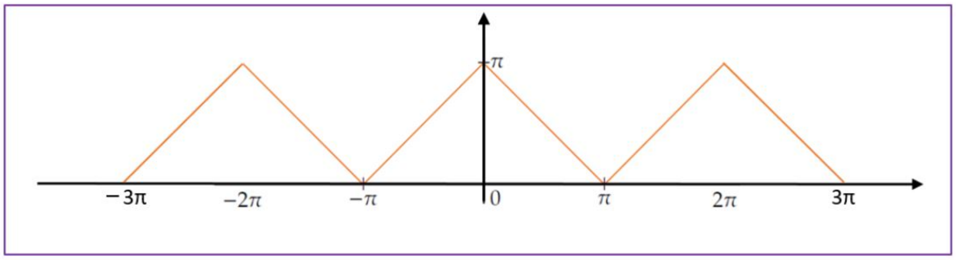


FIGURE 2 – Le graphe de f sur $[-3\pi, +3\pi]$.

2) The Fourier coefficients of f

f is expandable as a Fourier series (because f satisfies the conditions of Theorem 4.2.2). Moreover, f is an even function, so

$$b_n = 0, \, \forall n \geq 1,$$

and

$$a_0 = \frac{2}{\pi} \int_0^{\pi} f(x)\, dx \;\; et \;\; a_n = \frac{2}{\pi} \int_0^{\pi} f(x) \cos(nx)\, dx, \; \forall n \geq 1.$$

For a_0,

$$\begin{aligned} a_0 &= \frac{2}{\pi} \int_0^{\pi} f(x)\, dx \\ &= \frac{2}{\pi} \int_0^{\pi} (\pi - |x|)\, dx \\ &= \frac{2}{\pi} \int_0^{\pi} (\pi - x)\, dx = \pi. \end{aligned}$$

For a_n,

$$\begin{aligned}
a_n &= \frac{2}{\pi}\int_0^{\pi} f(x)\cos(nx)\,dx,\ \forall n \geq 1\\
&= \frac{2}{\pi}\int_0^{\pi} (\pi - x)\cos(nx)\,dx\\
&= \frac{2}{\pi}\left[\left[\frac{(\pi - x)}{n}\sin(nx)\right]_0^{\pi} + \frac{1}{n}\int_0^{\pi}\sin(nx)\,dx\right]\\
&= \frac{2}{\pi}\left[0 + \frac{1}{n}\left[-\frac{\cos(nx)}{n}\right]_0^{\pi}\right]\\
&= \frac{2}{\pi n^2}\left[-\cos(nx)\right]_0^{\pi} = \frac{2\left[1-(-1)^n\right]}{\pi n^2}.
\end{aligned}$$

We remark that

$$a_n = \begin{cases} 0 & si \;\; n = 2p \\ \frac{4}{\pi(2p+1)^2} & si \;\; n = 2p+1 \end{cases}$$

3) The Fourier series expansion of f

We have

$$S(x) = \frac{a_0}{2} + \sum_{n=1}^{+\infty} a_n \cos(nx) + \sum_{n=1}^{+\infty} b_n \sin(nx),$$

and since f *is continuous for all* $x \in \mathbb{R}$, *we have*

$$\begin{aligned}
S(x) &= f(x) = \pi - |x| = \frac{a_0}{2} + \sum_{n=1}^{+\infty} a_n \cos(nx) + \sum_{n=1}^{+\infty} b_n \sin(nx)\\
&= \frac{\pi}{2} + \sum_{n=1}^{+\infty} a_n \cos(nx)\\
&= \frac{\pi}{2} + \sum_{n=1}^{+\infty} a_{2n} \cos(2nx) + \sum_{n=0}^{+\infty} a_{2n+1} \cos((2n+1)x)\\
&= \frac{\pi}{2} + \sum_{n=0}^{+\infty} a_{2n+1} \cos((2n+1)x) \qquad (\textit{because } a_{2n} = 0)\\
&= \frac{\pi}{2} + \sum_{n=1}^{+\infty} \frac{4}{\pi(2n+1)^2} \cos((2n+1)x)\\
&= \frac{\pi}{2} + \frac{4}{\pi}\sum_{n=1}^{+\infty} \frac{\cos((2n+1)x)}{(2n+1)^2} \qquad (*).
\end{aligned}$$

4) ***Calculation of sums***

$A = \sum\limits_{n=0}^{+\infty} \frac{1}{(2n+1)^2}$.

We replace x by 0 in the equation $(*)$, we get

$$\begin{aligned} 0 &= \frac{\pi}{2} - \frac{4}{\pi}\sum_{n=0}^{+\infty}\frac{1}{(2n+1)^2} \\ &\Longrightarrow \sum_{n=0}^{+\infty}\frac{1}{(2n+1)^2} = \frac{\pi^2}{8}. \end{aligned}$$

$B = \sum\limits_{n=1}^{+\infty} \frac{1}{n^2}$.

From the Parseval's equality, we have:

$$\frac{1}{\pi}\int_{-\pi}^{\pi} |f(x)|^2\,dx = \frac{|a_0|^2}{2} + \sum_{n=1}^{+\infty}\left(|a_n|^2 + |b_n|^2\right),$$

then

$$\begin{aligned} \frac{1}{\pi}\int_{-\pi}^{\pi} (\pi - |x|)^2\,dx &= \frac{\pi^2}{2} + \sum_{n=0}^{+\infty}\frac{16}{\pi^2(2n+1)^4} \\ \frac{2}{\pi}\int_{0}^{\pi} (\pi - x)^2\,dx &= \frac{\pi^2}{2} + \frac{16}{\pi^2}\sum_{n=0}^{+\infty}\frac{1}{(2n+1)^4} \\ \frac{2\pi^2}{3} &= \frac{\pi^2}{2} + \frac{16}{\pi^2}\sum_{n=0}^{+\infty}\frac{1}{(2n+1)^4} \\ &\Longrightarrow \sum_{n=0}^{+\infty}\frac{1}{(2n+1)^4} = \frac{\pi^4}{96}. \end{aligned}$$

Example 4.3.2

Let f be a 2π-periodic function defined by:

$$f(x) = \begin{cases} 2 & si \quad -\pi < x \leq 0 \\ -2 & si \quad 0 < x < \pi \end{cases}$$

1) *Plot the graph of f over the interval $[-5\pi, 5\pi]$.*

2) *Calculate the Fourier coefficients of f.*

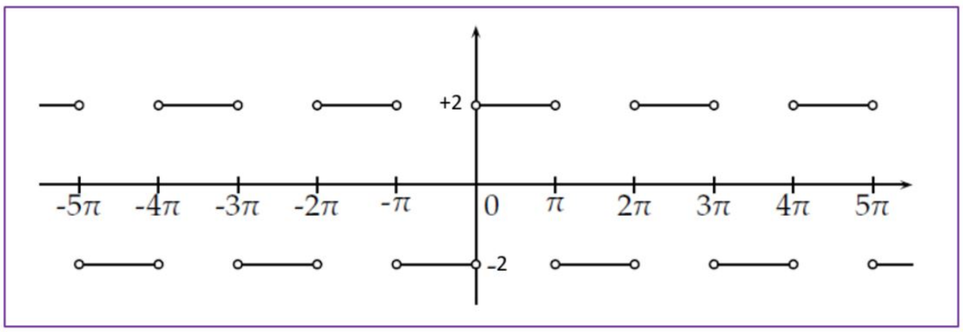
3) Obtain a Fourier series expansion of the function f.

4) Deduce the sums of the following infinite convergent series:

$$A = \sum_{n=0}^{+\infty} \frac{(-1)^n}{2n+1}, \ \ and \ B = \sum_{n=0}^{+\infty} \frac{1}{(2n+1)^2}.$$

Solution:

1) The graph of f



2) The Fourier coefficients of f

f is expandable in a Fourier series (because f satisfies the conditions of Theorem 4.2.2). Moreover, f is an odd function, so

$$a_0 = 0, \ and \ a_n = 0 \ \forall n \geq 1$$

and

$$b_n = \frac{2}{\pi} \int_0^{\pi} f(x) \sin(nx) \, dx,$$

then

$$\begin{aligned} b_n &= \frac{2}{\pi}\int_0^{\pi} f(x)\sin(nx)\,dx \\ &= \frac{2}{\pi}\int_0^{\pi} 2\sin(nx)\,dx \\ &= \frac{4}{n\pi}\left[-\cos(nx)\right]_0^{\pi} \\ &= \frac{4\left[-(-1)^n+1\right]}{n\pi}, \end{aligned}$$

We remark that

$$b_n = \begin{cases} 0 & si \;\; n = 2p \\ \frac{8}{\pi(2p+1)} & si \;\; n = 2p+1 \end{cases}$$

3) The Fourier series expansion of f

We have

$$S(x) = \frac{a_0}{2} + \sum_{n=1}^{+\infty} a_n\cos(nx) + \sum_{n=1}^{+\infty} b_n\sin(nx),$$

then

$$\begin{aligned} S(x) &= \frac{a_0}{2} + \sum_{n=1}^{+\infty} a_n\cos(nx) + \sum_{n=0}^{+\infty} b_n\sin(nx) \\ &= \sum_{n=0}^{+\infty} b_n\sin(nx) \\ &= \sum_{n=0}^{+\infty} \frac{8}{\pi(2n+1)}\sin(nx) \\ &= \frac{8}{\pi}\sum_{n=0}^{+\infty}\frac{\sin(nx)}{(2n+1)}. \qquad (1), \end{aligned}$$

since f *is not continuous for all* $x \in \mathbb{R}$*, then we have for* $x_0 \in \mathbb{R}$

$$S(x_0) = \begin{cases} f(x_0) & if\; x_0 \neq k\pi, \quad k \in \mathbb{Z} \\ 0 & if\; x_0 = k\pi, \quad k \in \mathbb{Z} \end{cases}$$

4) ***Calculation of sums***

$A = \sum_{n=0}^{+\infty} \frac{(-1)^n}{2n+1}$,

We replace x by $\frac{\pi}{2}$ in the equation (1),we get

$$\begin{aligned} S\left(\frac{\pi}{2}\right) &= f\left(\frac{\pi}{2}\right) = 2 = \frac{8}{\pi}\sum_{n=0}^{+\infty} \frac{\sin(nx)}{(2n+1)} \\ &\Longrightarrow \sum_{n=0}^{+\infty} \frac{(-1)^n}{2n+1} = \frac{\pi}{4}. \end{aligned}$$

$B = \sum_{n=0}^{+\infty} \frac{1}{(2n+1)^2}$,

From the Parseval's equality, we have:

$$\frac{1}{\pi}\int_{-\pi}^{\pi} |f(x)|^2 \, dx = \frac{|a_0|^2}{2} + \sum_{n=1}^{+\infty} \left(|a_n|^2 + |b_n|^2\right),$$

then

$$\begin{aligned} \frac{1}{\pi}\int_{-\pi}^{\pi} 2^2 dx &= \sum_{n=1}^{+\infty} |b_n|^2 \\ \frac{2}{\pi}\int_{0}^{\pi} 4dx &= \frac{64}{\pi^2}\sum_{n=1}^{+\infty} \frac{1}{(2n+1)^2} \\ 8 &= \frac{64}{\pi^2}\sum_{n=1}^{+\infty} \frac{1}{(2n+1)^2} \\ &\Longrightarrow \sum_{n=1}^{+\infty} \frac{1}{(2n+1)^2} = \frac{\pi^2}{8}. \end{aligned}$$

Chapter 5

Fourier Transform

The Fourier Transform is an extension of the Fourier Series expansion for periodic functions to non-periodic functions.

We denote by $L^1(\mathbb{R})$ the set of functions $f : \mathbb{R} \to \mathbb{R}$ that are integrable and for which $\int\limits_{-\infty}^{+\infty} |f(t)|\, dt$ converges.

5.1 Fourier Transform

Definition 5.1.1

For $f \in L^1(\mathbb{R})$. The Fourier transform of f denoted by $\mathcal{F}(f)$ is defined as follows: $F(f) : \mathbb{R} \longrightarrow \mathbb{C}$,

$$\mathcal{F}(f)(s) = \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{+\infty} f(t)\, e^{-ist} dt.$$

Since, we have

$$e^{-ist} = \cos(st) - i \sin(st),$$

then

$$\mathcal{F}(f)(s) = \frac{1}{\sqrt{2\pi}} \left[\int\limits_{-\infty}^{+\infty} f(t) \cos(st)\, dt - i \int\limits_{-\infty}^{+\infty} f(t) \sin(st)\, dt \right].$$

5.2 Properties of the Fourier transform

We have

$$\begin{aligned}\mathcal{F}(f)(s) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} f(t)\,e^{-ist}dt \\ &= \frac{1}{\sqrt{2\pi}}\left[\int_{-\infty}^{+\infty} f(t)\cos(st)\,dt - i\int_{-\infty}^{+\infty} f(t)\sin(st)\,dt\right].\end{aligned}$$

If f is an even function, then

$$f(\cdot)\cos(\cdot) \text{ is an even and } f(\cdot)\sin(\cdot) \text{ is an odd,}$$

and if f is an odd function, then

$$f(\cdot)\cos(\cdot) \text{ is an odd and } f(\cdot)\sin(\cdot) \text{ is an even.}$$

Using the privious remarks, we get the following proposition.

Proposition 5.2.1

For a function $f \in L^1(\mathbb{R})$, we have

i) If f is an even function, then

$$\mathcal{F}(f)(s) = \frac{2}{\sqrt{2\pi}}\int_{0}^{+\infty} f(t)\cos(st)\,dt.$$

ii) If f is an odd function,then

$$\mathcal{F}(f)(s) = -\frac{2i}{\sqrt{2\pi}}\int_{0}^{+\infty} f(t)\sin(st)\,dt.$$

Example 5.2.1

1) Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be a function defined by

$$f(t) = \begin{cases} 2 & \text{if } |t| \leq 3 \\ 0 & \text{if } |t| > 3. \end{cases}$$

We calculate the Fourier transform of f. It is clear that f is an even function, then

$$\begin{aligned}
\mathcal{F}(f)(s) &= \frac{2}{\sqrt{2\pi}} \int_0^{+\infty} f(t)\cos(st)\,dt \\
&= \frac{2}{\sqrt{2\pi}} \int_0^{+\infty} 2\cos(st)\,dt \\
&= \frac{4}{\sqrt{2\pi}} \int_0^{3} \cos(st)\,dt \\
&= \frac{4}{s\sqrt{2\pi}} \left[\sin(st)\right]_0^3, \quad s \neq 0 \\
&= \frac{4}{\sqrt{2\pi}} \frac{\sin(3s)}{s},
\end{aligned}$$

and if $s = 0$, we get

$$\mathcal{F}(f)(0) = \frac{2}{\sqrt{2\pi}} \int_0^3 2dt = \frac{12}{\sqrt{2\pi}}.$$

Then

$$\mathcal{F}(f)(s) = \begin{cases} \dfrac{4}{\sqrt{2\pi}} \dfrac{\sin(3s)}{s} & \text{if } s \neq 0 \\ \dfrac{12}{\sqrt{2\pi}} & \text{if } s = 0. \end{cases}$$

2) Let f a be function defined by.

$$f(t) = \begin{cases} 0 & \text{if } t < 0 \\ e^{-\alpha t} & \text{if } t \geq 0 \end{cases},$$

with $\alpha > 0$.

We calculate the Fourier transform of f,

$$\begin{aligned}
\mathcal{F}(f)(s) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(t)\, e^{-ist} dt \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{0} f(t)\, e^{-ist} dt + \frac{1}{\sqrt{2\pi}} \int_{0}^{+\infty} f(t)\, e^{-ist} dt \\
&= \frac{1}{\sqrt{2\pi}} \int_{0}^{+\infty} e^{-\alpha t} e^{-ist} dt, \\
&= \frac{1}{\sqrt{2\pi}} \int_{0}^{+\infty} e^{-(\alpha+is)t} dt \\
&= \frac{1}{\sqrt{2\pi}} \left[\frac{-1}{\alpha+is} e^{-(\alpha+is)t} \right]_0^{+\infty} \\
&= \frac{1}{\sqrt{2\pi}} \frac{1}{\alpha+is}, \qquad \left(because\ e^{-\alpha t} \longrightarrow 0\ as\ t \longrightarrow 0 \right),
\end{aligned}$$

then

$$\mathcal{F}(f)(s) = \frac{1}{\sqrt{2\pi}} \frac{1}{\alpha+is}, \qquad \alpha > 0.$$

Theorem 5.2.1

Let $f \in L^1(\mathbb{R})$. then

i) $\mathcal{F}(f)$ is continuous at $s_0 \in \mathbb{R} \Leftrightarrow \lim\limits_{s \longrightarrow s_0} \mathcal{F}(f)(s) = \mathcal{F}(f)(s_0)$.

ii) $\mathcal{F}(f)$ is bounded on $\mathbb{R} \Leftrightarrow \exists M > 0 : \quad |\mathcal{F}(f)(s)| \leq M, \ \forall s \in \mathbb{R}$.

Proposition 5.2.2 *Linearity of the Fourier transform*

Let f and g be two functions. Suppose that $\mathcal{F}(f)$ and $\mathcal{F}(g)$ exist, then

$$\mathcal{F}(\alpha f + \beta g)(s) = \alpha \mathcal{F}(f)(s) + \beta \mathcal{F}(g)(s), \quad for\ \alpha, \beta \in \mathbb{R}.$$

5.3 Inverse Fourier Transform

Definition 5.3.1

For $f \in L^1(\mathbb{R})$ a continuous function, the inverse Fourier transform of f denoted by $\mathcal{F}^{-1}$ is given by:

$$\mathcal{F}^{-1}(f)(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(s)\, e^{ist} ds.$$

Theorem 5.3.2

Let $f \in L^1(\mathbb{R})$ be a continuous function. Suppose that $\mathcal{F}(f) \in L^1(\mathbb{R})$, then

$$\mathcal{F}^{-1}(\mathcal{F}(f)) = \mathcal{F}\left(\mathcal{F}^{-1}(f)\right) = f,$$

and

$$f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \mathcal{F}(f)(s)\, e^{ist} ds.$$

Example 5.3.1

Let f be a function defined by

$$f(t) = \begin{cases} 0 & \text{if } t < 0 \\ e^{-\alpha t} & \text{if } t \geq 0 \end{cases},$$

with $\alpha > 0$.

Using Fourier transform and its inverse, we will calculate the following integral

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \left(\frac{1}{\sqrt{2\pi}} \frac{1}{\alpha + is} \right) e^{ist} ds,$$

from the previous example, we have

$$\mathcal{F}(f)(s) = \frac{1}{\sqrt{2\pi}} \frac{1}{\alpha + is}, \qquad \alpha > 0.$$

then

$$\begin{aligned} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \left(\frac{1}{\sqrt{2\pi}} \frac{1}{\alpha + is} \right) e^{ist} ds &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \mathcal{F}(f)\, e^{ist} ds \\ &= \mathcal{F}^{-1}(\mathcal{F}(f)(s)), \end{aligned}$$

since we have $\mathcal{F}^{-1}(\mathcal{F}(f)) = \mathcal{F}(\mathcal{F}^{-1}(f)) = f$*, then we get*

$$\begin{aligned} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \left(\frac{1}{\sqrt{2\pi}} \frac{1}{\alpha + is} \right) e^{ist} ds &= f(t) \\ &= \begin{cases} 0 & \textit{if } t < 0 \\ e^{-\alpha t} & \textit{if } t \geq 0 \end{cases}, \end{aligned}$$

with $\alpha > 0$.

Chapter 6

Laplace Transform

6.1 Laplace Transform

Definition 6.1.1

For a function f. The Laplace transform is given by

$$\mathcal{L}\left(f\left(t\right)\right) = F\left(s\right) = \int\limits_{0}^{+\infty} f\left(t\right) e^{-st} dt.$$

Example 6.1.1

1)For $f\left(t\right) = 2$,

$$\begin{aligned} \mathcal{L}\left(f\left(t\right)\right) &= \mathcal{L}\left(2\right) \\ &= \int\limits_{0}^{+\infty} 2e^{-st} dt \\ &= 2\left[\frac{-e^{-st}}{s}\right]_{0}^{+\infty} = \frac{2}{s}. \end{aligned}$$

2) For $f(t) = e^{\alpha t}$, $t \geq 0$ *and* $\alpha \in \mathbb{R}$

$$\begin{aligned}
\mathcal{L}(f(t)) &= \mathcal{L}\left(e^{\alpha t}\right) \\
&= \int_0^{+\infty} e^{\alpha t} e^{-st} dt \\
&= \int_0^{+\infty} e^{(\alpha - s)t} dt \\
&= \left[\frac{e^{(\alpha-s)t}}{\alpha - s}\right]_0^{+\infty} = \frac{1}{s-\alpha}, \quad (\textit{for } s > \alpha).
\end{aligned}$$

3) For $f(t) = t$, $t \geq 0$.

$$\begin{aligned}
\mathcal{L}(f(t)) &= \mathcal{L}(t) \\
&= \int_0^{+\infty} t e^{-st} dt \\
&= \left[\frac{-te^{-st}}{s}\right]_0^{+\infty} + \frac{1}{s}\int_0^{+\infty} e^{-st} dt \\
&= 0 + \frac{1}{s}\left[\frac{-e^{-st}}{s}\right]_0^{+\infty} = \frac{1}{s^2}. \quad (\textit{for } s > 0).
\end{aligned}$$

4) In general, for $f(t) = t^n$, *we have*

$$\mathcal{L}(f(t)) = \int_0^{+\infty} t^n e^{-st} dt = \frac{n!}{s^{n+1}}. \quad (\textit{for } s > 0).$$

6.2 Properties of Laplace transform

Proposition 6.2.1

Let f *and* g *be two functions. Suppose that* $\mathcal{L}(f(t))$ *and* $\mathcal{L}(g(t))$ *exist, then*

$$\mathcal{L}(\alpha f(t) + \beta g(t)) = \alpha\mathcal{L}(f(t)) + \beta\mathcal{L}(g(t)), \quad \textit{for } \alpha, \beta \in \mathbb{R}.$$

Example 6.2.1

Using this previous proposition, we can calculate the Laplace transform of $\sin(\alpha t)$ *and* $\sinh(\alpha t)$

1) For $f(t) = \sin(\alpha t)$, $t \geq 0$ *and* $\alpha \in \mathbb{R}$

$$\begin{aligned}
\mathcal{L}(f(t)) &= \mathcal{L}(\sin(\alpha t)) \\
&= \mathcal{L}\left(\frac{1}{2i}\left(e^{\alpha t} - e^{-\alpha t}\right)\right) \\
&= \frac{1}{2i}\mathcal{L}\left(e^{\alpha t}\right) - \frac{1}{2i}\mathcal{L}\left(e^{-\alpha t}\right) \\
&= \frac{1}{2i}\frac{1}{s - i\alpha} - \frac{1}{2i}\frac{1}{s + i\alpha}, \qquad (\textit{for } s > 0) \\
&= \frac{1}{2i}\frac{s + i\alpha - (s - i\alpha)}{(s - i\alpha)(s + i\alpha)}, \\
&= \frac{\alpha}{s^2 + \alpha^2}, \qquad (\textit{for } s > 0).
\end{aligned}$$

2) For $f(t) = \sinh(\alpha t)$, $t \geq 0$ *and* $\alpha \in \mathbb{R}$

$$\begin{aligned}
\mathcal{L}(f(t)) &= \mathcal{L}(\sinh(\alpha t)) \\
&= \mathcal{L}\left(\frac{1}{2}\left(e^{\alpha t} - e^{-\alpha t}\right)\right) \\
&= \frac{1}{2}\mathcal{L}\left(e^{\alpha t}\right) - \frac{1}{2}\mathcal{L}\left(e^{-\alpha t}\right) \\
&= \frac{1}{2}\frac{1}{s - \alpha} - \frac{1}{2}\frac{1}{s + \alpha}, \qquad (\textit{for } s > a) \\
&= \frac{\alpha}{s^2 - \alpha^2}. \qquad (\textit{for } s > \alpha).
\end{aligned}$$

Theorem 6.2.1

Let f *be a function. Suppose that* f,f' *and* f'' *are continuous and we suppose also that* $\mathcal{L}(f(t)) = F(s)$ *exists, then we have*

$$\mathcal{L}(f'(t)) = sF(s) - f(0),$$

and

$$\mathcal{L}(f''(t)) = s^2F(s) - sf(0) - f'(0).$$

6.3 Inverse Laplace Transform

Definition 6.3.1

Let $F(s)$ be the Laplace transform of a continuous function f. By applying $\mathcal{L}^{-1}$ the inverse Laplace Transform, we can determine the function f, i.e

$$\mathcal{L}(f(t)) = F(s) \Longrightarrow f(t) = \mathcal{L}^{-1}(F(s)).$$

Example 6.3.1

1) We have $\mathcal{L}(2) = \frac{2}{s}$, then

$$\mathcal{L}^{-1}\left(\frac{2}{s}\right) = 2,$$

2) For $t \geq 0$, we have $\mathcal{L}(t) = \frac{1}{s^2}$, (for $s > 0$), then

$$\mathcal{L}^{-1}\left(\frac{1}{s^2}\right) = t.$$

3) For $t \geq 0$, we have $\mathcal{L}\left(e^{\alpha t}\right) = \frac{1}{s-\alpha}$, (for $s > \alpha$), then

$$\mathcal{L}^{-1}\left(\frac{1}{s-\alpha}\right) = e^{\alpha t}, \quad \alpha \in \mathbb{R}$$

6.4 Applications to differential equations

For a function y with y, y' and y'' are continuous functions and $\mathcal{L}(y(t)) = F(s)$, we have

$$\mathcal{L}(y'(t)) = sF(s) - y(0),$$

and

$$\mathcal{L}(y''(t)) = s^2F(s) - sy(0) - y'(0).$$

Using these previous relations between y, y' and y'', we can solve an ordinary differential equations of the form

$$\begin{cases} ay''(t) + by'(t) + cy(t) = g(t) \\ y(0) = \beta \text{ and } y'(0) = \gamma \end{cases},$$

with $a, b, c, \beta, \gamma \in \mathbb{R}$ and g is a continuous function.

Example 6.4.1

let F be a function defined by

$$F(s) = \frac{s-1}{(s-2)(s^2-2s+1)}.$$

1) Show that

$$\frac{s-1}{(s-2)(s^2-2s+1)} = \frac{1}{s-2} - \frac{1}{s-1}.$$

2) Using the Laplace transform, solve the system (E) :

$$(E)\begin{cases} y'' - 2y' + y = e^{2t} \\ y(0) = 0 \;\; et \;\; y'(0) = 1 \end{cases}.$$

Solution:

1) Let us show that:

$$\frac{s-1}{(s-2)(s^2-2s+1)} = \frac{1}{s-2} - \frac{1}{s-1}.$$

We have

$$\begin{aligned} \frac{s-1}{(s-2)(s^2-2s+1)} &= \frac{s-1}{(s-2)(s-1)^2} \\ &= \frac{1}{(s-2)(s-1)} \\ &= \frac{1}{s-2} - \frac{1}{s-1}. \end{aligned}$$

2) Solving the system (E) :

$$\begin{aligned} y'' - 2y' + y &= e^{2t} \Rightarrow \mathcal{L}(y'') - 2\mathcal{L}(y') + \mathcal{L}(y) = \mathcal{L}\left(e^{2t}\right) \qquad (\mathcal{L} \text{ is Laplace transform}) \\ &\Rightarrow \overbrace{s^2\mathcal{L}(y) - sy(0) - y'(0)}^{\mathcal{L}(y'')} - 2\overbrace{(s\mathcal{L}(y) - y(0))}^{\mathcal{L}(y')} + \mathcal{L}(y) = \overbrace{\frac{1}{s-2}}^{\mathcal{L}\left(e^{2t}\right)} \\ &\Rightarrow \mathcal{L}(y)\left(s^2 - 2s + 1\right) - 1 = \frac{1}{s-2} \\ &\Rightarrow \mathcal{L}(y)\left(s^2 - 2s + 1\right) = \frac{1}{s-2} + 1 \\ &\Rightarrow \mathcal{L}(y)\left(s^2 - 2s + 1\right) = \frac{s-1}{s-2} \\ &\Rightarrow \mathcal{L}(y) = \frac{s-1}{(s-2)(s^2-2s+1)} \\ &\Rightarrow \mathcal{L}(y) = \frac{1}{s-2} - \frac{1}{s-1}, \end{aligned}$$

then

$$\begin{aligned} y(t) &= \mathcal{L}^{-1}\left(\frac{1}{s-2}-\frac{1}{s-1}\right) \qquad \left(\mathcal{L}^{-1} \textit{ is the inverse Laplace transform}\right) \\ &\Rightarrow\ y(t)=\mathcal{L}^{-1}\left(\frac{1}{s-2}\right)-\mathcal{L}^{-1}\left(\frac{1}{s-1}\right),\ \left(\textit{we have } \mathcal{L}^{-1}\left(\frac{1}{s-a}\right)=e^{\alpha t},\ \textit{for } s>a\right) \\ &\Rightarrow\ y(t)=e^{2t}-e^{t}, \quad t\geq 0. \end{aligned}$$

Bibliography

[1] Margo Kondratieva and Justin Rowsell, Sequaences and series. Published by the Canadian Mathematical Society, 2015.

[2] Béatrice de Tilière, Analyse vectorielle. 2019.

[3] Malcolm R. Adams, Sequences and Series: An Introduction to Mathematical Analysis.

[4] Belattar Zokha, Handout: Les séries numériques, les séries de fonctions, les séries entières et les séries de Fourier. University Center of El Hadj, Institute of Sciences, 2018-2019.

[5] P. Dyke, An introduction to Laplace transforms and Fourier series. Springer, 2014.

[6] K. Allab, Eléments d'Analyse: fonction d'une variable réelle. Office des publications universitaires, Algeria, 2002.

[7] Hamza Tabti, Handout of Mathematics 03 Lessons and exercises. Ibn Khaldoun University of Tiaret, 2024-2025.

[8] J.Colin, J.Morvan and R.Morvan, Formules de Taylor, Développement limité: Exercices corrigées avec rappels de cours, Cépadués, 2008.

[9] Roger Dupont, Mathématiques. Séries, séries entières, séries de Fourier, 64 exercices corrigés, rappels de cours, formulaires pratiques, Vuibert, 1991.