People's Democratic Republic of Algeria
Ministry of Higher Education and Scientific Research
Ibn Khaldoun University of Tiaret
Faculty of Applied Sciences
Department of Mechanical Engineering

Specialty: Mechanical Engineering

Handout for:

Strength of Materials II Courses and Solutions of Problems

Presented at

Ibn Khaldoun University of Tiaret

by:

Dr. BALTACH Abdelghani

Associate Professor A

Mechanical Engineering Department

Intended for Mechanical Engineering

Bachelor's Degree Students

Experts: - Zagane Mohammed El Sallah
- Khelil Foudil

Ibn Khaldoun University of Tiaret – 2024

TABLE OF CONTENTS

FOREWORD

This work is a course on Strength of Materials II (SOM02), primarily designed for third-year undergraduate students (LMD) in the Mechanical Engineering field and potentially for students in other specialties. This handout is developed with the aim of facilitating students in comprehending and assimilating the taught material.

The course is structured around four chapters. The first chapter is divided into two parts; the first part provides a general introduction to Strength of Materials (SOM), covering the objectives, the studied components, the assumptions considered in the calculation of construction elements, as well as some general concepts in SOM. The second part is dedicated to the study of the displacement of symmetrical beams in plane bending. Three methods for calculating deflection and rotation are proposed: the double integration method, the method of initial parameters, and the method of Area Moments.

The second chapter focuses on presenting general theorems of elastic systems based on the elastic deformation energy of bodies subjected to tension, torsion, shear, and bending, along with applications. The purpose is to utilize these theorems for determining deflection and slope.

In the third chapter, composite loads such as deflected bending, compound bending, and torsion bending are studied. In this section, normal and tangential stresses due to composite loading are determined.

Finally, the last chapter addresses the solution of hyper static systems, specifically the determination of support reactions using the force method.

With the theoretical foundation provided, students can refer to key questions illustrated by examples and simple applications that are, however, detailed in their treatment.

CHAPTER 1: MOVEMENT OF SYMMETRIC BEAMS IN PLANE BENDING

INTRODUCTION

In general, mechanics is the study of the effects of external actions on solids and fluids (dynamic study of the movement of a pendulum: mechanics of rigid bodies). In this chapter, we begin with the definition of the strength of materials.

Strength of materials is the study of deformations, displacements, and stresses in objects of simple shape. In the context of this course, we focus on beams, and in the mechanics of deformable solids, we study the relative displacements between points of a solid (notion of deformations) and the associated internal forces (notion of stresses).

This chapter is divided into two parts.

The first part provides a review of the strength of materials, where we discuss the purpose, the studied components, types of loading, and the assumptions considered in the calculation of structural elements.

The second part is dedicated to the study of simple bending. In this section, we determine the internal forces due to bending, namely, the bending moment and shear force, normal and tangential stresses, as well as the methods used for determining deflection and rotation.

PART A

GENERALITIES

1. PURPOSE OF STRENGTH OF MATERIALS (SOM)

The objective is to determine, through calculation, machine components, and structural elements:

- Size these components (economic objectives)
- Verify their mechanical strength (deformations / imposed stress limits)

It is derived from the more general theory of the Mechanics of Continuous Media.

2. SCOPE OF STRENGTH OF MATERIALS (SOM)

It deals not only with the engineering methods used to calculate the capacity of structures and their elements to withstand applied loads without self-destruction or significant deformation but also aims to present the basic criteria for the design of structures (shape, dimensions, etc.) and the use of materials under the best conditions of safety and economy.

2.1. CALCULATION OF MECHANICAL COMPONENTS

- Transmission shafts

2.2. CALCULATION OF STRUCTURES

- Buildings, frameworks, metal structures...
- Bridge Civil Engineering
- Structural framework of various systems

3. ASSUMPTIONS

The main assumptions of the strength of materials are as follows:

3.1. SLENDERNESS

The transverse dimensions of the beam are small compared to the longitudinal dimensions.

Note:

Otherwise, other theories such as plates and shells, or elasticity are used to solve the problem.

3.2. RADIUS OF CURVATURE

The radii of curvature must be limited.

3.3. SECTION VARIATIONS

Section variations should be slow and continuous.

3.4. CONSTRAINTS WITHIN THE FRAMEWORK OF THIS COURSE

- Straight beams and problems in the plane.
- Constant section.
- Straight sections symmetrical with a plane of symmetry.
- Conclusion: a beam defined by:

A midline A cross-section

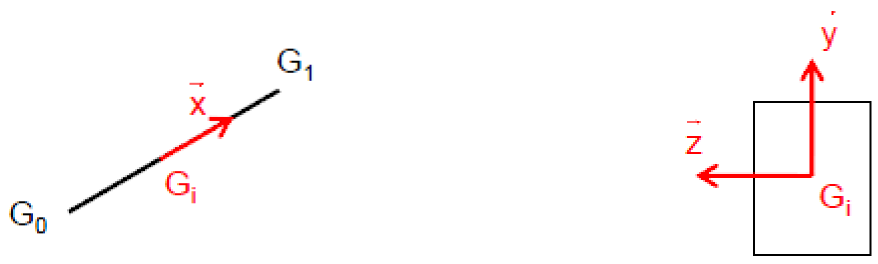
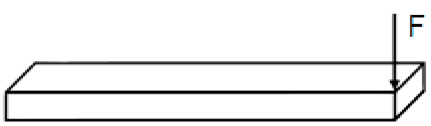


Figure 1.1: Cross-sections remain flat and perpendicular to the midline fiber during deformation.

4. MECHANICAL ACTIONS

Two types of mechanical actions:

- Localized
- Distributed



- Loading must be referred to the midline level.

Figure 1.2: Mechanical Actions

5. MATERIALS

The material is assumed to:

5.1. LINEAR ELASTICITY

It is assumed that at every point, stresses and strains are proportional, and after deformation, the element returns to its initial state.

6. DEFORMATION

Deformations are proportional to stresses.

6.1. HYPOTHESIS OF SMALL DEFORMATIONS

Only the elastic behavior zone of materials is considered:

- Strains and displacements remain small.
- Calculations are based on the undeformed structure.

6.2. NAVIER-BERNOULLI HYPOTHESIS

Straight and planar cross-sections remain straight and planar after deformation: the midline deforms, but the straight sections are "rigid."

7. LOADING

Resistance calculations are simplified by applying a principle stated by Adhémar Barré, Count of Saint-Venant, often experimentally verified.

7.1. SAINT-VENANT'S PRINCIPLE

Stresses and strains in a cross-section far from the points of force application depend only on the resultant and resultant moment at the center of gravity of the section associated with the system of forces.

Consequence

• The results of Strength of Materials (SOM) are valid far from the points of force application.

• Regardless of the nature of a force system, only the resultant torque at the center of gravity of the section determines its state.

In practice

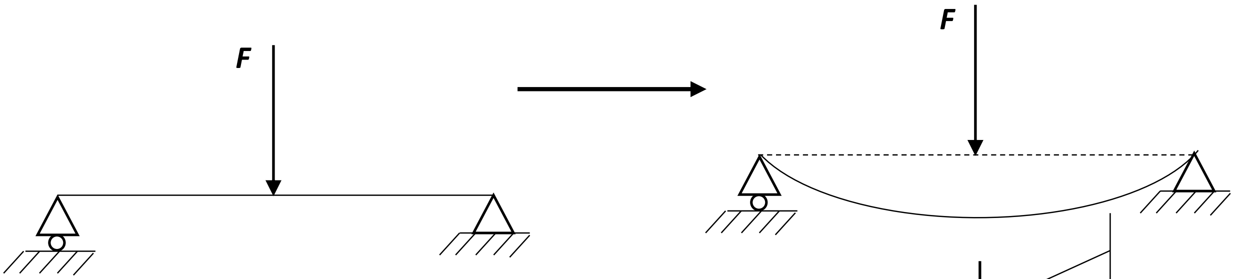
• It is considered that beyond 2-3 times the largest transverse dimension, results are valid.

PART B

MOVEMENT OF SYMMETRICAL BEAMS IN PLANE BENDING

1. DEFINITION

A beam is subjected to bending when the applied forces tend to change its curvature.



There are two components of internal forces: the bending moment M_{fz} (or M_{fy}) and the shear force T_y (or T_z), and the beam has a plane of symmetry, and the bending forces act in this plane perpendicular to the major axis of the beam.

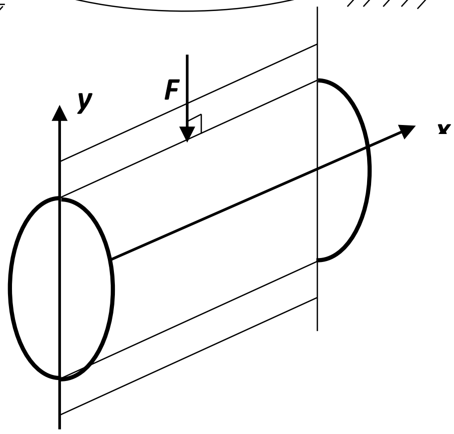
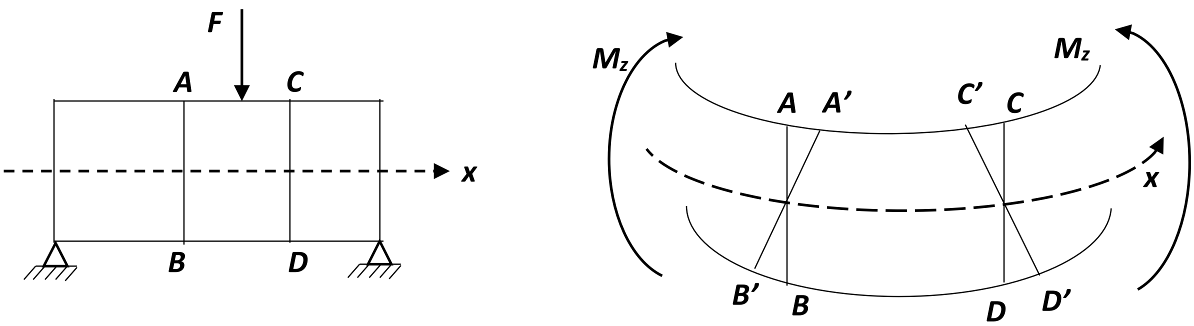


Figure 1.3 Simple Bending.

2. STUDY OF LONGITUDINAL DEFORMATIONS

Figure 1.4 shows the tensile and compressive external fibers of a bent beam section.

2.1. STUDY OF A SECTION UNDER PURE BENDING



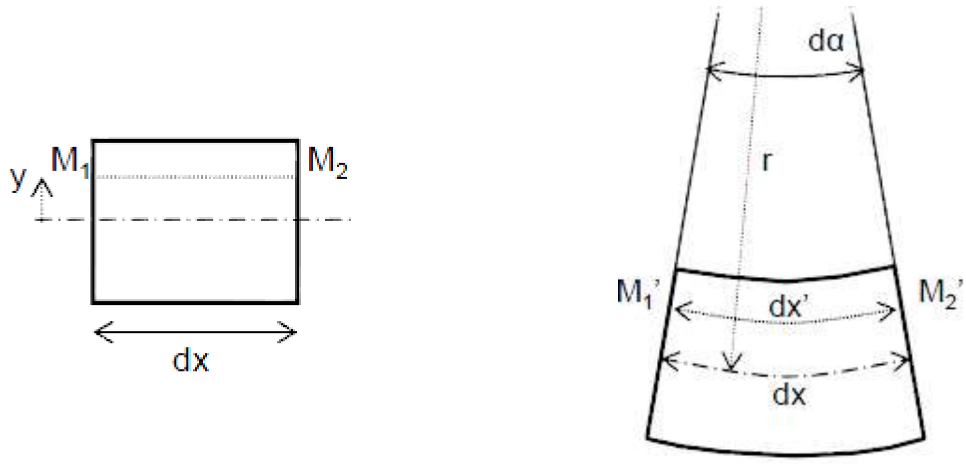


Figure 1.4 Geometry of the deformation of a beam under pure bending (M>0)

2.2. LONGITUDINAL STRAINS

$$\left.\begin{array}{l} \varepsilon_x(y) = \dfrac{dx' - dx}{dx} \\ d\alpha = \dfrac{dx}{r} = \dfrac{dx'}{r-y} \end{array}\right\} \Rightarrow \varepsilon_x(y) = \frac{r-y}{r} - 1 = -\frac{y}{r} \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1.1)$$

It can be expressed as follows:

$$\varepsilon_x(x,y) = -\frac{y}{r(x)} = -y\frac{d\alpha}{dx} \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1.2)$$

The intuition (and experimental observations) confirms:

1/ a) Fibers with y > 0 shorten; they are compressed ($\varepsilon_x < 0$).

b) Fibers with y < 0 elongate; they are tensed ($\varepsilon_x > 0$).

c) The neutral fiber does not experience normal stresses.

2/ Length variations between fibers also induce shearing deformations.

3. STUDY OF NORMAL STRESSES

Normal stresses develop in the cross-sections of a beam subjected to a bending moment.

3.1. EXPRESSION OF NORMAL STRESS UNDER PURE BENDING

According to Hooke's Law:

$$\sigma_x = E.\varepsilon_x = -E\frac{y}{r} \quad \text{……………………………………………….…(1.3)}$$

Note:

Linear distribution of normal stresses in the cross-section.

Tension/compression on either side of the neutral axis.

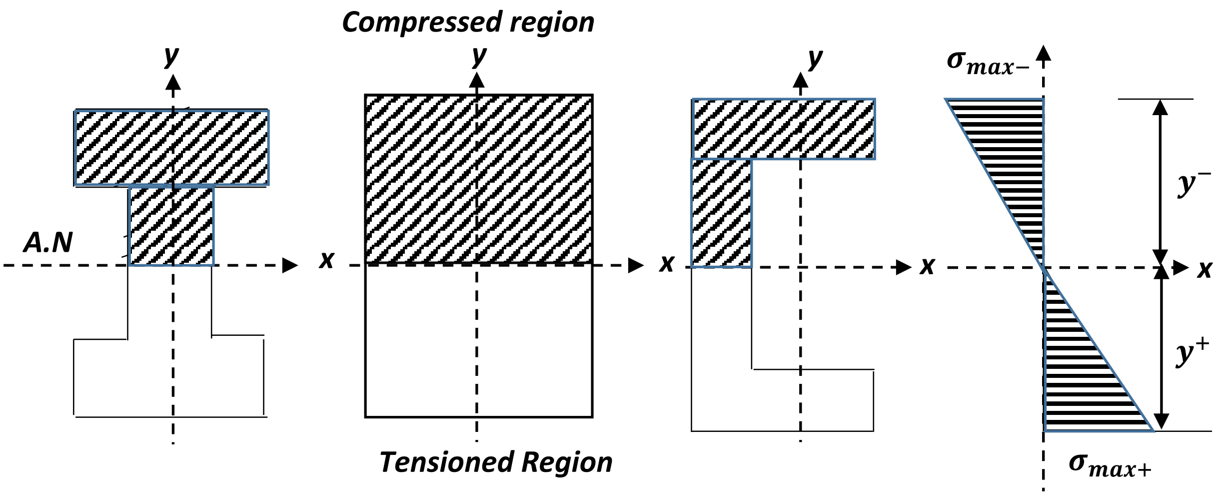


Figure 1.5 Compressed and tensioned regions of sections with a horizontal axis of symmetry.

3.2. RELATIONSHIP WITH BENDING MOMENT:

The above relations are introduced into the general relationship $M_{fz} = f(\sigma x)$

$$M_{fz} = -\int_s y.\sigma_x.ds$$

$$= \int_s y^2.E.\frac{d\alpha}{dx}.ds = E.\frac{d\alpha}{dx}.\int_s y^2.ds = -\frac{\sigma_x}{y}.\int_s y^2.ds \quad \text{……………..(1.4)}$$

$$\Rightarrow \sigma_x(x,y) = -\frac{M_{fz}(x).y}{\int_s y^2.ds} = -\frac{M_{fz}(x)}{I_{GZ}(s)}.y$$

With: $\iint_{(s)} y^2.ds = I_{GZ}(s)$ …………………………………………....(1.5)

I_{GZ} **:** sectional moment of inertia (S) about (G, Z).

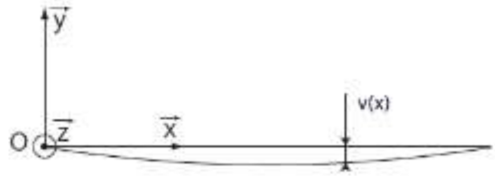
This formula allows us to determine normal stresses at any point on the beam, based on the bending moment. The linear distribution of normal stresses and the maximum stresses will be located at y_{max} or y_{min}.. If the section is symmetrical, they are at y_{max} and y_{min}.

$$\sigma_{x\max} = -\frac{M_{fz}}{I_{GZ}}.y_{\max} \quad \text{……………………………………….(1.6)}$$

4. STUDY OF DEFORMATION

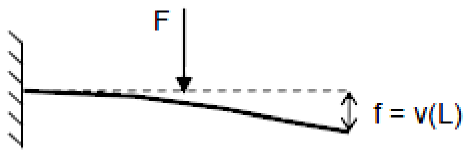
Under bending actions, the neutral axis deforms. The deformed shape, represented by the equation v(x) of the neutral axis curve, is called the deflection. The value of the deflection at a point is called the sag.

4.1. EXPRESSION



$$V'(x)=\tan(\alpha)\Rightarrow V'(x)=\alpha(x)\text{..........}(1.7)$$

Therefore :



$$V''(x)=\frac{d\alpha}{dx}=\frac{M_{fz}(x)}{E.I_{Gz}}\text{.............}(1.8)$$

Deflection equation

By integration, and with boundary conditions, the deflection V(x) is obtained.

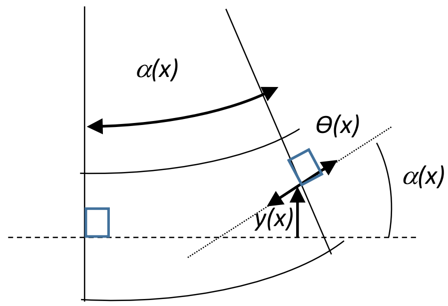


Figure 1.6 Curvature and radius of curvature.

5. TANGENTIAL STRESSES (RELATED TO T_Y)

5.1. RECIPROCITY PRINCIPLE

Equilibrium (PFS) of the elemental volume.

$$dv\rightarrow\tau_{xy}=\tau_{yx}\text{.....................}(1.9)$$

Transverse shear ↔ longitudinal shear

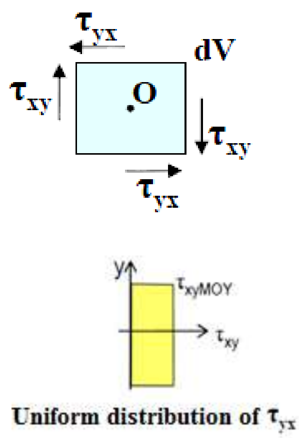


Figure 1.7 Average Shear Stresses at Cross Sections

5.2. AVERAGE SHEAR STRESSES

$$T_y(x)=\iint_{(s)}\tau_{xy}.ds=\tau_{xy\,MOY}=\frac{T_y(x)}{s}\text{.......}(1.10)$$

5.3. VALUES OF TANGENTIAL STRESSES

Let's isolate a portion of the beam (elementary section)

STRESSES ON THE SHADED PORTION

On S1 : $\sigma_1 = \frac{M_{fz}}{I_{Gz}}.y$ et $\tau_1(y)$ ……................(1.11)

On S2 : $\sigma_2 = -\frac{M_{fz} + dM_{fz}}{I_{Gz}}.y$ et $-\tau_1(y)$ …..(1.12)

On S3 : $\tau(y)$

By projection onto ***x***

$$|\tau(y)| = \frac{T_y.A(y)}{I_{Gz}.b(y)} \text{ où } A(y) = \int_{S_1} y.ds \text{ …………(1.13)}$$

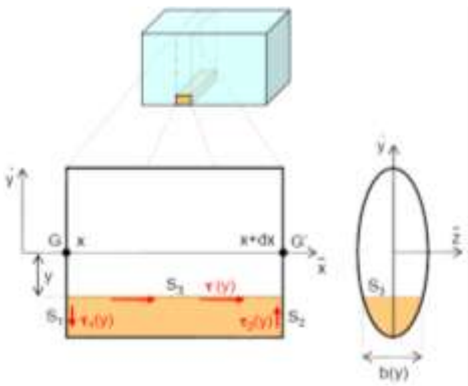


Figure 1.8 Stresses in a Bending Section

Remarks

- A(y) is called the static moment of S1 with respect to the z-axis.
- This expression provides a better approximation of τ in the cross-section.

In particular, a zero stress is found on the upper and lower faces.

Order of magnitude of normal stresses / tangential stresses

It can be shown that: $\frac{o(\tau)}{o(\sigma)} = \frac{a}{l}$ ……………(1.14)

- The order of magnitude ratio of tangential/normal stresses corresponds to the slenderness ratio of the beam a/l.
- Considering *the assumption about slenderness, only normal stresses are critical in flexion.*

6. DEFLECTION OF BEAMS WITH CONSTANT CROSS-SECTION

Various methods are employed for determining deflection and rotation; among the most commonly used methods are the double integration method, the method of initial parameters, the method of deformation superposition, and the method of area moments. In this section, we begin with the method of initial parameters:

6.1. INITIAL PARAMETERS METHOD (MACAULAY)

Consider the bi-articulated beam with a constant cross-section as depicted in Figure 2.5. The applied loads divide the beam into five segments, and a direct application of the integration method would lead to the determination of ten integration constants.

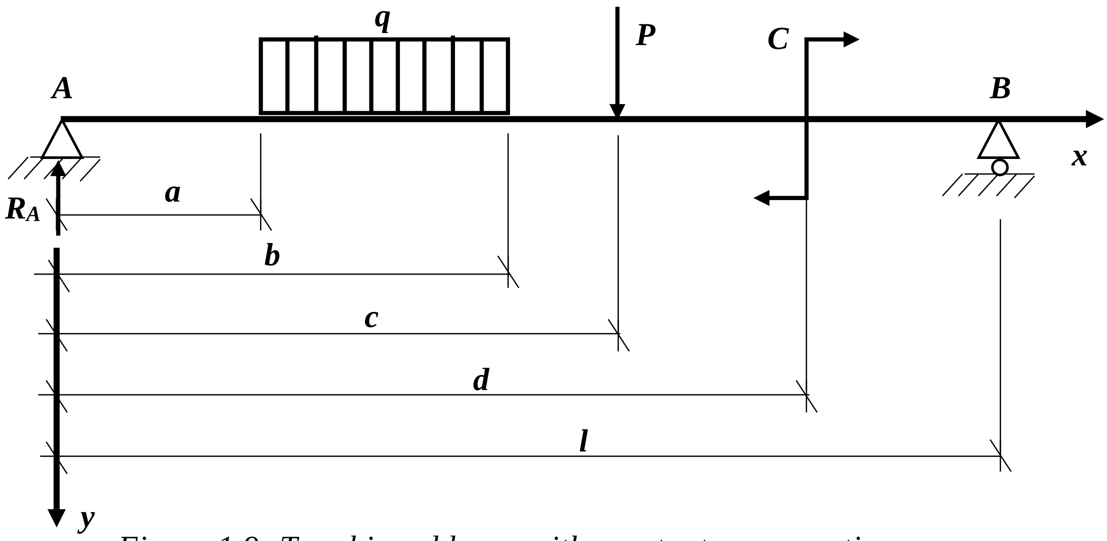


Figure 1.9: Two-hinged beam with constant cross-section.

.

The Clebsch method allows, through a calculation artifice, to reduce the constants to just two, regardless of the number of segments. Moreover, the method provides a unique expression for the deformation that is valid for all segments. The expression for the rotation is naturally obtained by differentiating the deformation function.

The uniqueness of the method lies in its specific presentation of calculations. The fundamental idea of the method is to express the moment on a segment by adding new terms (at least one term) to the moment expression on the preceding segment while keeping the same origin of the x coordinates (see rule 1).

Let's apply this artifice to the considered example. For each segment, let's write the expression for the moment, the differential equation of the elastic, and then perform the two successive derivations.

1^st^ Rule: It consists of placing the origin of the coordinates ***x, y*** at the center of gravity of an extreme section of the beam, for example, the left end.

- Section 1 : $0 \le x \le a$

$M_z = R_A.x$

$$M_z = R_A.x$$
$$EI_z y'' = -R_A.x$$
$$EI_z y' = -R_A.\frac{x^2}{2} + C_1$$
$$EI_z y = -R_A.\frac{x^3}{6} + C_1.x + D_1$$

By making $x = 0$ in the last two expressions, we obtain:

$$C_1 = EI_z.y'_0 = EI_z.\theta_0$$
$$D_1 = EI_z.y_0 = EI_z.y_0$$

In other words, C1 and D1 represent respectively the rotation and the deflection, multiplied by the bending rigidity of the beam (EIz), of the initial section.

- Section 2 $a \le x \le b$

$$M_z = R_A.x - \frac{q}{2}.(x-a)^2$$
$$EI_z y'' = -R_A.x + \frac{q}{2}.(x-a)^2$$
$$EI_z y' = -R_A.\frac{x^2}{2} + \frac{q}{6}(x-a)^3 + C_2$$
$$EI_z y = -R_A.\frac{x^3}{6} + \frac{q}{24}(x-a)^4 + C_2.x + D_2$$

By making ***x = a*** in the last two equations, we deduce that: ***C2 = C1*** *and* ***D2 = D1***.

- Section 3 $b \le x \le c$

2nd Rule: We assume the distributed load applied across the remaining length of the beam and apply an equal and opposite load to balance the added load (this artifice allows for general expressions that hold true throughout the entire length of the beam).

$$M_z = R_A.x - \frac{q}{2}.(x-a)^2 + \frac{q}{2}.(x-b)^2$$

$$EI_z y'' = -R_A.x + \frac{q}{2}.(x-a)^2 - \frac{q}{2}.(x-b)^2$$

$$EI_z y' = -R_A.\frac{x^2}{2} + \frac{q}{6}(x-a)^3 - \frac{q}{6}.(x-b)^3 + C_3$$

$$EI_z y = -R_A.\frac{x^3}{6} + \frac{q}{24}(x-a)^4 - \frac{q}{24}(x-b)^4 + C_3.x + D_3$$

When comparing the deflections and rotations in the junction section at ***x = b***, we find : ***C3 = C2*** et ***D3 = D2***.

- Section 4 $c \le x \le d$

$$M_z = R_A.x - \frac{q}{2}.(x-a)^2 + \frac{q}{2}.(x-b)^2 - P(x-c)$$

$$EI_z y'' = -R_A.x + \frac{q}{2}.(x-a)^2 - \frac{q}{2}.(x-b)^2 + P(x-c)$$

$$EI_z y' = -R_A.\frac{x^2}{2} + \frac{q}{6}(x-a)^3 - \frac{q}{6}.(x-b)^3 + \frac{P}{2}(x-c)^2 + C_4$$

$$EI_z y = -R_A.\frac{x^3}{6} + \frac{q}{24}(x-a)^4 - \frac{q}{24}(x-b)^4 + \frac{P}{6}(x-c)^3 + C_4.x + D_4$$

By comparing once again the displacements and rotations to the left and right of the section ***x = c***, it is demonstrated that: ***C4 = C3 et D4 = D3.***

- Section 5 $d \le x \le l$

3rd Rule: The concentrated couple is multiplied by ***(x-d)^0*** to mark the section where its influence begins and to maintain the generality of the expressions.

$$M_z = R_A.x - \frac{q}{2}.(x-a)^2 + \frac{q}{2}.(x-b)^2 - P(x-c) + C.(x-d)^0$$

$$EI_z y'' = -R_A.x + \frac{q}{2}.(x-a)^2 - \frac{q}{2}.(x-b)^2 + P(x-c) - C.(x-d)^0$$

$$EI_z y' = -R_A.\frac{x^2}{2} + \frac{q}{6}(x-a)^3 - \frac{q}{6}.(x-b)^3 + \frac{P}{2}(x-c)^2 - C.(x-d) + C_5$$

$$EI_z y = -R_A.\frac{x^3}{6} + \frac{q}{24}(x-a)^4 - \frac{q}{24}(x-b)^4 + \frac{P}{6}(x-c)^3 - C.(x-d)^2 + C_5.x + D_5$$

Comparing once again the rotations and deflections at the junction section ***(x = d)***, obtained using the valid relationships for segments 4 and 5, it is demonstrated that: ***C5 = C4*** and ***D5 = D4***.

Thus, it is demonstrated that ultimately there are only two integration constants for the entire beam:

$$C_1 = C_2 = C_3 = C_4 = C_5 = EI_z.y'_0 = EI_z.\theta_0$$
$$D_1 = D_2 = D_3 = D_4 = D_5 == EI_z.y_0 = EI_z.y_0$$

These two constants characterize the displacements (rotation and deflection) of the initial section of the beam, hence their designation as initial parameters. They are determined based on the support conditions of the considered beam. In a simple or double support, the deflection is zero ***f = 0***, while in a fixed support, we have: ***f = θ = 0.***

The total number of equations can be reduced to four by adopting the following writing mode:

$$\begin{array}{r|l|l|l|l|}
 & x<a & x<a & x<a & x<a \quad x<a \\
M_z = R_A.x & -\dfrac{q}{2}.(x-a)^2 & +\dfrac{q}{2}.(x-b)^2 & -P.(x-c) & +C.(x-d)^0 \\
EI_z y'' = -R_A.x & +\dfrac{q}{2}.(x-a)^2 & -\dfrac{q}{2}.(x-b)^2 & +P.(x-c)^2 & -C.(x-d)^0 \\
EI_z y' = EI_z\theta_0 - R_A.\dfrac{x^2}{2} & +\dfrac{q}{6}.(x-a)^3 & -\dfrac{q}{6}.(x-b)^3 & +\dfrac{P}{2}.(x-c)^2 & -C.(x-d) \\
EI_z y = EI_z y_0 + EI_z\theta_0.x - R_A.\dfrac{x^3}{6} & +\dfrac{q}{24}.(x-a)^4 & -\dfrac{q}{24}.(x-b)^4 & +\dfrac{P}{6}.(x-c)^3 & -\dfrac{C}{2}.(x-c)^2
\end{array}$$

To calculate a quantity *(M_s , y'', y'* or *y)* for a given section, only the terms to the left of the section limit should be considered.

In the treated example, the boundary conditions are written as follows: $y = 0$ at $x = 0$ and at $x = l$. The first condition yields $f_0 = 0$, and from the second, we determine the value of θ_0. The expressions for *y(x)* and *θ(x)* are given by the equations:

$$EI\theta(x) = EI\theta_0 + \sum M \frac{(x-a)}{1!} + \sum P \frac{(x-b)^2}{2!} + \sum q_c \frac{(x-c)^3}{3!} + \sum q_d \frac{(x-d)^3}{3!} +$$
$$\sum q_c^{'} \frac{(x-c)^4}{4!} + \sum q_d^{'} \frac{(x-d)^4}{4!} + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1.15)$$

Where:

M : Concentrated external moments or at the embedding.

a : Distance between the origin of coordinates and the points of application of moments M

p : Concentrated forces including reactions

b : Distances between the origin of coordinates and the points of application of forces P

q_c , q_d : Respectively, the intensities at the beginning and at the end of the distributed load

q'_c , q'_d : Respectively, the values of the derivatives of q at the points $x= c$ et $x = d$

The directions of the charges are positive as shown below :

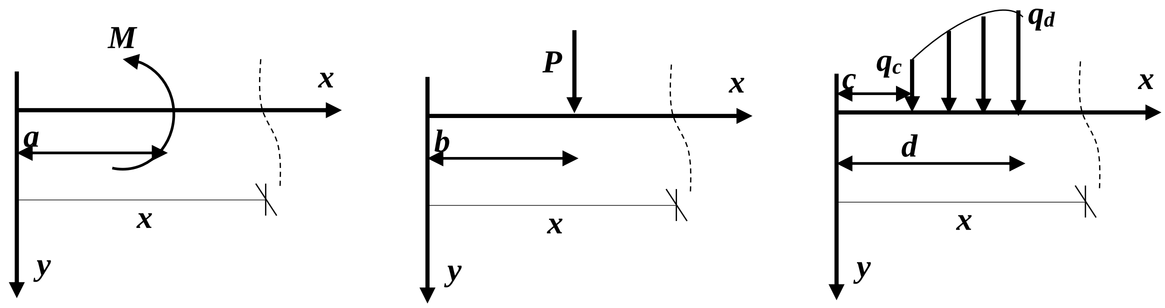


Figure 1.10 Positive Load Directions

The two initial parameters y_0 and θ_0 are determined by the support conditions of the beam.

6.1.1 APPLICATION

Determine the maximum deflection and rotations at the supports of the beam depicted in the figure below.

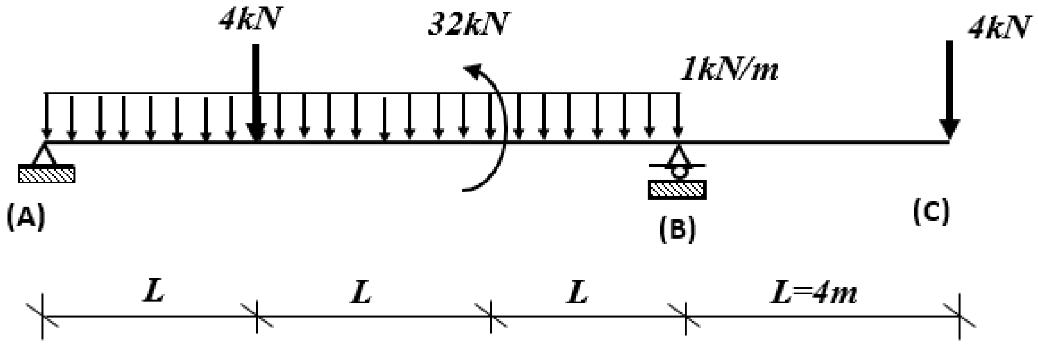
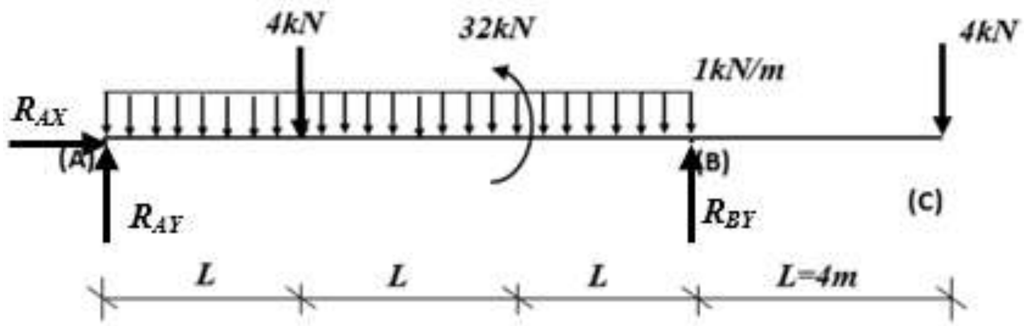


Figure 1.11 Simply supported two-hinged beam subjected to simple bending

6.1.2 Solution

Calculation of support reactions



In static equilibrium:

$$\sum \vec{F}_x = \sum \vec{F}_y = \sum \vec{M}_z = \vec{0} \text{ So:}$$

$$\sum \mathrm{F_x} = 0 \Rightarrow \mathrm{R_{AX}} = 0$$
$$\sum \mathrm{F_y} = 0 \Rightarrow \mathrm{R_{AY}} + \mathrm{R_{BY}} \text{ -4-(1.12)-4} = 0$$
$$\Rightarrow \mathrm{R_{AY}} = 4 + 12 + 4\text{-}\mathrm{R_{BY}}..........(*)$$
$$\Rightarrow \sum M_z / A = 0 \Rightarrow (-4.16) + (\mathrm{R_{BY}}.12) + 32 - (12.\frac{12}{2}) - (4.4) = 0$$
$$\Rightarrow \mathrm{R_{BY}} = \frac{120}{12} = 10kN...........(**)$$
$$\textit{we substitute } (**) \textit{ on } (*) \textit{ and find}:$$
$$\mathrm{R_{AY}} = \mathrm{R_{BY}} = 10kN$$

Using the initial parameters method, one can determine the maximum deflection and rotations at the supports:

$$EI\theta(x) = EI\theta_0 - \left[\frac{10}{2}.x^2\right] + \left[\frac{x^3}{6} - \frac{(x-12)^3}{6}\right] + \left[2.(x-4)^2\right] + \left[32.(x-8)\right] - \left[\frac{10}{2}.(x-12)^2\right]$$

$$EIy(x) = EIy_0 + EI\theta_0.x - \left[\frac{5}{3}.x^3\right] + \left[\frac{x^4}{24} - \frac{(x-12)^4}{24}\right] + \left[\frac{2}{3}.(x-4)^3\right] + \left[\frac{32}{2}.(x-8)^2\right]$$
$$-\left[\frac{5}{3}.(x-12)^3\right]$$

Initial conditions:

$$Support\ A:$$
$$x = 0 \Rightarrow EIy(0) = 0 \ \Rightarrow EIy_0 = 0 \Rightarrow y_0 = 0$$

$$Support\ B:$$
$$x = 12 \Rightarrow EIy(12) = 0$$
$$\Rightarrow EIy_0 + EI\theta_0.12 - \left[\frac{5}{3}.12^3\right] + \left[\frac{12^4}{24} - \frac{(12-12)^4}{24}\right] + \left[\frac{2}{3}.(12-4)^3\right] + \left[\frac{32}{2}.(12-8)^2\right]$$
$$-\left[\frac{5}{3}.(12-12)^3\right] = 0$$

$$\Rightarrow 12EI\theta_0 = 2880 - 864 - 256 - \frac{1024}{3}$$
$$\Rightarrow EI\theta_0 = \frac{4256}{36} = 118{,}22$$
$$\Rightarrow \theta_0 = \frac{118{,}22}{EI}$$

$$We\ have:$$
$$x = 12 \Rightarrow \theta(0) = \theta_0 = \frac{118{,}22}{EI}$$
$$So:$$

$$EI\theta(12) = EI\theta_0 - \frac{10}{2} + 12^2 + \frac{12^3}{6} - \frac{(12-12)^3}{6} + 2.(12-4)^2 + 32.(12-8) - 5.(12-12)^2$$
$$EI\theta(12) = \frac{18.22}{EI}.EI - 720 + 288 + 128 + 128$$
$$EI\theta(12) = -57.78$$
$$\Rightarrow \theta(12) = \frac{-57.78}{EI}$$

The maximum deflection

$\theta(x) = 0 \Rightarrow 3^{rd}$ degree polynomial equation

x	*0*	*4*	*8*	*12*	*16*
EIΘ(x)	*118.22*	*48.89*	*-84.4*	*-57.78*	*-25.78*

So $\theta(x) = 0$ *for* $x \in \left]4,8\right[$

By employing the ***Dichotomy*** method, convergence is achieved at *x=5,48*

$$y(5.48) = \frac{414}{EI}$$

6.2. SUPERPOSITION OF DEFORMATIONS

The differential equations of the deformation are linear equations, meaning that all terms of ***y, y***', and ***y***'' are of the first order. Deformations resulting from multiple load cases can therefore be superimposed or accumulated. This method is especially useful when the loading consists of several elementary load cases, or when deformations are provided in the reference documents of the Strength of Materials.

6.2.1. SUPERPOSITION PRINCIPLE:

The effect produced by multiple mechanical actions is equal to the sum of the effects produced by these mechanical actions taken separately. The term "effect of mechanical actions" refers to the state of stress generated by these actions as well as the associated deformations.

Applying the previously stated principle of superposition allows us to state: "If a beam is subjected to several simple loads, the state of stress and deformation is the sum of the states of stress and deformation due to each of these simple loads taken separately."

6.2.2. LIMITATIONS OF THE SUPERPOSITION THEOREM:

- The elastic limit must not be reached.

- The sum of external actions from different simple loading problems must be equal to that of the complex problem.

6.2.3. APPLICATION

Determine the maximum deflection of the beam below.

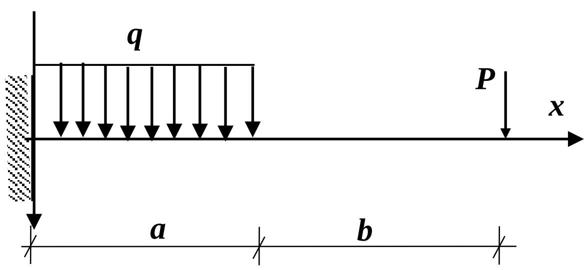
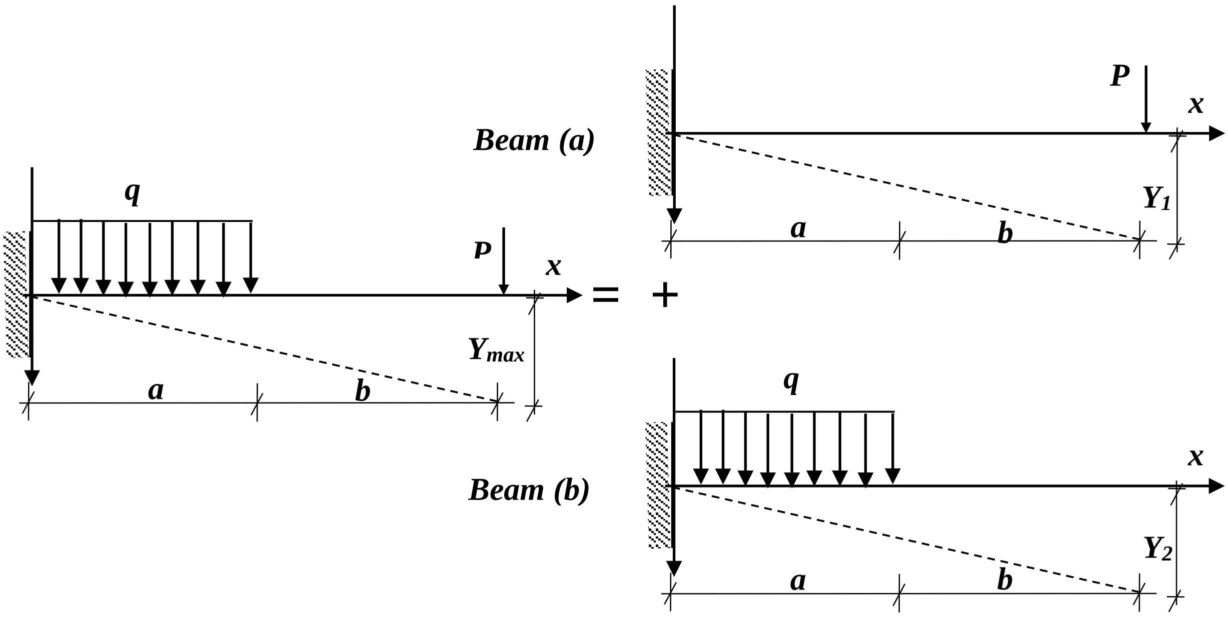


Figure 1.12 embedding beam subjected to a distributed and concentrated load

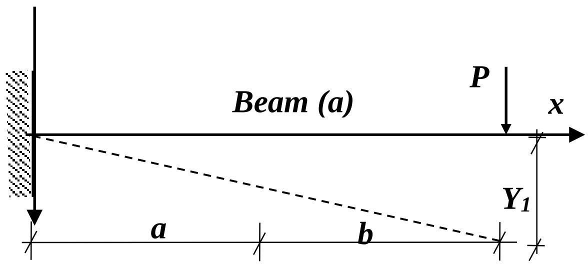
6.2.4 Solution

We divide this beam into two segments (a) and (b) as shown in the figure below:



For beam (a):

The maximum deflection due to the concentrated load ***P*** is given by:

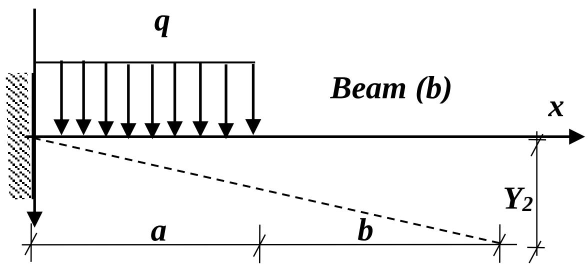


$$Y_1 = \frac{Pl^3}{3EI}$$

For beam (b):

The maximum deflection due to the distributed load ***q*** is given by:

$$Y_2 = \frac{qa^3(4l - a)}{24EI}$$



The maximum deflection Y_{max} is obtained by summing the two deflections Y_1 and Y_2:

$$Y_{\max} = Y_1 + Y_2$$

$$Y_{\max} = \frac{Pl^3}{3EI} + \frac{qa^3(4l-a)}{24EI}$$

6.3 AREA-MOMENT METHOD

The calculation of the slope (rotation) and deflection of a beam using the "area-moment method" also involves concepts studied earlier in this chapter, except that integration is performed geometrically, based on the bending moment diagram. We will see that this method is particularly suitable for the analysis of beams with varying bending stiffness ***EI*** along their length.

Indeed, we observed that the double integration method with singularity functions is especially well-suited for cases where ***EI*** is constant.

6.3.1 THEOREMS

From equations 1.17 and 1.18 of the deformation:

$$\frac{1}{\rho} = \frac{d^2 v}{dx^2} = \frac{M}{EI} \quad \text{..(1.17)}$$

$$\frac{1}{\rho} = \frac{d\varphi}{ds} \approx \frac{d\varphi}{dx} = \frac{d\ \ v}{dx^2} \quad \text{..(1.18)}$$

On peut récrire :

$$\frac{1}{\rho} = \frac{d\varphi}{dx} = \frac{M}{EI} \quad \text{...(1.19)}$$

$$d\varphi = \frac{M}{EI}dx \quad \text{...(1.20)}$$

By integrating equation 1.20, we can determine the slope variation, φ_{AB}, between two points ***A*** and ***B*** on the elastic curve of the beam (fig. 1.13); thus:

$$\varphi_{AB} = \int_{\varphi_A}^{\varphi_B} d\varphi = \int_{x_A}^{x_B} \frac{M}{EI} dx \ldots\ldots\ldots\ldots\ldots(1.21)$$

Moreover, the right-hand side of equation 1.21 represents the area under the ***M/EI*** curve between x_A and x_B (fig.1.13b). Therefore:

$$\varphi_{AB} = \left[aire\ sous \frac{M}{EI} \right]_A^B \ldots\ldots\ldots\ldots\ldots(1.22)$$

This equation 1.22 can be expressed in theorem form.

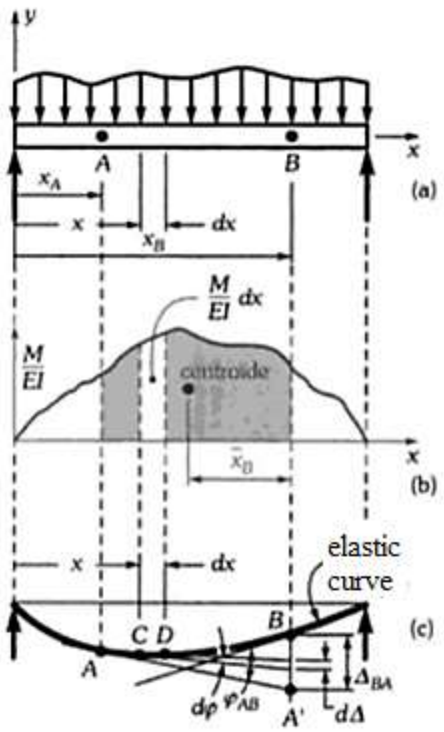


Figure 1.13 Area-Moment Method

Theorem 1 : Slope Variation.

The angle between the tangents to the elastic curve at points ***A*** and ***B*** is equal to the area (between these two points) under the bending moment curve divided by the flexural rigidity (***EI***). Additionally, in Figure 1.13c, it can be observed that the distance $d\Delta$, on the vertical line passing through point ***B*** and bounded by $d\varphi$, is given by the equation:

$$d\Delta = (x_B - x_A) d\varphi \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1.23)$$

By integrating Equation 1.23, the vertical distance $\Delta_{BA} = BA'$ between point ***B*** and the tangent at point ***A*** is obtained:

$$\Delta_{BA} = \int_{x_A}^{x_B} (x_B - x) \frac{M}{EI} dx \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1.24)$$

We define Δ_{BA} as the tangential deflection at point ***B*** with respect to point A. By convention, , the first subscript represents the point on the elastic curve, and the second represents the point from which the tangent originates. The integral of Equation 1.24 yields the first moment of the area under the curve ***M/EI*** between x_A and x_B, This first moment is evaluated with respect to a vertical axis passing through point ***B***. Denoting $\bar{x}_B$ as the distance between point ***B*** and the centroid of this area, Equation 1.24 can be rewritten as follows:

$$\varphi_{BA} = \left[aire\ sous \frac{M}{EI} \right]_A^B \bar{x}_B \quad \text{...(1.25)}$$

Equation 1.25 can also be expressed as a theorem, the second of the method of areas moments.

Theorem 2: Tangential Deflection

The tangent deflection at any point ***B*** (on the elastic curve) on the tangent passing through another point ***A*** of the elastic curve is equal to the first moment, with respect to point ***B***, of the area under the ***M/EI*** curve between ***A*** and ***B***.

In Figure 1.14, the sign and index conventions for tangent deflection and slope variation are provided. When the bending moment is positive (Fig. 1.14a), the intersection point of the tangents is below the beam, as are the tangent deflections. The opposite occurs when the bending moment is negative (Fig. 1.14b). It should be noted that there is a significant difference between Δ_{BA} et Δ_{AB}

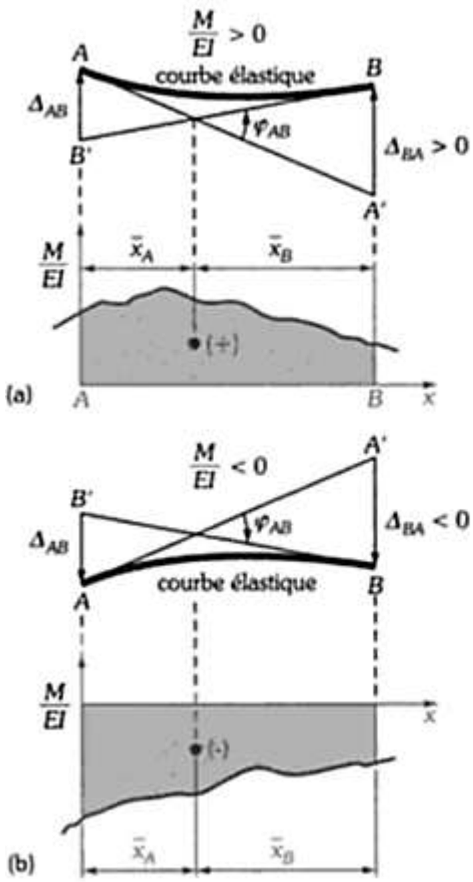
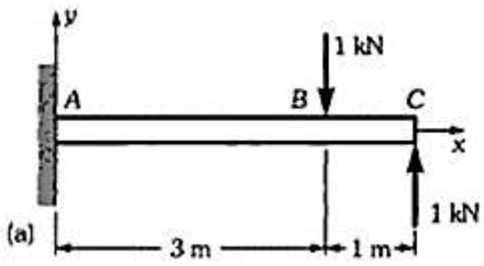
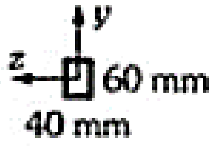


Figure 1.14 Moment of areas: sign and index convention: a) positive bending moment; b) Negative bending moment

6.3.2 APPLICATION

The beam in the figure below is fixed at its left end. You are required to calculate the slope and deflection at point ***C*** (right end) using the method of areas moments (***E = 200 GPa***).





6.3.3 Solution

Calculation of Support Reactions

In static equilibrium

$\sum \vec{F}_x = \sum \vec{F}_y = \sum \vec{M}_z = \vec{0}$ so :

$$\sum F_x = 0 \Rightarrow R_{AX} = 0\,kN$$
$$\sum F_y = 0 \Rightarrow R_{AY} + 1 - 1 = 0$$
$$\Rightarrow R_{AY} = 0\,kN$$
$$\Rightarrow \sum M_Z / A = 0 \Rightarrow M_A + (1.3) - (1.4) = 0$$
$$\Rightarrow M_A = 1kN.m$$

Using the method of sections, we can plot the diagrams of internal forces:

1st *section:* $0 \le x \le 3$

$$M_f(x) = 1 \quad \begin{cases} x = 0 \Rightarrow M_f(0) = 1kN.m \\ x = 3 \Rightarrow M_f(3) = 1kN.m \end{cases}$$

1nd *section:* $3 \le x \le 4$

$$M_f(x) = -.x + 4 \quad \begin{cases} x = 3 \Rightarrow M_f(3) = 1kN.m \\ x = 4 \Rightarrow M_f(4) = 0kN.m \end{cases}$$

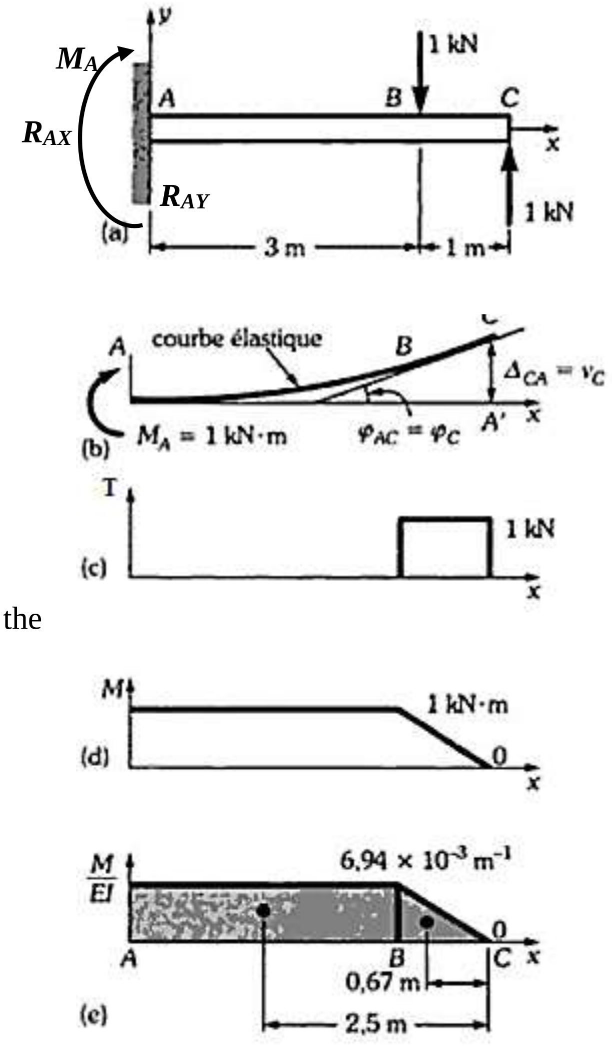


Figure 1.15Application Example.

Figure 1.15b illustrates the predictable elastic curve: the fixed support requires the curve to be perfectly horizontal at ***A***; we will see later that achieving this condition significantly simplifies the resolution using the method of moment areas. Figures 1.15c and 1.15d show the diagrams of shear forces and bending moments, and Figure 1.15e shows the ***M/EI*** diagram.

Since, in this case, the flexural rigidity ***EI*** is constant, the ***M*** and ***M/EI*** diagrams are similar.

1. Calculation of the Slope at Point C

Since the elastic curve is horizontal at ***A,*** we can evaluate the slope φ_C by calculating φ_{AC} from the equation of theorem 1

$$\varphi_{AC} = aire\, sous \frac{M}{EI} entre\, A\, et\, C$$

Moment of Inertia:

$$I = \frac{bh^3}{12} = \frac{40.60^3}{12} = 720.000\, mm^4$$

$$\frac{M}{EI} = \frac{1.1000.1000}{200000.720000} = 6{,}94.10^{-3}\, m$$

$$\varphi_{AC} = S_1 + S_2 = \left[\left(6{,}94.10^{-3}\right).3\right] + \left[\left(\frac{6{,}94.10^{-3}}{2}\right).1\right]$$
$$\varphi_{AC} = 24{,}94.10^{-3}\, rad$$
$$\varphi_{AC} \approx 1{,}39^\circ$$

2. Calculation of Deflection:

Again, the fact that the tangent at ***A*** is horizontal allows for the direct calculation of the deflection v_C, as it is equal to the tangential deflection Δ_{CA}. Therefore, we have (theorem 2):

$\Delta_{CA} =$ First moment with respect to ***C*** of the area ***M/EI*** enclosed between ***A*** and ***C***

$$\Delta_{CA} = \left[\left(6{,}94.10^{-3}\right).3.2{,}5\right] + \left[\left(\frac{6{,}94.10^{-3}}{2}\right).1.0{,}67\right]$$
$$\Delta_{CA} = 54.4.10^{-3}\, m$$
$$\Delta_{CA} = v_C = 54{,}4 mm$$

The fact that Δ_{CA} is positive indicates that the displacement of point ***C*** occurs upward, relative to the tangent at ***A***.

CHAPTER 2 GENERAL THEOREMS OF ELASTIC SYSTEMS

INTRODUCTION

In this chapter, we will examine the relationships that exist between the stresses acting on a system and the displacements they produce.

1. DEFORMATION OF ELASTIC STRUCTURES

1.1 CONCEPTS OF WORK AND COMPLEMENTARY WORK:

To illustrate, let's consider the case of a prismatic bar subjected to axial tension force $\boldsymbol{F_1}$, resulting in elongation $\boldsymbol{\delta_1}$ (Fig.2.1a).

We assume that the force $\boldsymbol{F_1}$ is applied gradually, in a slow manner, so as not to produce any inertia force. Under these conditions, the loading (force $\boldsymbol{F_1}$ here) is said to be applied statically, and the resulting displacement (elongation in this case) is related to the applied force by a relationship represented by the *"F- δ"* diagram in (Fig. 2.1b).

Let $\boldsymbol{F}$ be an intermediate value and $\boldsymbol{\delta}$ the corresponding elongation. An increase $\boldsymbol{dF}$ in the load corresponds to an additional elongation $\boldsymbol{d\delta}$. The elemental work produced by $\boldsymbol{F}$ during the increase in $\boldsymbol{d\delta}$ is defined by:

$\boldsymbol{d\tau_e = Fd\delta}$ ……………………………………………………………(2.1)

It is represented by the shaded area (inclined hatching) of the $\boldsymbol{F\text{-}\delta}$ diagram (Fig. 2.1b).

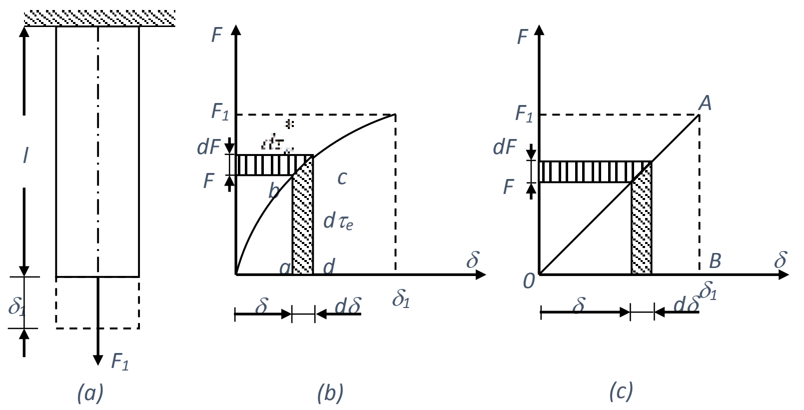


Figure 2.1 Force-Displacement Diagram

Note:

In the previous figure (Fig. 2.1), $Fd\delta$ more precisely represents the rectangle ***"abcd."*** In other words, the work done by dF during the displacement $d\delta$, which is an infinitely small quantity of higher order than 1, is neglected. The total work done by the force F_1 during the displacement δ_1 is obtained by summing up the elemental works, i.e.,

$$\tau_e = \int_0^{\delta_1} F d\delta \quad \dots\dots(2.2)$$

It is represented by the area bounded by the F-δ curve and the δ axis up to δ_1. Similarly, the *elemental complementary work* of the displacement δ during the increase in load dF is given by:

$$d\tau_e^* = \delta dF \quad \dots\dots(2.3)$$

The total complementary work done by F_1, applied gradually from **0** to F_1, during the displacement δ_1 is given by:

$$\tau_e^* = \int_0^{F_1} \delta dF \quad \dots\dots(2.4)$$

This is the area to the left of the F-δ curve.

1.2 ENERGY AND COMPLEMENTARY DEFORMATION ENERGY

Consider a body subjected to external loads. Under the action of external loads, the body deforms, and the internal forces (stresses) perform work that opposes the work of external loads. This internal work, changed in sign, is designated as the potential energy of deformation: (W) ($-\tau_i = W$).

Isolate an element $dv = dxdydz$ of the considered body. The elementary energy stored in dv is calculated as the work done by the forces acting on the faces of the element dv. Thus, the work done by the elemental force $\sigma_x.dydz$ during the variation of $d\varepsilon_x$ of the deformation ε_x, which produces the displacement $\Delta dx = d\varepsilon_x.dx$, is given by :

$$dW = \sigma_x.dydz.d\varepsilon_x dx = \sigma_x d\varepsilon_x dv \quad \dots\dots(2.5)$$

Considering all stress components and using index notation, we obtain for the element dv:

$$dW = \sigma_{ij} d\varepsilon_{ij} dv \quad \text{...(2.6)}$$

The energy stored in the entire volume of the body (***v***) is :

$$W = \int_v \sigma_{ij} d\varepsilon_{ij} dv \quad \text{...2.7)}$$

Consider a unidirectional stress-strain diagram (unidimensionnel) (Fig. 2.2b).

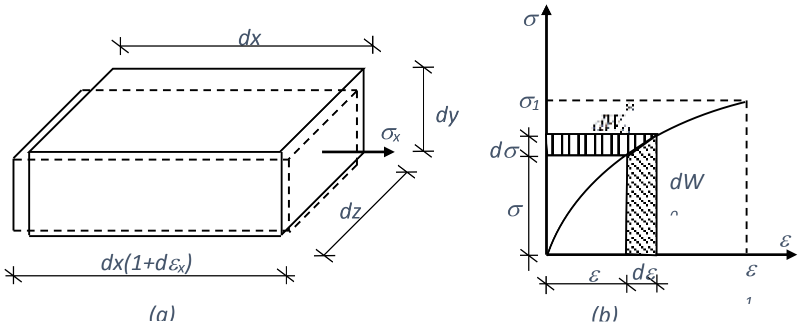


Figure 2.2 Stress-Strain diagram

We have: $dW_0 = \sigma d\varepsilon$

This quantity has the unit of energy per unit volume. The integral: $W_0 = \int_0^{\varepsilon_1} \sigma d\varepsilon$ is called the density of deformation energy and is represented by the area between the σ-ε curve and the stress axis $\boldsymbol{\varepsilon}$. Note that we have :

$$W = \int_v dW_0 dv \quad \text{...(2.8)}$$

Similarly, the elemental complementary energy produced by an increase in $\boldsymbol{d\sigma_{ij}}$ stresses during the displacements produced by the corresponding deformations $\boldsymbol{\sigma_{ij}}$ is given by:

$$dW^* = \varepsilon_{ij} d\sigma_{ij} dv \quad \text{...(2.9)}$$

And for the entire volume of the body:

$$W^* = \int_v \varepsilon_{ij} d\sigma_{ij} dv \quad \text{...(2.10)}$$

We also have: $dW_0^* = \varepsilon d\sigma$ and $W_0^* = \int_0^{\sigma_1} \varepsilon d\sigma$

2. WORK AND ENERGY IN THE LINEAR ELASTIC DOMAIN:

2.1 WORK OF A FORCE:

If the relation between $\boldsymbol{F}$ and $\boldsymbol{\delta}$ is linear, within the scope of Hooke's Law (and small displacements), i.e., when the relation (Fig. 2.1c) holds at any moment during loading:

$\boldsymbol{F = k\delta}$ ($\boldsymbol{k}$ = constant) ……………………………(2.11)

the total work becomes :

$$\tau_e = \int_0^{\delta_1} k\delta d\delta = \frac{1}{2} k\delta_1^2 \quad \text{……………………………………………(2.12)}$$

and since: $F_1 = k\delta_1$, it follows that :

$$\tau_e = \frac{1}{2} F_1 \delta_1 \quad \text{…………………………………………………(2.13)}$$

The total work is represented by the area of the triangle ***OAB*** (Fig. 2.1c).

Note that in the case of linear elasticity, we have: $\tau_e = \tau_e^*$.

2.2 GENERALIZATION:

If a system in equilibrium is subjected to an overall load $\boldsymbol{F\ (F_1, F_2, \ldots F_i, \ldots, F_n)}$ and the points of application of these forces undergo displacements, whose projections on the directions of these same loads are $\boldsymbol{\delta_1, \delta_2, \ldots, \delta_n}$, the work done during the loading of the system (transition from the initial equilibrium state to the final equilibrium state) is given by:

$$\tau_e = \frac{1}{2} \sum_{i=1}^{n} F_i \delta_i \quad \text{………………………………………………(2.14)}$$

It should be recalled that we assume:

- the loading is static (loadings are slow),
- the material has a linear elastic behavior (Hooke's Law is satisfied),
- the displacements do not affect the action of the loads (hypothesis of small displacements, no second-order effects).

2.3 POTENTIAL ENERGY OF DEFORMATION:

In the linear elastic domain, the stress-strain relation $\boldsymbol{(\sigma_{ij}\text{-}\varepsilon_{ij})}$ is linear, and as in work, the factor 1/2 appears in the expression of energy (Fig. 2.3).

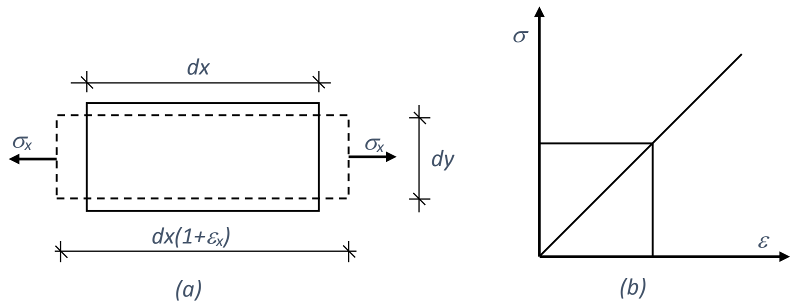


Figure 2.3 Potential Energy of Deformation in the Linear Elastic Range

Thus, the work done by the force $\sigma_x dydz$ during the deformation ε_x which causes a change in length $\Delta dx = \varepsilon_x dx$ is:

$$dW = \frac{1}{2}\sigma_x dydz\varepsilon_x dx = \frac{1}{2}\sigma_x \varepsilon_x dv \quad \text{.......................................(2.15)}$$

For all stresses acting on **dv** (in index notation)

$$\begin{cases} dW = \frac{1}{2}\sigma_{ij}.\varepsilon_{ij}dv \\ \quad and \\ W = \frac{1}{2}\int_v \sigma_{ij}\varepsilon_{ij}dv \end{cases} \quad \text{..(2.16)}$$

Note:

In the context of linear elasticity, we have***:*** $W = W^*$**.**

3. Deformation Work Of Simple Loads In The Case Of Beams:

We will separately calculate the deformation work (deformation energy) based on the forces N, M, T and M_t in a beam (straight or curved) of length l. Consider a beam segment dx (ds) small enough to assume that the forces do not vary over dx.

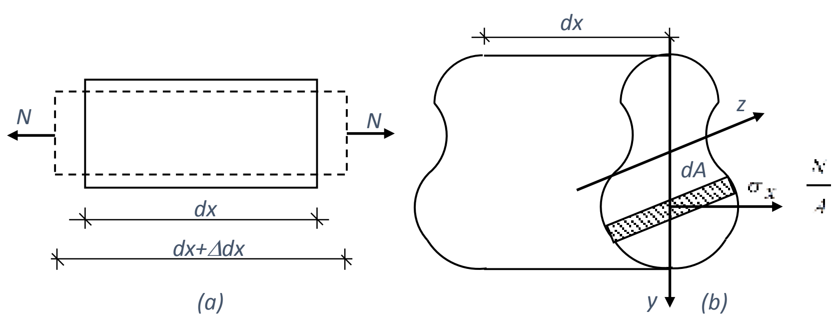


Figure 2.4 Deformation Work of Simple Stresses (Tension)

3. 1 NORMAL FORCE:

Under the influence of normal stress, the segment dx undergoes a change in length Δ***dx*** defined by:

$$\frac{\Delta dx}{dx} = \varepsilon_x \Rightarrow \Delta dx = \varepsilon_x dx = \frac{\sigma_x}{E} dx \quad \ldots\ldots(2.17)$$

As in the case of normal force where $\boldsymbol{\sigma_x = N/A}$, we have :

$$\boldsymbol{\Delta dx = (N/EA)dx} \quad \ldots\ldots(2.18)$$

The energy stored in the layer ***dA.dx*** is calculated as the work done by the force $\boldsymbol{\sigma_x . dA}$ during the displacement $\boldsymbol{\Delta dx}$, giving:

$$d^2W = \frac{1}{2}(\sigma_x dA)\Delta dx = \frac{1}{2}(\frac{N}{A}dA)\frac{N}{EA}dx = \frac{1}{2}\frac{N^2}{EA^2}dAdx \quad \ldots\ldots(2.19)$$

Note:

The notation $\boldsymbol{d^2W}$ is used to denote a quantity smaller than the elemental energy. The elemental energy stored in the segment ***dx*** is obtained by integrating over the area ***A*** of the section:

$$dW = \frac{dx}{2}\int_A \frac{N^2}{EA^2}dA = \frac{1}{2}\frac{N^2 dx}{EA^2}\int_A dA = \frac{N^2}{2EA}dx \quad \ldots\ldots(2.20)$$

And for the entire beam:

$$W = \frac{1}{2}\int_l \frac{N^2}{EA}dx \quad \ldots\ldots(2.21)$$

3.2 BENDING MOMENT:

Consider the layer ***dAdx***. Under the influence of bending stresses, the layer undergoes a change in length:$\boldsymbol{\Delta dx = \varepsilon_x dx = (\sigma_x/E)dx}$**.** Considering the Navier's relation, it follows::

$$\sigma_x = \frac{M_z y}{I_z} \Rightarrow \Delta dx = \frac{M_z y}{EI_z}dx \quad \ldots\ldots(2.22)$$

The energy stored in the layer ***dAdx*** is :

$$d^2W = \frac{1}{2}(\sigma_x dA)\Delta dx = \frac{1}{2}(\frac{M_z y}{I_z}dA)\frac{M_z y}{EI_z}dx = \frac{1}{2}\frac{M_z^2 y^2}{EI_z^2}dAdx \quad \ldots\ldots(2.23)$$

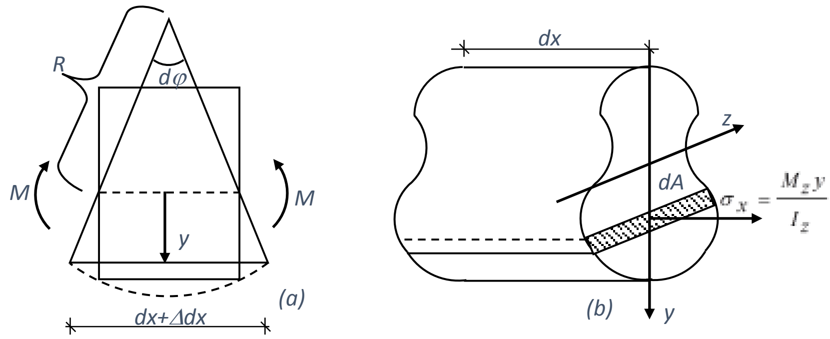


Figure 2.5 Deformation Work of Simple Stresses (Bending)

By integrating over the surface, we obtain the stored energy in the section ***dx*** :

$$dW = \frac{dx}{2}\int_A \frac{M_z^2 y^2}{EI_z^2} dA = \frac{1}{2}\frac{M_z^2 dx}{EI_z^2}\int_A y^2 dA = \frac{M_z^2}{2EI_z}dx \quad (2.24)$$

Thus, the deformation energy of the beam, calculated by integration over ***l*** *is*:

$$W = \frac{1}{2}\int_l \frac{M_z^2}{EI_z} dx \quad (2.25)$$

In the case of left bending, we have a relationship similar to the previous equation for each bending moment, and for both moments, we have:

$$W = \frac{1}{2}\int_l \left(\frac{M_z^2}{EI_z} + \frac{M_y^2}{EI_y}\right)dx \quad (2.26)$$

3.3 SHEAR FORCE:

The stored energy in a section *dx* subjected to a shear force T_y is given by:

$$dW = \frac{\kappa_y T_y^2}{2GA} dx \quad (2.27)$$

And for the entire beam:

$$W = \frac{1}{2}\int_l \frac{\kappa_y T_y^2}{GA} dx \quad (2.28)$$

If the beam is subjected to T_y and T_z, we will have:

$$W = \frac{1}{2}\int_l \left(\frac{\kappa_y T_y^2}{GA} + \frac{\kappa_z T_z^2}{GA}\right)dx \quad (2.29)$$

3.4 TORSION MOMENT::

The angle through which the extreme sections of the section ***dx*** rotate relative to each other under a torsion moment $\boldsymbol{M_t}$ is given by (Fig. 2.6): $d\varphi_t = \dfrac{qM_t}{GI_P}dx$

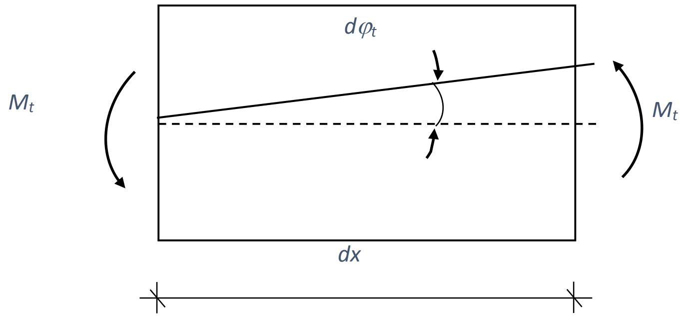


Figure 2.6 Deformation work of simple loads (torsion)

Where:

- q is a constant depending on the shape and dimensions of the section, referred to as the torsion coefficient ($\boldsymbol{q \approx 40I_P^2/A^4}$). This factor equals ***1*** for a circular section and is greater than ***1*** for other cases.

- The quantity $\boldsymbol{C = GI_P/q}$ is designated as torsional stiffness.

The energy stored in the differential section ***dx*** is calculated as the work done by $\boldsymbol{M_t}$ during the rotation $\boldsymbol{d\varphi_t}$:

$$dW = \frac{1}{2}M_t d\varphi_t = \frac{qM_t^2}{2GI_P}dx \qquad (2.30)$$

And for the entire beam:

$$W = \frac{1}{2}\int_l \frac{qM_t^2}{GI_P}dx \qquad (2.31)$$

4. GENERAL EXPRESSION OF THE POTENTIAL ENERGY OF DEFORMATION:

Consider within an elastic body a small element ***dv = dxdydz*** small enough to assume that stresses do not vary on the faces of the element.

Calculate the energy stored in the element ***dv*** when subjected to all stresses (Fig. 2.7a).

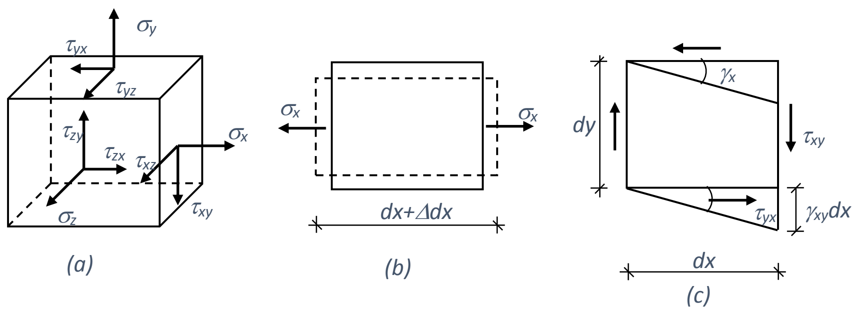


Figure 2.7 Stresses in the volume element dv

The deformation work of the force $\sigma_x dydz$ during the displacement $\Delta dx = \varepsilon_x dx$ (Fig. 2.7b) is given by :

$$dW = \frac{1}{2}(\sigma_x dydz)\varepsilon_x dx = \frac{1}{2}\sigma_x \varepsilon_x dxdydz \quad \text{...(2.32)}$$

For all three normal stresses:

$$dW = \frac{1}{2}(\sigma_x \varepsilon_x + \sigma_y \varepsilon_y + \sigma_z \varepsilon_z)dxdydz \quad \text{..(2.33)}$$

Where $\varepsilon_x, \varepsilon_y$ and ε_z are the longitudinal strains and can be expressed in terms of normal stresses from the generalized Hooke's law.

The strains induced by normal and tangential stresses being independent, if there are tangential stresses in addition to normal stresses, their effect can be simply added. The work of the force $\tau_{xy}dydz$ during the displacement $\gamma_{xy}dx$ (Fig. 2.7c) is given by:

$$dW = \frac{1}{2}(\tau_{xy} dydz)\gamma_{xy} dx = \frac{1}{2}\tau_{xy}\gamma_{xy} dxdydz \quad \text{...(2.34)}$$

In the presence of all stresses, it follows :

$$dW = \frac{1}{2}(\sigma_x\varepsilon_x + \sigma_y\varepsilon_y + \sigma_z\varepsilon_z + \tau_{xy}\gamma_{xy} + \tau_{yz}\gamma_{yz} + \tau_{zx}\gamma_{zx})dxdydz \quad \text{...............................(2.35)}$$

The potential energy of deformation for the entire body is obtained by summing over the entire volume:

$$W = \frac{1}{2}\int_v (\sigma_x\varepsilon_x + \sigma_y\varepsilon_y + \sigma_z\varepsilon_z + \tau_{xy}\gamma_{xy} + \tau_{yz}\gamma_{yz} + \tau_{zx}\gamma_{zx})dv \quad \text{..............................(2.36)}$$

The expression for ***W*** can be formulated in terms of stresses only or strains only by using the expressions of stresses in terms of strains given by the generalized Hooke's law. In the case of a beam subjected to ***N, M, T,*** and ***Mt*** loads, the expression for ***W*** is:

$$W = \frac{1}{2}\int_l \frac{M^2}{EI}dx + \frac{1}{2}\int_l \frac{N^2}{EA}dx + \frac{1}{2}\int_l \frac{\kappa T^2}{GA}dx + \frac{1}{2}\int_l \frac{qM_t^2}{GI_P}dx \quad \text{..................................(2.37)}$$

Note that this last expression does not result from the application of the superposition principle, which is not applicable since the energy is not linearly related to the loads.

5. CASTIGLIANO'S THEOREM:

5.1 FIRST FORM OF THE THEOREM:

Consider an elastic system subjected to a load ***F*** ***(F_1, F_2,…, F_n)***. During loading, the system deforms, and the points of application of the forces undergo displacements ***δ_1, δ_2,…, δ_n*** (***δ_i*** measured in the direction of ***F_i***).

The stored energy ***W*** in the system during loading can be expressed in terms of the forces or the displacements at their point of application.

$$W = W(F_1, F_2, \dots, F_n) = W(\delta_1, \delta_2, \dots, \delta_n) \quad \text{(2.38)}$$

Let's give the force ***F_i*** an increment ***dF_i***. This results in a variation in energy defined by the quantity ***($\partial W/\partial F_i$)dF_i*** and the total energy, under ***F*** ***(F_1, F_2, …, F_n)*** and ***dF_i***, is written as :

$$W + \frac{\partial W}{\partial F_i} dF_i \quad \text{(2.39)}$$

Since the work of forces does not depend on the order in which they are applied, let's first apply ***dF_i*** and then the overall load ***F*** ***(F_1, F_2, …, F_n).***

The infinitesimal force ***dF_i*** produces an infinitesimal displacement ***$d\delta_i$*** as well, so that the work done can be considered an infinitely small quantity of order 2 that is legitimate to neglect: ***(1/2) $dF_i d\delta_i \approx 0$.***

Now, let's apply the overall load ***F*** ***(F_1, F_2, …, F_n)***. The work done is equal to ***W*** **:**

τ_e = W. Furthermore, the force ***dF_i***, whose point of application has undergone a displacement ***δ_i***, produces work equal to ***$dF_i\delta_i$***.

Hence the total work::

$$\tau_e + dF_i\delta_i = W + dF_i\delta_i \quad \text{(2.40)}$$

The first form of Castigliano's theorem,

$$\frac{\partial W}{\partial F_i} = \delta_i \quad \text{(2.41)}$$

Can be stated as follows:

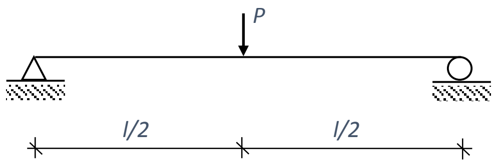
In an elastic system with undeformable supports, the derivative of the deformation energy with respect to one of the forces acting on the system is equal to the projection, in the direction of that force, of the elastic displacement at its point of application.

6. APPLICATIONS :

6.1 Example 1:

Consider a two-hinged beam with a constant section loaded in the middle by a concentrated force ***P***.

Calculate the deflection at mid-span (f).



6.1.2 Solution

The deflection value is obtained by directly applying the following formula:

$$f = \partial W/\partial P$$

$$W = \frac{1}{2}\int_l \frac{M_z^2}{EI_z}dx + \frac{1}{2}\int_l \frac{\kappa T_y^2}{GA}dx \text{ ....(1)}$$

Calculation of support reactions

In static equilibrium:

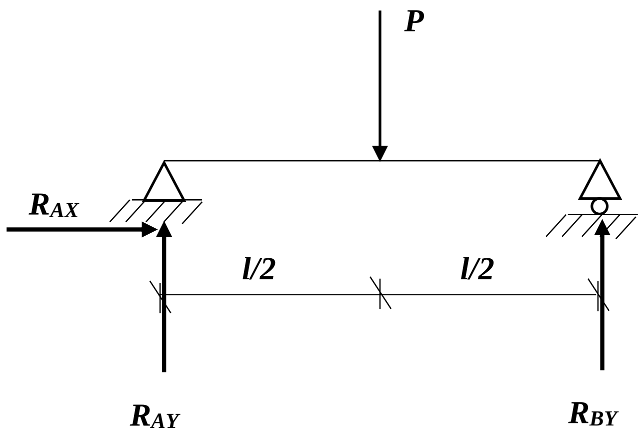
$$\sum \vec{F}_x = \sum \vec{F}_y = \sum \vec{M}_z = \vec{0} \text{ so :}$$

$$\sum F_x = 0 \Rightarrow R_{AX} = 0$$

$$\sum F_y = 0 \Rightarrow R_{AY} + R_{BY} = P\text{.........(*)}$$

$$\Rightarrow \sum M_z / A = 0 \Rightarrow R_{BY}.l - P.\frac{l}{2} = 0$$

$$\Rightarrow R_{BY} = \frac{P.l}{2.l} = \frac{P}{2}\text{........................(**)}$$

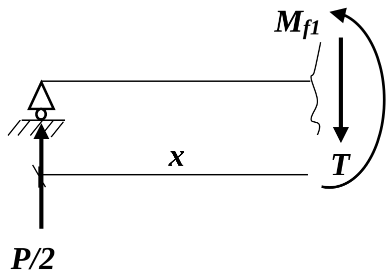


Replacing (**) into (*), we find:

$$R_{AY} = R_{BY} = \frac{P}{2}$$

By the method of sections, we can determine the internal forces:

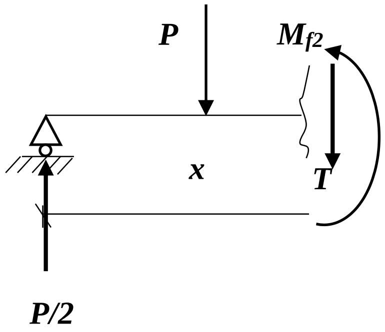
1st section: $0 \le x \le \frac{l}{2}$



$$M_{f1}(x) = \frac{p}{2}..x$$

$$T_1(x) = \frac{P}{2}$$

2nd section: $\frac{l}{2} \le x \le l$



$$M_{f2}(x) = \frac{P}{2}.(l - x)$$

$$T(x) = -\frac{P}{2}$$

Substituting these values into (1), we obtain:

$$W = \frac{1}{2EI_z}[\int_0^{1/2}(\frac{P}{2}x)^2 dx + \int_{1/2}^{l}(\frac{P}{2}(l-x))^2 dx] + \frac{\kappa}{2GA}[\int_0^{1/2}(\frac{P}{2})^2 dx + \int_{1/2}^{l}(-\frac{P}{2})^2 dx]$$

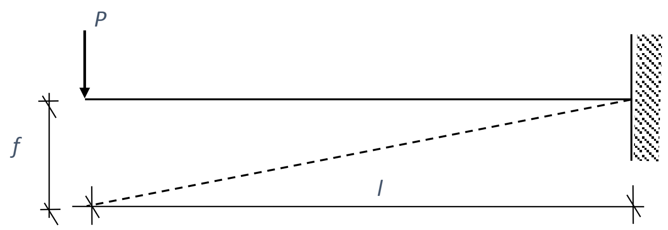
$$W = \frac{P^2 l^3}{96 EI_z} + \frac{\kappa P^2 l}{8GA}$$

Therefore:

$$f = \frac{Pl^3}{48 EI_z} + \frac{\kappa Pl}{4GA}$$

6.2 Example 2:

Calculate the displacement of the point of application of the load P (assuming constant flexural stiffness).



6.2.1 Solution

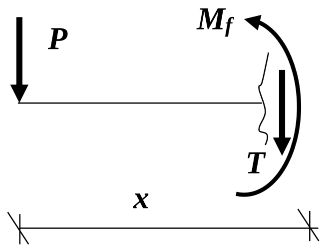
The value of the deflection is obtained by the direct application of the following formula:

$f = \partial W/\partial P$

$$W = \frac{1}{2}\int_l \frac{M_z^2}{EI_z}dx + \frac{1}{2}\int_l \frac{\kappa T_y^2}{GA}dx \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots.(1)$$

By the method of sections, we can determine the internal forces:

1st section: $0 \le x \le l$



$$M_f(x) = -P.l$$
$$T = -P$$

Substituting these values into (1), we obtain:

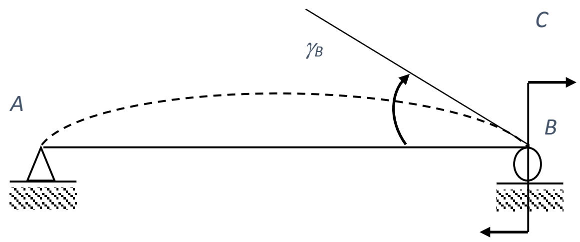
$$W = \frac{l}{2EI_z}\int_0^l (-Px)^2\,dx + \frac{\kappa}{2GA}\int_0^l (-P)^2\,dx$$
$$W = \frac{P^2 l^3}{6EI_z} + \frac{\kappa P^2 l}{2GA}$$

Therefore:

$$f = \frac{Pl^3}{3EI_z} + \frac{\kappa Pl}{GA}$$

6.3 Example 3:

Calculate the rotation of the end B of the beam shown.



6.3.1 Solution

$$W = \frac{1}{2EI_z}\int_0^l (-\frac{C}{l}x)^2\,dx + \frac{\kappa}{2GA}\int_0^l (-\frac{C}{l})^2\,dx = \frac{C^2 l}{6EI_z} + \frac{\kappa C^2}{2GAl}$$

Therefore:

$$\gamma_B = \frac{\partial W}{\partial C} = \frac{Cl}{3EI_z} + \frac{\kappa C}{GAl}$$

CHAPITRE 3 COMPOUND STRESSES.

GENERALITIES

A stress is considered compound if there is more than one internal force in a cross-section of a part

Let ***ox***, ***oy*** and ***oz*** be the principal axes (figure 3.1)

In this section, we focus on the most common compound stresses:

- Deviated bending: (M_z , T_y , M_y , T_z) ;
- Compound bending: (M_z , T_y, M_y , T_z , N_x) ;
- Bending-torsion: (M_z , T_y, M_x) ou (M_y , T_z , M_x).
-

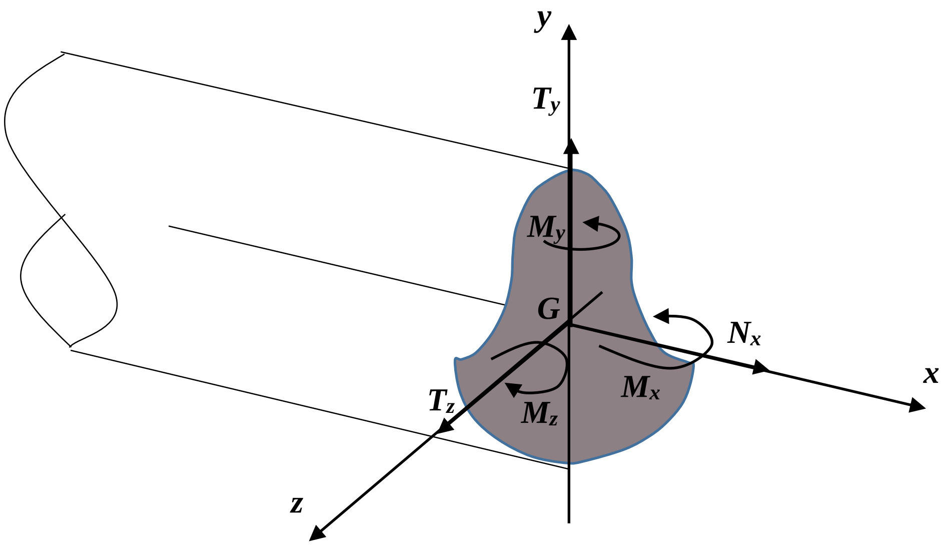


Figure 3.1 Internal forces.

1. DEVIATED BENDING

Bending is considered deviated if, in a cross-section of the part, the internal ***forces (Mz, Ty, My,*** and ***Tz)*** acting on the principal axes are not zero (Figure 3.2).

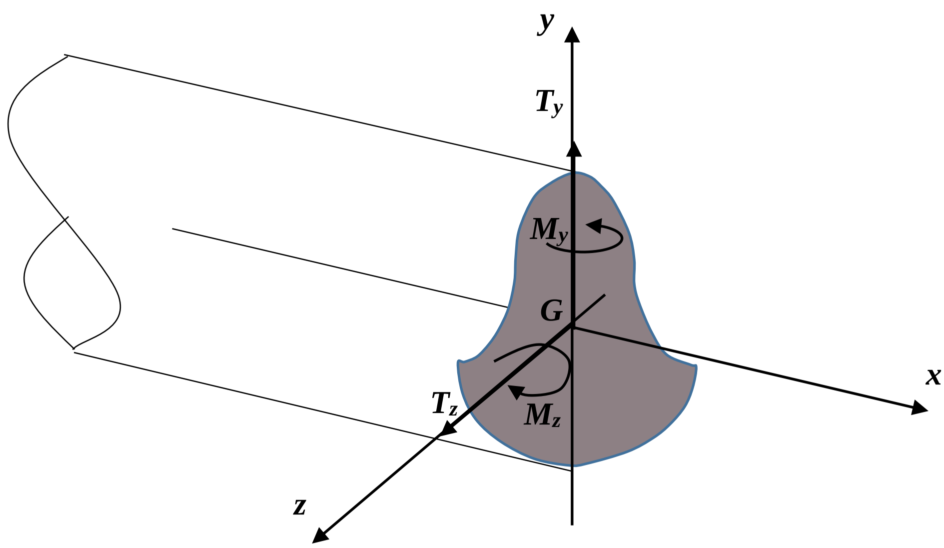


Figure 3.2 Bending Deflection.

1.1 NORMAL STRESS

The normal stress is given by:

$$\sigma_x = \frac{M_z}{I_z} y + \frac{M_y}{I_y} z \quad \text{..(3.1)}$$

The neutral axis (AN) is defined as the set of points where the normal stress is zero.

For a point *P(x₀ ; y₀)* belonging to the neutral axis, we have:

M_y and ***M_z*** are the moments components ***M***. Therefore:

$$\begin{cases} M_z = M\cos\alpha \\ M_y = M\sin\alpha \end{cases} \Rightarrow \tan\alpha = \frac{M_y}{M_z} \quad \text{...(3.3)}$$

By substituting equation [4.3] into [4.2], we get:

$$\tan\beta = \frac{I_z}{I_y}\tan\alpha \quad \text{..(3.4)}$$

$$\frac{M_z}{I_z} y_0 + \frac{M_y}{I_y} z_0 = 0 \Rightarrow \frac{y_0}{z_0} = -\frac{M_y}{M_z}\frac{I_z}{I_y} = \tan\beta \quad \text{..(3.2)}$$

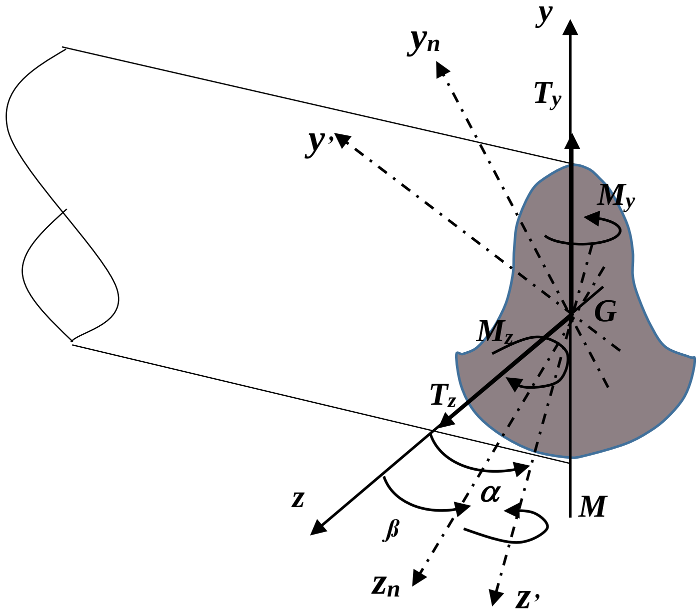


Figure 3.3 Coordinates of a point belonging to the neutral axis.

1.2 SHEAR STRESS

The shear stress at a point is given by:

$$\tau = \sqrt{\tau_{xy}^2 + \tau_{xz}^2} \quad \text{......(3.5)}$$

With:

$$\begin{cases} \tau_{xy} = \dfrac{T_y}{b}\dfrac{S_z}{I_z} \\ \tau_{xz} = \dfrac{T_z}{b}\dfrac{S_y}{I_y} \end{cases} \quad \text{......(3.6)}$$

1.3 STRENGTH CALCULATION

The steps for calculating strength are:

- - Determine the critical sections (areas where internal forces are maximum) and the critical points in the section (points farthest from the neutral axis).
- - Calculate the value of normal stress and verify that it is less than the allowable stress:

$$\sigma_x \leq \overline{\sigma} \quad \text{......(3.7)}$$

$\overline{\sigma}$: Material's allowable stress.

- Check for a failure criterion (for ductile materials): :

$$\sqrt{\sigma_x^2 + 3\tau^2} \leq \overline{\sigma} \qquad \text{(Von Mises Criterion)} \ldots\ldots(3.8)$$

1.4 STIFFNESS CALCULATION:

The deflection is calculated using the differential equation of the elastic curve:

$$EI_z \frac{d^2 f}{dx^2} = M_z \ldots\ldots(3.9)$$

With:

$$f = \sqrt{[v(x)]^2 + [w(x)]^2} \ldots\ldots(3.10)$$

$v(x)$ and $w(x)$ are the displacements along y and z, respectively.

The maximum deflection must satisfy the condition:

$$f \leq \overline{f} \ldots\ldots(3.11)$$

$$\overline{f} = \frac{l}{150} \div \frac{l}{1000}$$

With:

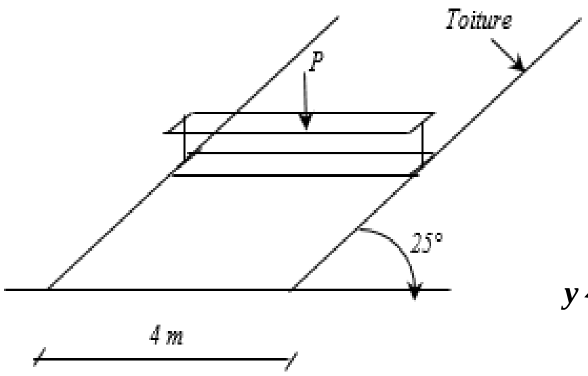
l : the beam span.

$\overline{f}$: the allowable deflection.

The deflection check is performed after the strength check.

1.5 APPLICATION :

Calculate the maximum load ***P*** that an ***IPE140*** beam can withstand while adhering to the strength conditions ***[σ]=120 MPa.***



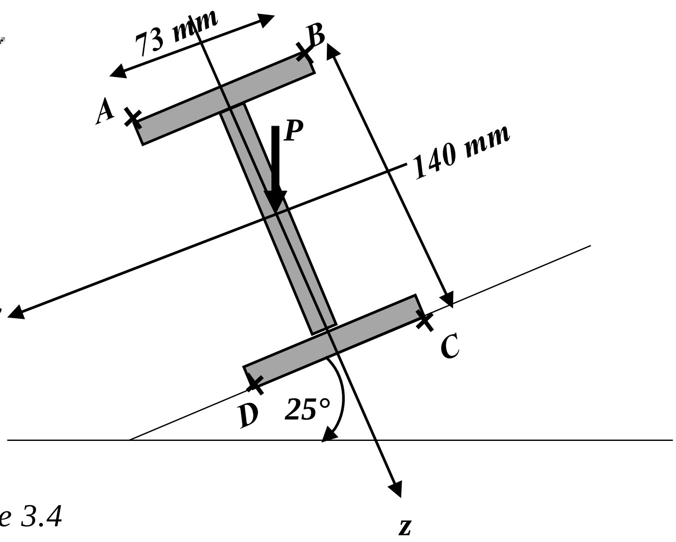
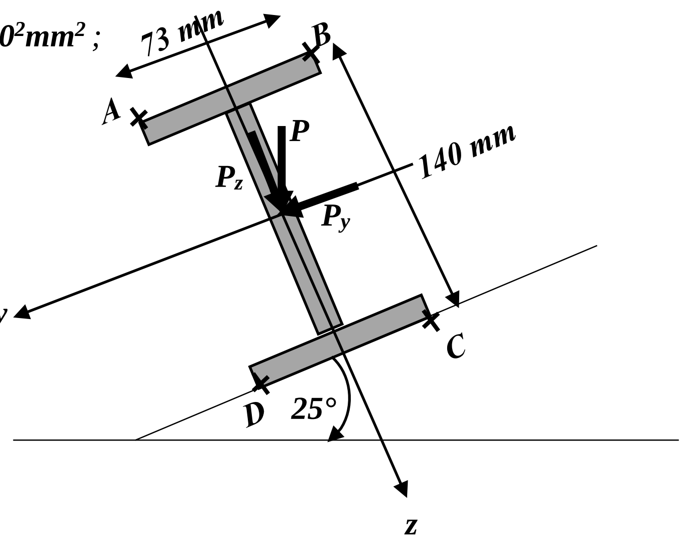


Figure 3.4

a) Position of the IPE140 beam in space *b) Position of the IPE140 beam in the plane*

1.5.1 Solution

Geometric characteristics of an ***IPE140 :A=16,4.10²mm² ;*** ***I_y=541,2.10⁴mm⁴ ;I_z=44,9.10⁴mm⁴***

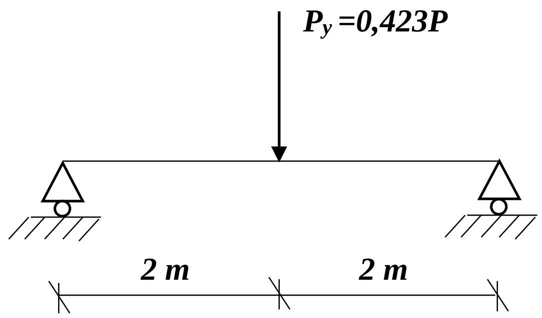


The loading vector ***P*** can be decomposed as:

$\vec{P} = P_y.\vec{j} + P_z.\vec{k}$ With :

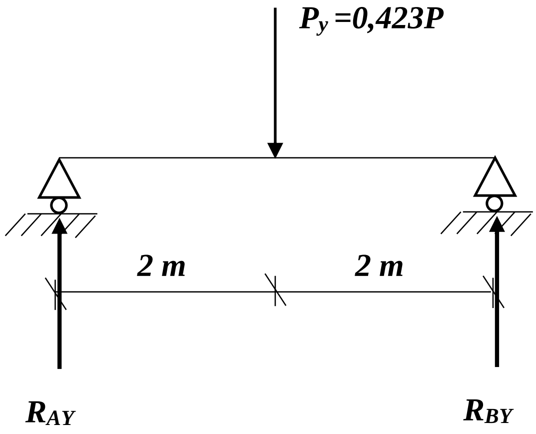
$$\begin{cases} P_y = P\sin 25 = 0{,}423P \\ P_y = P\cos 25 = 0{,}908P \end{cases}$$

a) Force balance for the "xy" plane:



Calculation of support reactions

In static equilibrium:



$$\sum \vec{F}_x = \sum \vec{F}_y = \sum \vec{M}_z = \vec{0}$$

thus:

$$\sum \mathrm{F_y} = 0 \Rightarrow \mathrm{R_{AY}} + \mathrm{R_{BY}} = \mathrm{P_y} = 0,423P.........(*)$$

$$\Rightarrow \sum M_z / A = 0 \Rightarrow \mathrm{R_{BY}}.4 - \mathrm{P_y}.2 = 0$$

$$\Rightarrow \mathrm{R_{BY}} = \frac{\mathrm{P_y}.2}{4} = \frac{0,423P.2}{4} = 0,212P...........(**)$$

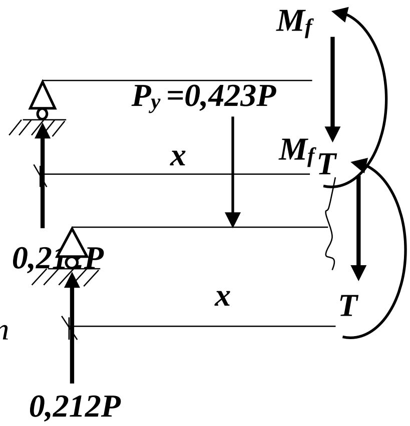
Replacing (**) into (*), we find:

$$\mathrm{R_{AY}} = \mathrm{R_{BY}} = 0,212P$$

By the method of sections, we can plot the diagrams of internal forces:

1st section: $0 \le x \le 2$

$$M_f(x) = 0,212P.x \quad \begin{cases} x = 0 \Rightarrow M_f(0) = 0\,Kn.m \\ x = 2 \Rightarrow M_f(2) = 0,423P\,Kn.m \end{cases}$$



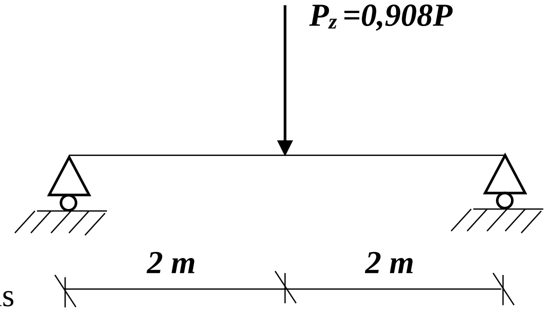
$$T(x) = 0,212P \quad \begin{cases} x = 0 \Rightarrow T(0) = 0,212P\,Kn \\ x = 2 \Rightarrow T(2) = 0,212P\,Kn \end{cases}$$

2nd section : $2 \le x \le 4$

$$M_f(x) = 0,212P.x - 0,423P(x-2) \quad \begin{cases} x = 2 \Rightarrow M_f(0) = 0,423P\,Kn.m \\ x = 4 \Rightarrow M_f(2) = 0\,Kn.m \end{cases}$$

$$T(x) = 0,212P \quad \begin{cases} x = 0 \Rightarrow T(0) = 0,212P\,Kn \\ x = 2 \Rightarrow T(2) = 0,212P\,Kn \end{cases}$$

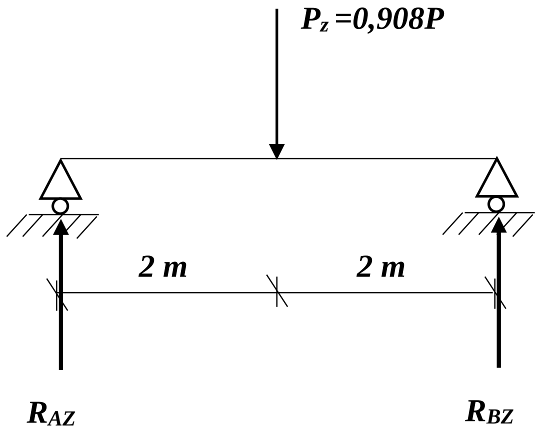
b) Force balance for the "xz" plane:



Calculation of support reactions

In static equilibrium:

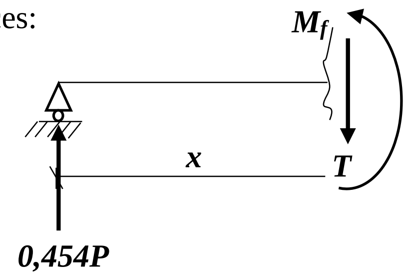
$$\sum \vec{F}_x = \sum \vec{F}_y = \sum \vec{M}_y = \vec{0}$$



thus :

$$\sum \mathrm{F_y} = 0 \Rightarrow \mathrm{R_{AZ}} + \mathrm{R_{BZ}} = \mathrm{P_z} = 0{,}908P.........(*)$$
$$\Rightarrow \sum M_z / A = 0 \Rightarrow \mathrm{R_{BZ}}.4 - \mathrm{P_z}.2 = 0$$
$$\Rightarrow \mathrm{R_{BZ}} = \frac{\mathrm{P_z}.2}{4} = \frac{0{,}908P.2}{4} = 0{,}454P...........(**)$$
On remplace (**) dans (*) on trouve :
$$\mathrm{R_{AZ}} = \mathrm{R_{BZ}} = 0{,}454P$$

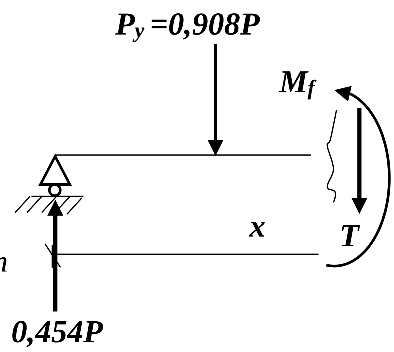
By the method of sections, we can plot the diagrams of internal forces:



1^{st}*section :* $0 \le x \le 2$

$$M_f(x) = 0{,}454P.x \quad \begin{cases} x = 0 \Rightarrow M_f(0) = 0\,Kn.m \\ x = 2 \Rightarrow M_f(2) = 0{,}908P\,Kn.m \end{cases}$$

$$T(x) = 0{,}454P \quad \begin{cases} x = 0 \Rightarrow T(0) = 0{,}454P\,Kn \\ x = 2 \Rightarrow T(2) = 0{,}454P\,Kn \end{cases}$$



2^{nd} *section :* $2 \le x \le 4$

$$M_f(x) = 0{,}454P.x - 0{,}908P(x-2) \quad \begin{cases} x = 2 \Rightarrow M_f(0) = 0{,}908P\,Kn.m \\ x = 4 \Rightarrow M_f(2) = 0\,Kn.m \end{cases}$$

$$T(x) = 0{,}454P \quad \begin{cases} x = 0 \Rightarrow T(0) = 0{,}454P\,Kn \\ x = 2 \Rightarrow T(2) = 0{,}454P\,Kn \end{cases}$$

c) Diagrams of internal forces

Plane « xoy » **Plane « xoz »**

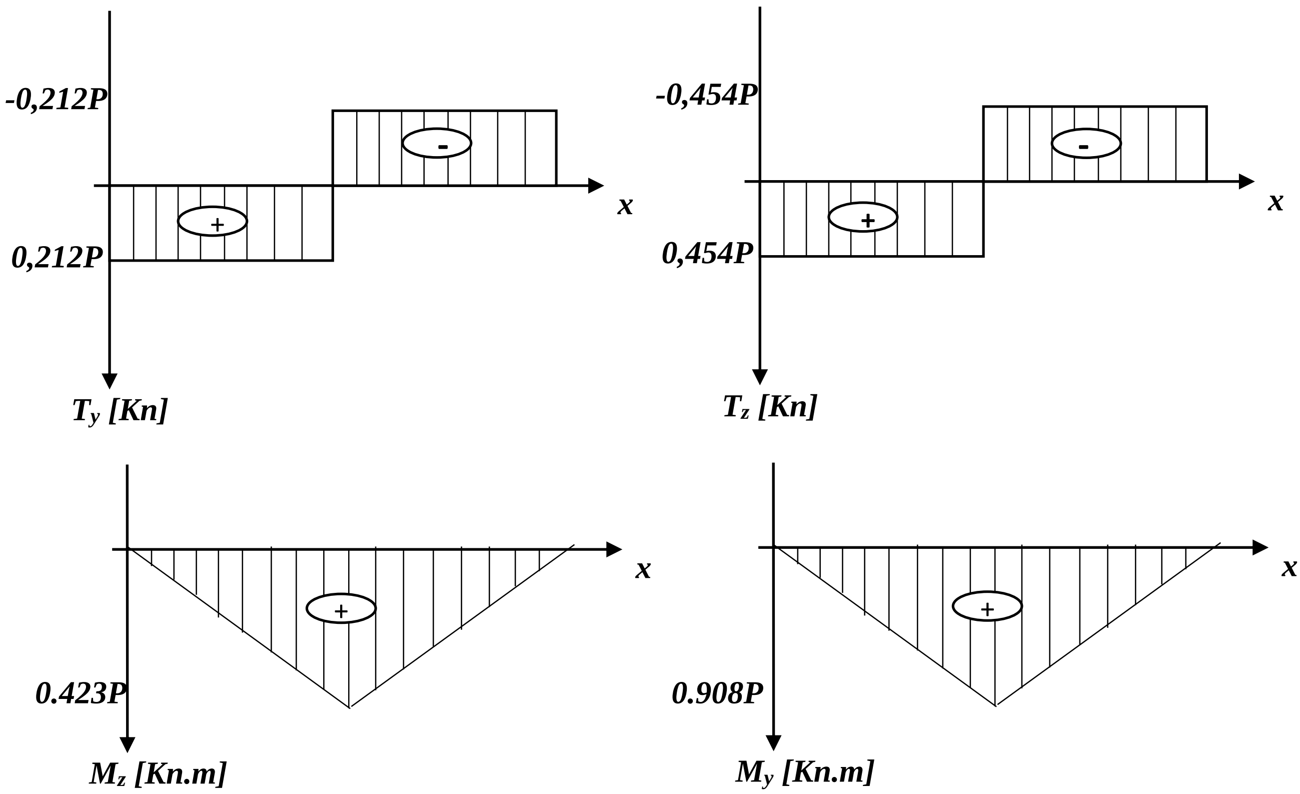


Figure 3.5 Diagrams of internal forces.

d) Neutral axis equation

We have: $\sigma(x) = \dfrac{M_z}{I_z} y + \dfrac{M_y}{I_y} z$

So for *x=o*

$$\sigma(0) = \frac{M_z}{I_z} y_0 + \frac{M_y}{I_y} z_0 = 0 \Rightarrow \frac{y_0}{z_0} = -\frac{M_y}{M_z}\frac{I_z}{I_y} = \tan\beta$$

$$-\frac{M_y}{M_z}\frac{I_z}{I_y} = \frac{-0{,}908P}{0{,}423P} . \frac{44{,}9.10^4}{541{,}2.10^4} \approx -0{,}17 \Rightarrow \beta = -9{,}64°$$

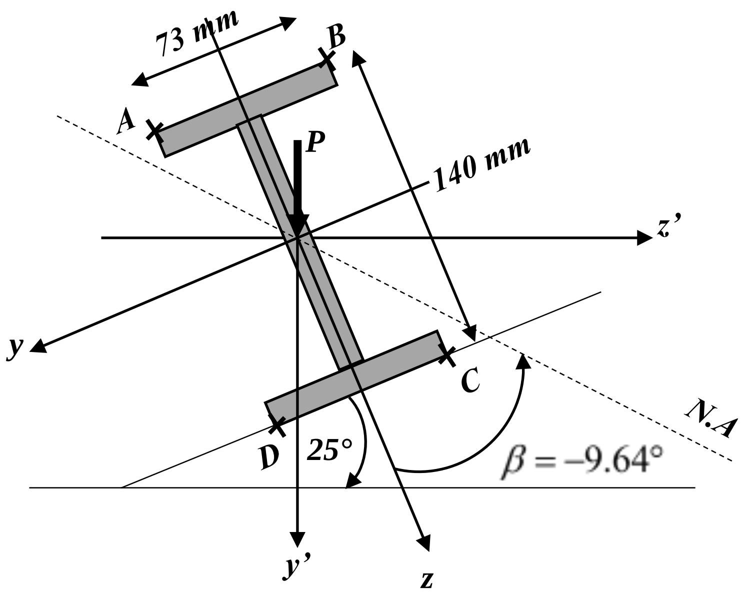


Figure 3.6 Position of the neutral axis in the plane.

This is the equation of a line passing through the center of gravity of the section. The farthest point from the neutral axis is point *B (-36,5 ;-70)* so the maximum stress at this point is given by:

$$\sigma_x^B = \frac{M_z}{I_z} y_B + \frac{M_y}{I_y} z_B = \frac{0{,}423P}{44{,}9.10^4}.(-36{,}5) + \frac{0{,}908P}{501{,}2.10^4}.(-70)$$

$$\Rightarrow \sigma_x^B = -0{,}047P\, Kn/mm^2 = 47{,}06P \text{ MPa}$$

e) Strength calculation

$$\sigma^{\max} \le [\sigma] \Rightarrow \sigma_x^B \le [\sigma] \Rightarrow 47{,}06P \le 120 \Rightarrow P \le 2.6Kn$$

2. COMBINED BENDING

Bending is considered combined if, in a cross-section of the piece, the internal forces (***N****x*, ***M****z* , ***T****y* , ***M****y* and ***T****z*) acting on the principal axes are not zero (figure 3.7.)

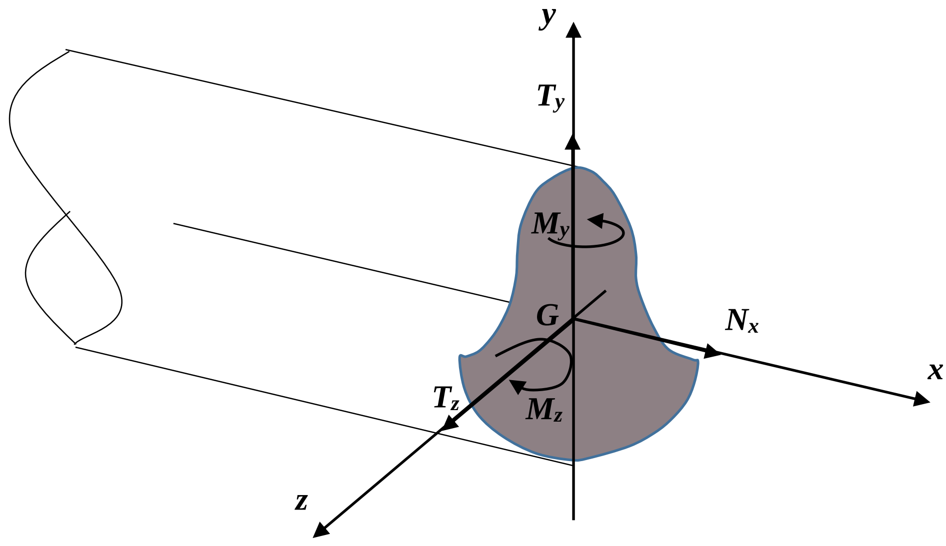


Figure 3.7 Combined bending.

2.1 NORMAL STRESS

The normal stress is given by:

$$\sigma_x = \frac{N_x}{A} + \frac{M_z}{I_z} y + \frac{M_y}{I_y} z \quad \text{………………………………………………………...…….(3.12)}$$

A : being the area of the cross-section of the piece.

Neutral Axis Equation

Let a point ***P(x₀ ; y₀)*** belong to the ***NA*** axis. Then:

$$\sigma_x = \frac{N_x}{A} + \frac{M_z}{I_z} y_0 + \frac{M_y}{I_y} z_0 = 0 \quad \text{…………………………………..………………(3.13)}$$

On the other hand (fig 3.7)

$$\begin{cases} N = P \\ M_z = P.y_P \\ M_y = P.z_P \end{cases} \quad \text{………………………………………………………….………..….(3.14)}$$

By substituting equation [3.13] into [3.14], we obtain:

$$1+\frac{y_P}{i_z^2}y_0+\frac{z_P}{i_y^2}z_0=0 \quad (3.15)$$

With:

$$i_y^2=\frac{I_y}{A} \quad \text{et} \quad i_z^2=\frac{I_z}{A} \quad (3.16)$$

i_y and i_z are the radii of gyration.

2.2 SHEAR STRESS

The shear stress at a point is given by:

$$\tau=\sqrt{\tau_{xy}^2+\tau_{xz}^2} \quad (3.17)$$

With:

$$\begin{cases}\tau_{xy}=\dfrac{T_y}{b}\dfrac{S_z}{I_z}\\ \tau_{xz}=\dfrac{T_z}{b}\dfrac{S_y}{I_y}\end{cases} \quad (3.18)$$

2.3 STRENGTH CALCULATION

The steps for calculating strength are as follows:

- Identify critical sections (zones where internal forces are maximum) and critical points within the section (points furthest from the neutral axis).
- Calculate the value of normal stress and verify that it is less than the allowable stress value:

$$\sigma_x \le \overline{\sigma} \quad (3.19)$$

$\overline{\sigma}$: Allowable stress of the material.

2.4 STIFFNESS CALCULATION:

The maximum deflection must satisfy the condition:

$$f \le \overline{f} \quad (3.20)$$

Note:

- In the case of compound bending, the neutral axis does not pass through the centroid of the cross-sectional area.
- σ_x is maximum for points furthest from the neutral axis.
- For strength calculation, as in the case of deflected bending, it is sufficient to verify that: $\sigma_x \le \overline{\sigma}$ .The term $\tau = \sqrt{\tau_{xy}^2 + \tau_{xz}^2}$ is negligible.

2.5 APPLICATION :

Calculate the maximum load ***P*** that an ***IPE140*** beam can support while meeting strength conditions. ***[σ]=120 MPa.***

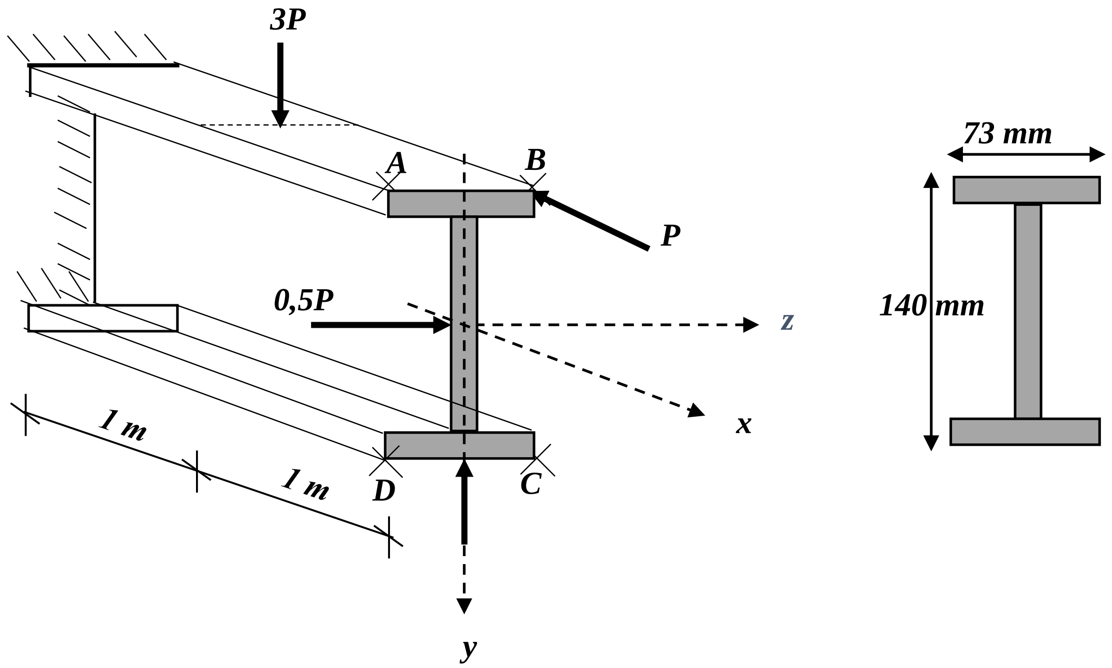


Figure 3.8 Embedded Console at One End.

2.5.1 Solution

Geometric characteristics of an ***IPE140 :*** $A=16,4.10^2 mm^2$ ***;*** $I_y=541,2.10^4 mm^4$ ***;*** $I_z=44,9.10^4 mm^4$.

Force balance for the "xy" plane:

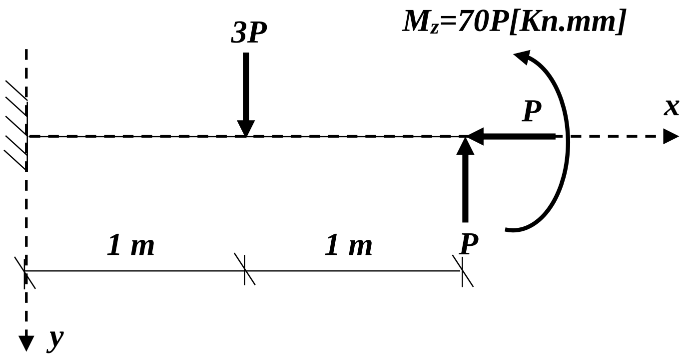


Figure 3.9 Force balance for the "xy" plane

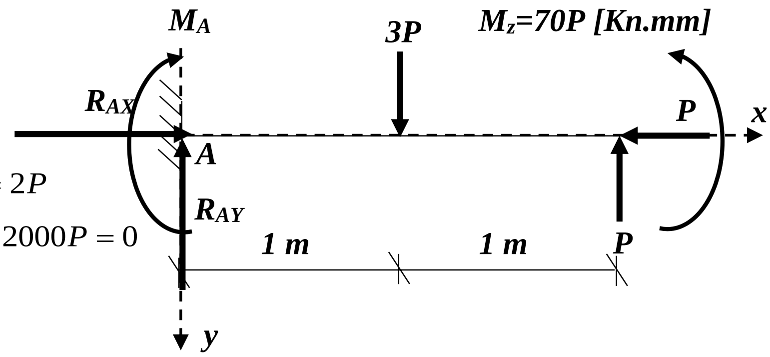
Calculation of moments:

$$M_z = P.\frac{140}{2} \Rightarrow M_z = 70P[Kn.mm]$$

Calculation of support reactions (embedding)

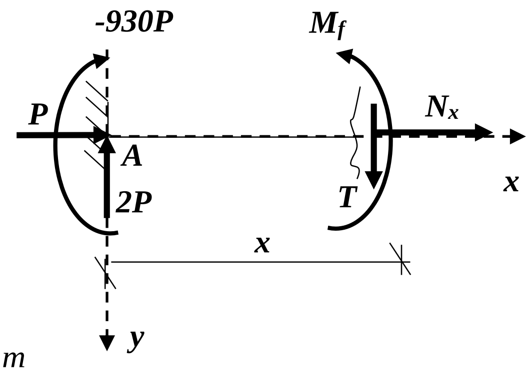
In static equilibrium::

$\sum \vec{F}_x = \sum \vec{F}_y = \sum \vec{M}_z = \vec{0}$ thus:



$$\sum F_x = 0 \Rightarrow R_X - P = 0 \Rightarrow R_X = P$$
$$\sum F_y = 0 \Rightarrow R_{AY} - 3P + P = 0 \Rightarrow R_{AY} = 2P$$
$$\sum M_z / A = 0 \Rightarrow M_A - 70P + 3000P - 2000P = 0$$
$$\Rightarrow M_A = 2000P + 70P - 3000P$$
$$\Rightarrow M_A = -930P$$

By the method of sections, we can plot the diagrams of internal forces:

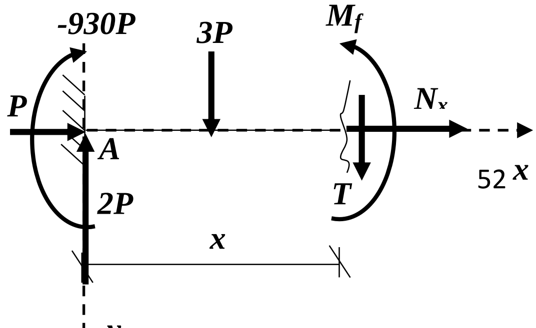


1st section : $0 \le x \le 1000$

$$\sum F / x = 0 \Rightarrow N + P = 0 \Rightarrow N = -P$$

$$M_f(x) = 2P.x - 930P \quad \begin{cases} x = 0 \Rightarrow M_f(0) = -930P\ Kn.mm \\ x = 1000 \Rightarrow M_f(1000) = 1070P\ Kn.mm \end{cases}$$

2nd section : $1000 \le x \le 2000$



$$\sum F / x = 0 \Rightarrow N + P = 0 \Rightarrow N = -P$$

$$M_f(x) = 2P.x - 930P - 3P(x - 1000)$$

$$\begin{cases} x = 0 \Rightarrow M_f(1000) = 1070P \; Kn.mm \\ x = 1000 \Rightarrow M_f(2000) = 70P \; Kn.mm \end{cases}$$

a) Force balance for the "xz" plane:

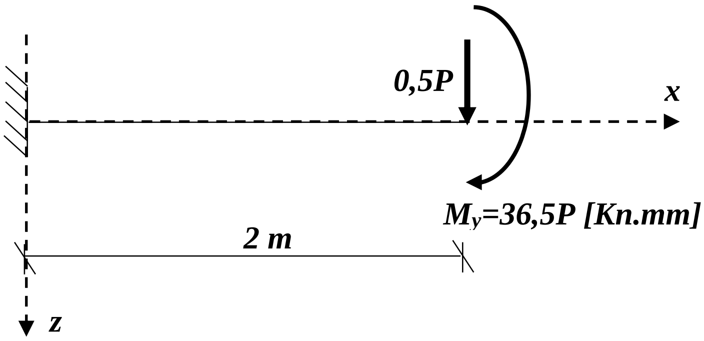


Figure 3.10 Force balance for the "xy" plane

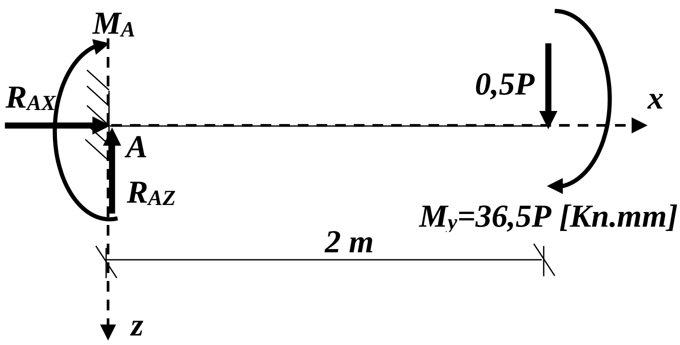
Calculation of moments:

$$M_y = P.\frac{73}{2} \Rightarrow M_z = 36{,}5P[Kn.mm]$$

Calculation of support reactions (embedding)

In static equilibrium:

$\sum \vec{F}_x = \sum \vec{F}_z = \sum \vec{M}_y = \vec{0}$; Therefore:

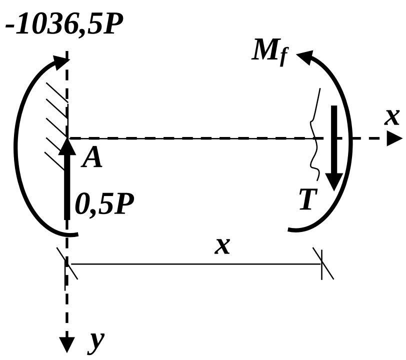


$$\sum F_x = 0 \Rightarrow R_{AX} = 0$$

$$\sum F_z = 0 \Rightarrow R_{AZ} - 0{,}5P = 0 \Rightarrow R_{AZ} = 0{,}5P$$

$$\sum M_y / A = 0 \Rightarrow M_A + 36{,}5P + 0{,}5P.2000 = 0$$

$$\Rightarrow M_A = -1036{,}5P$$



By the method of sections, we can plot the diagrams of internal forces:

1st section : $0 \leq x \leq 2000$

$$M_f(x) = -1036P + 0{,}5P.x \quad \begin{cases} x = 0 \Rightarrow M_f(0) = -1036{,}5P\ Kn.mm \\ x = 2000 \Rightarrow M_f(2000) = -36{,}5P\ Kn.mm \\ x = 1000 \Rightarrow M_f(1000) = -536{,}5P\ Kn.mm \end{cases}$$

a) Diagrams of internal forces

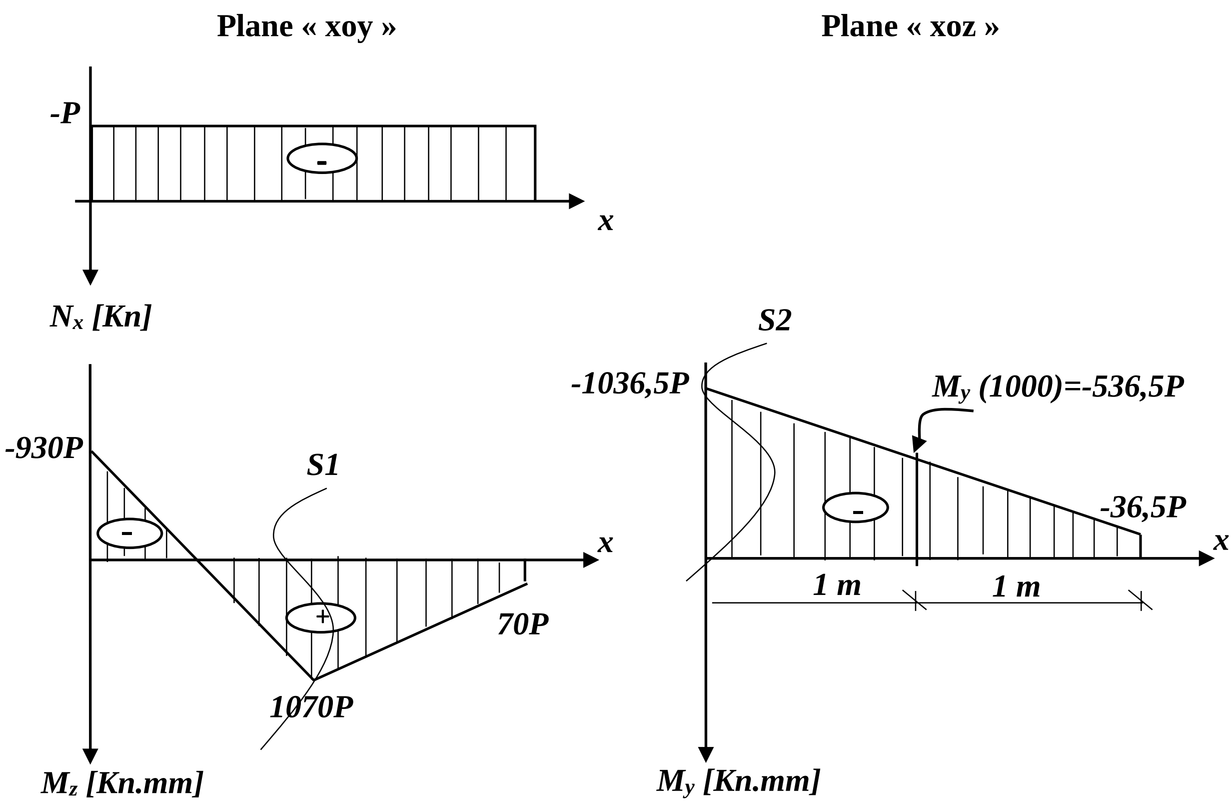


Figure 3.11 Diagrams of internal forces

b) Neutral axis equation

From the internal forces diagrams, we observe that there are two (02) critical sections: ***S1*** and ***S2***.

$$S_1 : \begin{cases} N_x = -P \\ M_z = -930P \\ M_y = -1036{,}5P \end{cases} \qquad S_2 : \begin{cases} N_x = -P \\ M_z = 1070P \\ M_y = -536{,}5P \end{cases}$$

For section ***S1 :***

We have: $\sigma_x = \frac{N_x}{A} + \frac{M_z}{I_z} y + \frac{M_y}{I_y} z$

So for *x=0:*

$$\sigma(0)=0 \Rightarrow \frac{N_x}{A}+\frac{M_z}{I_z}y_0+\frac{M_y}{I_y}z_0=0 \Rightarrow \frac{-P}{A}-\frac{930P}{I_z}y_0-\frac{1036{,}5P}{I_y}z_0=0$$

To plot the neutral axis, it is sufficient to define two points belonging to this axis:

- Take point *E* $(y_0\,;\,z_0)$ with $y_0=0$ and calculate the value of z_0. Then,

$$\sigma=0 \Rightarrow \frac{-P}{A}-\frac{930P}{I_z}(0)-\frac{1036{,}5P}{I_y}z_0=0 \Rightarrow z_0=\frac{P}{A}.\frac{-I_y}{1036{,}5P}=\frac{-I_y}{1036{,}5.A}$$

$$z_0=\frac{-541{,}2.10^4}{16{,}4.10^2.1036{,}5}=-3{,}18$$

So, *E (0;-3.18).*

- We take the point *G* $(y_0\,;\,z_0)$ with $z_0=0$ and calculate the value of y_0.Thus,

$$\sigma=0 \Rightarrow \frac{-P}{A}-\frac{930P}{I_z}y_0-\frac{1036{,}5P}{I_y}(0)=0 \Rightarrow y_0=\frac{P}{A}.\frac{-I_z}{930P}=\frac{-I_z}{930.A}$$

$$y_0=\frac{-44.9.10^4}{16{,}4.10^2.930}=-0{,}29$$

So, *G (-0,29 ;0).*

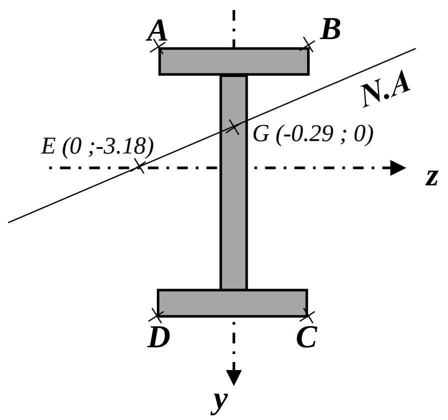


Figure 3.12 Position of the neutral axis for S1.

The furthest point from the neutral axis is point *C (-70 ;-36,5)* so the maximum stress is located at this point and is given by:

$$\sigma_x^C=\frac{-P}{16{,}4.10^2}-\frac{930P}{44{,}9.10^4}(70)-\frac{1036{,}5P}{541{,}2.10^4}(36{,}5)$$

$$\sigma_x^C=0{,}152589P\approx 0{,}153P$$

c) Strength condition

$$\sigma^{\max}\le[\sigma] \Rightarrow \sigma_x^C\le 120 \Rightarrow 0{,}153P\le 120 \Rightarrow P\le 784{,}31N$$

$P \leq 0{,}78KN$ ………………….(1)

For section ***S2 :***

We have: $\sigma_x = \dfrac{N_x}{A} + \dfrac{M_z}{I_z} y + \dfrac{M_y}{I_y} z$

So, for *x=0*

$$\sigma(0) = 0 \Rightarrow \frac{N_x}{A} + \frac{M_z}{I_z} y_0 + \frac{M_y}{I_y} z_0 = 0 \Rightarrow \frac{-P}{A} + \frac{1070P}{I_z} y_0 - \frac{536{,}5P}{I_y} z_0 = 0$$

To plot the neutral axis, it is sufficient to define two points belonging to this axis:

- Take point *E' (y_0 ; z_0)* with $y_0 = 0$ and calculate the value of z_0. Then,

$$\sigma = 0 \Rightarrow \frac{-P}{A} + \frac{1070P}{I_z}(0) - \frac{536{,}5P}{I_y} z_0 = 0 \Rightarrow z_0 = \frac{P}{A} \cdot \frac{-I_y}{536{,}5P} = \frac{-I_y}{536{,}5.A}$$

$$z_0 = \frac{-541{,}2.10^4}{16{,}4.10^2.536{,}5} = -6{,}15$$

So, *E' (0 ;-6,15).*

- We take the point *G' (y_0 ; z_0)* with $z_0 = 0$ and calculate the value of y_0.Thus,

$$\sigma = 0 \Rightarrow \frac{-P}{A} + \frac{1070P}{I_z} y_0 - \frac{536{,}5P}{I_y}(0) = 0 \Rightarrow y_0 = \frac{P}{A} \cdot \frac{I_z}{1070P} = \frac{I_z}{1070.A}$$

$$y_0 = \frac{44.9.10^4}{16{,}4.10^2.1070} = 0{,}26$$

Donc *G' (0,26 ;0).*

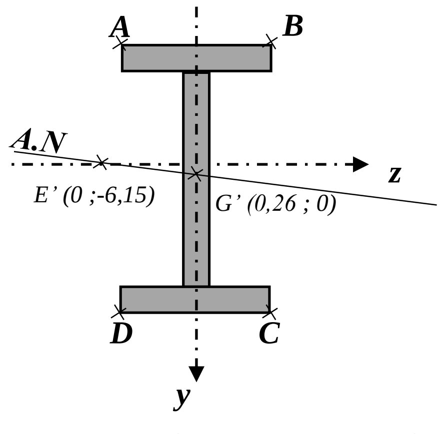


Figure 3.13 Position of the neutral axis for S2.

The furthest point from the neutral axis is point *B (-70 ; 36,5)* so the maximum stress is located at this point and is given by:

$$\sigma_x^B = \frac{-P}{16{,}4.10^2} + \frac{1070P}{44{,}9.10^4}(-70) - \frac{536{,}5P}{541{,}2.10^4}(36{,}5)$$

$$\sigma_x^B = 0{,}17P \approx 0{,}171P$$

a) Strength condition

$$\sigma^{\max} \le [\sigma] \Rightarrow \sigma_x^B \le 120 \Rightarrow 0{,}171P \le 120 \Rightarrow P \le 701N$$

$P \le 0{,}701KN$ ………………….(2)

From equations (1) and (2), we find that:

$P \le 0{,}701KN$

3. BENDING-TORSION

A component is in bending-torsion if it is simultaneously subjected to pure bending and pure torsion.

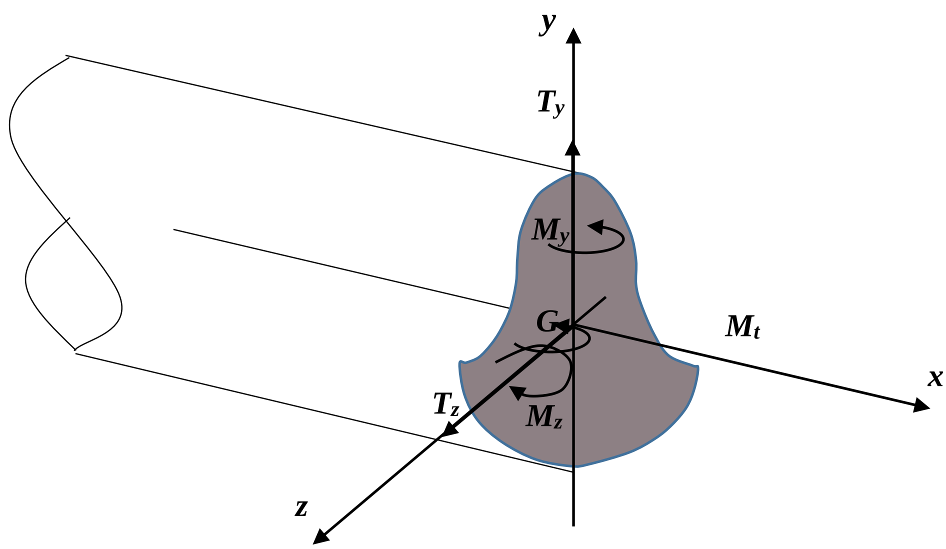


Figure 3.14 Bending-torsion.

3.1 NORMAL STRESS

The normal stress is given by::

$$\sigma_x = \frac{M_z}{I_z} y + \frac{M_y}{I_y} z \quad \text{………………………………………….(3.21)}$$

Where:

A: is the cross-sectional area of the component.

3.2 SHEAR STRESS

The shear stress at a point is given by:

$$\tau = \sqrt{\tau_{xy}^2 + \tau_{xz}^2 + \tau_t^2} \quad \text{………………………………………..………………}(3.22)$$

Where:

$$\begin{cases} \tau_{xy} = \dfrac{T_y}{b}\dfrac{S_z}{I_z} \\ \tau_{xz} = \dfrac{T_z}{b}\dfrac{S_y}{I_y} \\ \tau_t = \dfrac{M_t}{I_0} R \end{cases} \quad \text{…………………………………………………………………..….}(3.23)$$

$\boldsymbol{M_t}$: torsional moment.

$\boldsymbol{I_0}$: polar moment of inertia.

R : radius of the bar.

3.3 STRENGTH CALCULATION

The steps for calculating strength are:

- Identify the critical sections (areas where internal forces are maximum) and critical points within the section (points farthest from the neutral axis).
- Evaluate the values of normal stress and shear stress due to torsion and check if the calculated values are less than the allowable stress of the material:

$$\sigma_x \le \overline{\sigma} \quad \text{et} \quad \tau_t \le \overline{\tau} \quad \text{……………………………………………………….…..}(3.24)$$

$\overline{\tau}$: Allowable shear stress of the material.

Note:

- The strength condition is satisfied if the conditions at the points furthest from the neutral axis are met.
- The shear stress due to torsion is not negligible.
- The calculation of hollow circular sections is similar to that of solid sections.

3.4 APPLICATION :

Calculate the diameter of the component while satisfying the strength conditions.

$\overline{\sigma} = 120MPa$ and $\overline{\tau} = 90MPa$

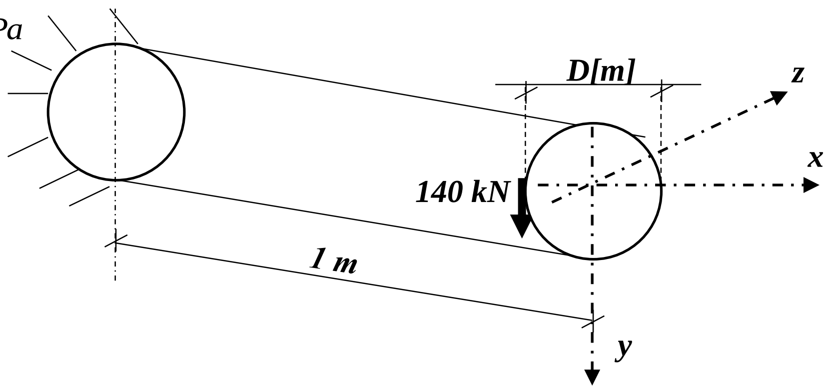


Figure 3.15 Bending-torsion console.

3.4.1 Solution

a) Equivalent System

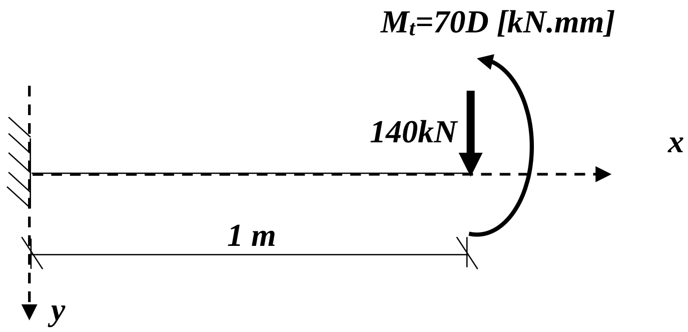


Figure 3.16 Equivalent System

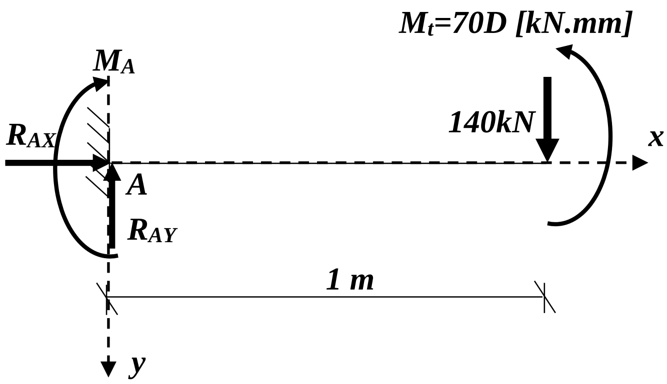
Calculation of moments:

$$M_t = 140.\frac{D}{2} \Rightarrow M_t = 70D[Kn.mm]$$

Calculation of support reactions (embedding)

In static equilibrium:

$\sum \vec{F}_x = \sum \vec{F}_z = \sum \vec{M}_y = \vec{0}$; thus:

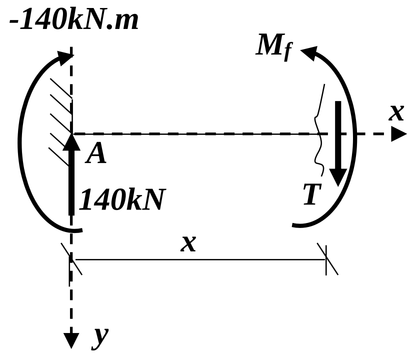


$\sum F_x = 0 \Rightarrow R_{AX} = 0$

$\sum F_y = 0 \Rightarrow R_{AY} - 140 = 0 \Rightarrow R_{AY} = 140kN$

$\sum M_z / A = 0 \Rightarrow M_A + 140.1 = 0$

$\Rightarrow M_A = -140kN.m$



By the method of sections, we can plot the diagrams

of internal forces:

1st section : $0 \le x \le 1$

$$M_f(x) = 140.x - 140 \quad \begin{cases} x = 0 \Rightarrow M_f(0) = -140\,kN.m \\ x = 1 \Rightarrow M_f(1) = 0kN.m \end{cases}$$

f) Diagrams of internal forces

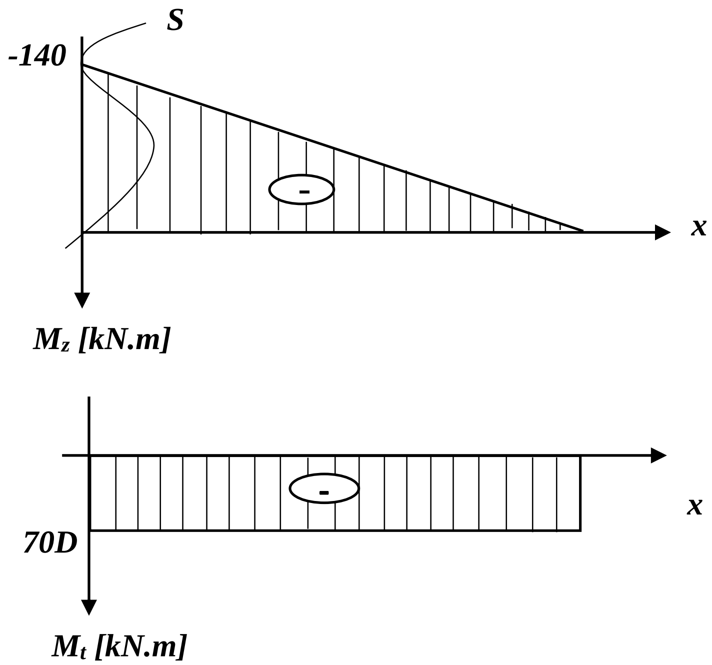


Figure 3.17 Diagrams of internal forces.

b) NORMAL STRESS:

The maximum normal stress is given by the equation:

$$\sigma_x^{\max} = \frac{M_z}{I_z} y = \frac{140.10^6}{\frac{\pi}{64} D^4} . \frac{D}{2} = \frac{1,426.10^9}{D^3} \le 120 \Rightarrow D \ge 228,2mm$$

c) Shear stress

The shear stress due to torsion is given by the equation:

$$\tau_t = \frac{M_t}{I_0} R$$

$$M_t^{\max} = 70D.10^6\, kN.m;$$

$$R^{\max} = \frac{D}{2}; \quad I_0 = \frac{\pi}{32} D^4$$

$$\tau_t^{\max} = \frac{M_t^{\max}}{I_0} R^{\max} \Rightarrow \tau_t^{\max} = \frac{70D.10^3}{\frac{\pi}{32} D^4} \frac{D}{2} = \frac{356,507.10^3}{D^2} \leq 90$$

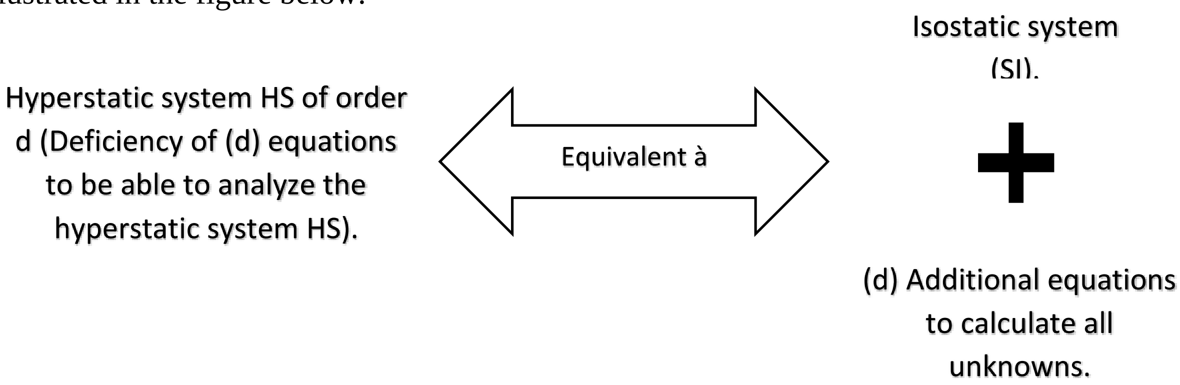
$$\Rightarrow D \geq 62,95mm$$

$$D'ou\, D \geq 228,2mm \Rightarrow D \geq 230mm$$

CHAPITRE 4 RESOLUTION OF HYPERSTATIC SYSTEMS

INTRODUCTION

A system is termed hyperstatic if the number of unknowns in the connections exceeds the number of equations derived from statics. This difference is called the hyperstaticity degree of the system. To study and analyze a structure with a hyperstaticity degree of ***d***, it is necessary to establish additional equations (known as compatibility equations). The methods involve choosing a base system from which the simplest isostatic system (***SI***) is determined, as illustrated in the figure below.



Due to the interdependence between forces and displacements, there are two ways to approach the calculation of hyperstatic structures, namely: either by focusing on forces (in redundant connections) or by focusing on displacements (the displacement method).

1. CALCULATION OF THE DEGREE OF HYPERSTATISM::

Generally, the number of redundant supports represents the degree of hyperstatism. There are two approaches to calculate this degree:

1.1 SUPPORT-BASED REASONING:

The degree of hyperstatism $d_h = n - 2$. If the beam has a fixed support, then the degree of hyperstatism is increased by one unit $d_h = d_h \, calculé + 1$

. Where: ***n*** is the number of supports

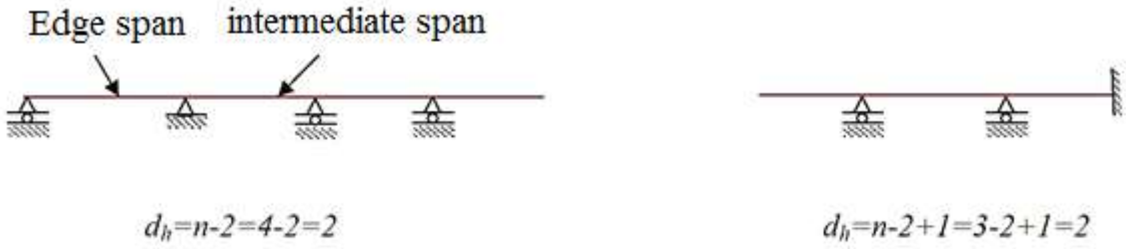


Figure 4.1 Degree of hyperstatism (support-based reasoning)

1.2 SPAN-BASED REASONING:

The degree of hyperstatism $d_h = n - 1$. If the beam has a fixed support, then the degree of hyperstatism is increased by one unit $d_h = d_h\, calculé + 1$

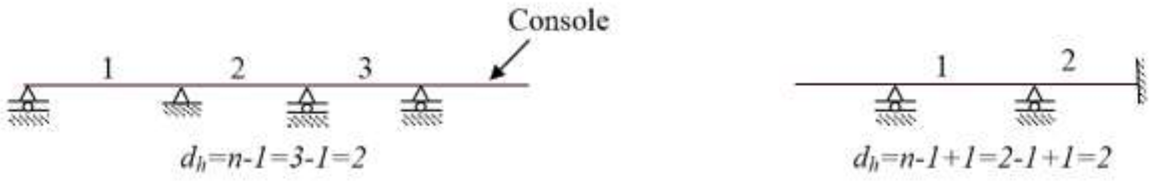


Figure 4.2 Degree of hyperstatism (span-based reasoning)

2. FORCE METHOD (OR CUT METHOD)

The force method (or cut method) is one of the general methods for analyzing statically indeterminate systems. It involves selecting and determining hyperstatic unknowns that, once calculated, allow the determination of forces at any point in the structure, turning it into a statically determinate system. This method is based on the principle of superposition of the effects of actions and constitutes a global elastic analysis, limited in the context of these notes to plane structures loaded in their plane, where the beams' cross-section has the torsion center coinciding with the center of gravity.

2.1 DEFINITIONS

A structure is said to be statically determinate or statically determinate when all support reactions and internal forces can be determined using only static equilibrium equations. In contrast, when there are too few equations, the structure is said to be statically indeterminate or hyperstatic. The degree of static indeterminacy or degree of hyperstaticity ***h*** of a structure is then equal to the number of simple cuts required to make it statically determinate.

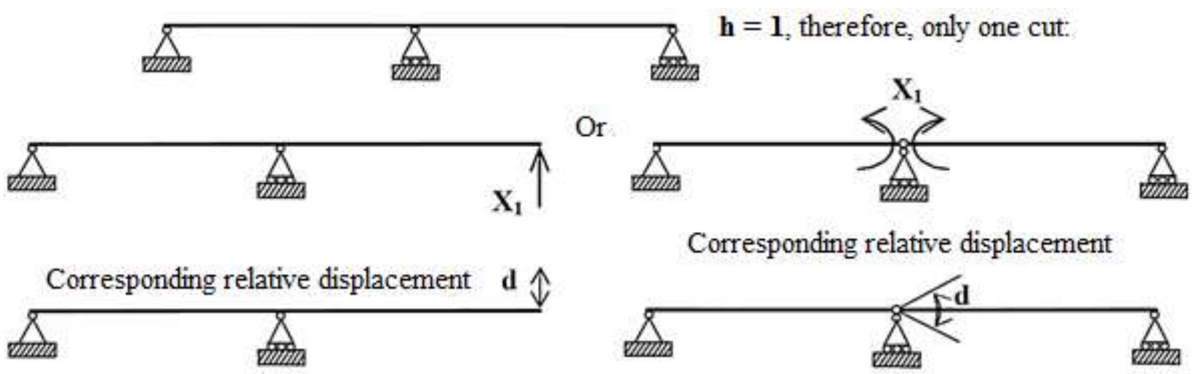
It's important to note that hyperstaticity in a structure can arise from redundant connections with the external world (external hyperstaticity), in which case the simple cuts will relate to the supports. Hyperstaticity can also arise from redundant connections within the structure itself (internal hyperstaticity), in which case the simple cuts will relate to internal forces ***M, N, V.***

The total degree of hyperstaticity h is, of course, equal to the sum of the internal degree h_i and external degree h_e.

Each simple cut i modifies the system by removing the connection related to an unknown reaction component or an unknown internal force (actually two equal and opposite forces). This eliminates the component or force corresponding to a hyperstatic unknown. The total number h of simple cuts required to make the structure statically determinate is therefore equal to the number of hyperstatic unknowns in the problem.

The statically determinate structure derived from the actual structure will be called the reference statically determinate structure S_0. There are obviously several possible reference statically determinate structures for the same initial structure since the simple cuts can be made in any sections. However, it is necessary to pay particular attention when making cuts to avoid resulting in a mechanism.

We will call X_j $(j=1, 2, \dots h)$ the hyperstatic unknowns. Associated with the cuts related to these forces, under a load on the reference statically determinate structure S_0, There may be displacements d_i $(i=1, 2, \dots h)$ called relative displacements of the edges of cut i. The objective of the force method (or cut method) is to determine the h hyperstatic unknowns X_j of a structure with a degree of hyperstaticity h.



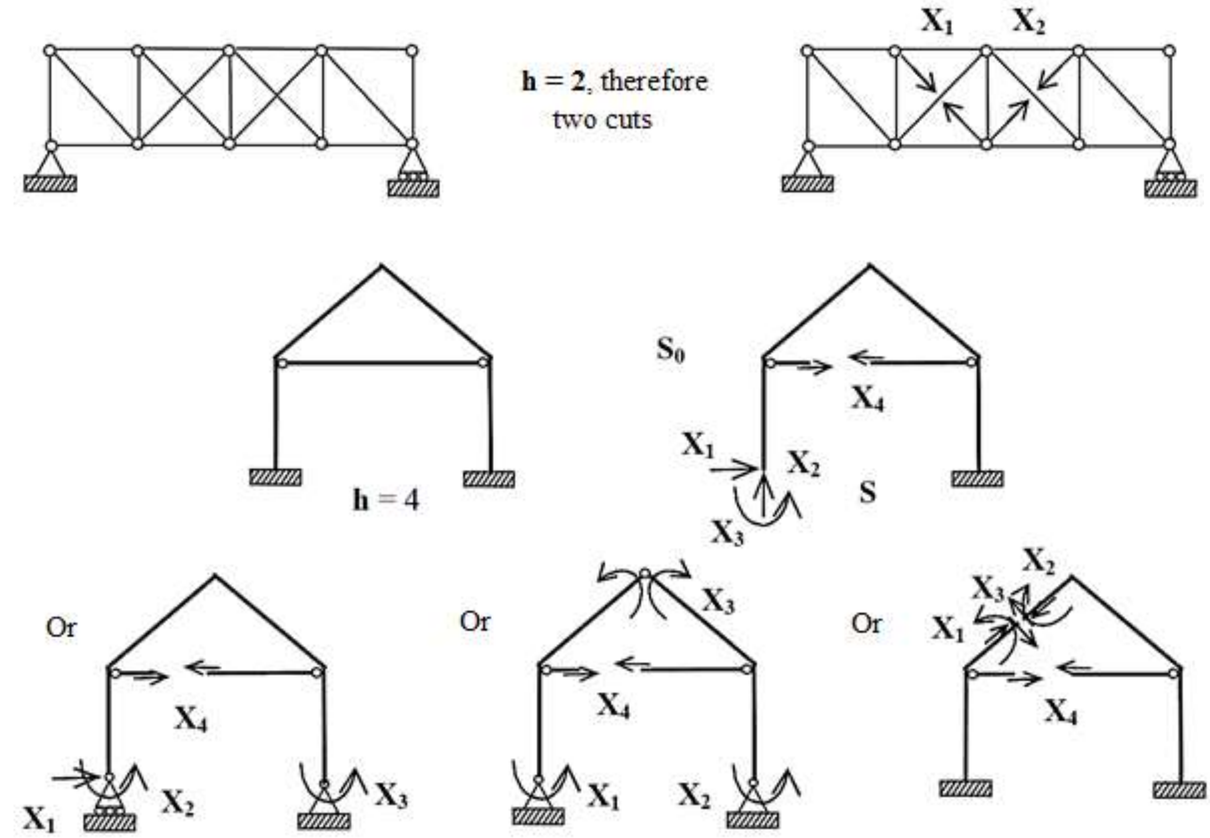


Figure 4.3 Representation of the force method

2.2 FLEXIBILITY COEFFICIENTS F_{IJ} AND F_{IP}

The flexibility coefficient f_{ij}, is defined as the relative displacement of the cut edges ***i*** in the ***i*** direction due to a unit force $X_j = 1$ acting on the cut ***j***, in the ***j*** direction.

In this definition, the terms force and displacement must be considered in the generalized sense. Thus, in the case of a cut related to a support,X_j can represent either a unit force or a unit moment, and in the case of an internal cut, X_j can represent a pair of unit internal forces (***M, N,*** or ***V***). Similarly, f_{ij} can represent either the actual displacement or rotation associated with a support cut, or the actual relative displacement or relative rotation associated with an internal cut.

Example :

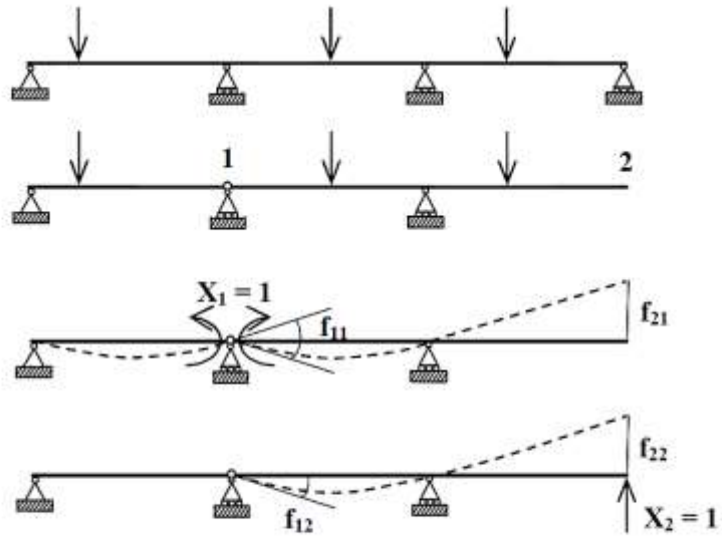


Figure 4.4 Flexibility coefficient

The flexibility coefficient fij representing a displacement in the reference isostatic structure can be calculated using the unit force theorem:

$$f_{ij} = \int_0^{str} \frac{N_i N_j}{EA} dx + \int_0^{str} \frac{M_i M_j}{EI} dx + \int_0^{str} \frac{V_i V_j}{GA'} dx \quad \text{.....(4.1)}$$

Where : $\boldsymbol{N_i}$, $\boldsymbol{M_i}$, $\boldsymbol{V_i}$ are the equations of internal forces in the reference isostatic structure under a virtual unit load $\boldsymbol{X_i}$ applied at ***i*** (to obtain the relative displacement); and: $\boldsymbol{N_j}$, $\boldsymbol{M_j}$, $\boldsymbol{V_j}$ are the equations of internal forces in the reference isostatic structure under the unit hyperstatic unknown $\boldsymbol{X_j}$ applied at ***j***.

It is important to note that, according to the reciprocity theorem of ***Betty-Maxwell***,

$$f_{ij} = f_{ji} \quad \text{.....(4.2)}$$

The flexibility coefficient $\boldsymbol{f_{ip}}$, is defined as the relative displacement of the cut edges ***i*** in the ***i***, direction produced by the applied external forces.

In this definition, the terms force and displacement must be considered in the generalized sense. Thus, $\boldsymbol{f_{ip}}$ can represent either the actual displacement or rotation associated with a support cut, or the actual relative displacement or relative rotation associated with an internal cut.

Example :

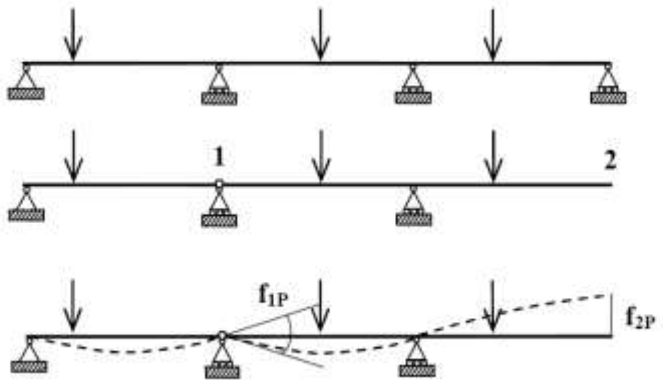


Figure 4.5 Displacement represented by the flexibility coefficient f_{ip}

The flexibility coefficient $\boldsymbol{f_{ip}}$ representing a displacement in the reference isostatic structure can be calculated using the unit force theorem:

$$f_{ip} = \int_0^{str} \frac{N_i N_p}{EA} dx + \int_0^{str} \frac{M_i M_p}{EI} dx + \int_0^{str} \frac{V_i V_p}{GA'} dx \quad \text{.....(4.3)}$$

Where : $\boldsymbol{N_i}$, $\boldsymbol{M_i}$, $\boldsymbol{V_i}$ are the equations of internal forces in the reference isostatic structure under a virtual unit load $\boldsymbol{X_i}$ applied a $\boldsymbol{i}$ (to obtain the relative displacement); and $\boldsymbol{N_P}$, $\boldsymbol{M_P}$, $\boldsymbol{V_P}$ are the equations of internal forces in the reference isostatic structure due to the entire applied external forces.

2.3 GENERAL EQUATION OF THE FORCE METHOD

Consider, now, the reference isostatic structure successively and separately loaded by each hyperstatic unknown $\boldsymbol{X_j}$ and by the applied actual forces:

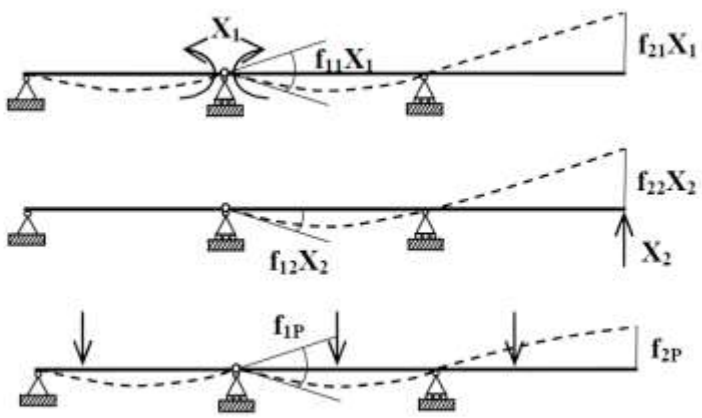


Figure 4.6 Reference isostatic structure

According to the principle of superposition, we can consider the actual hyperstatic structure (of degree ***h***) as the superposition of (***h+1***) states of the reference isostatic structure, provided that the total relative displacements of the cut edges are zero, as there are no cuts in the actual structure. If we consider cut ***i*** in the ***i*** direction, it will undergo a total relative displacement di, the sum of displacements produced by each unknown force X_j, i.e., $f_{ij}.X_j$, and the displacement due to the applied external forces, i.e., f_{ip}. This total relative displacement being zero, we can write:

$$\sum_{j=1}^{h} f_{ij}.X_j + f_{ip} = 0 \text{ for all : } i=1,2,\ldots h \qquad (4.4)$$

By this condition, we obtain a system of ***h*** equations for ***h*** inconnues X_j. These equations reflect the compatibility condition of displacements at the ***h*** cuts and constitute the equations of the force method.

These equations can also be written in matrix form: ***[F].[X] = [A]*** , with :

- ***[F]***: the flexibility matrix of the structure (array of f_{ij} coefficients), square ***hxh*** and symmetric;

- ***[X]*** : the vector of hyperstatic unknowns X_j ;

- ***[A]*** : the vector of displacements f_{ip} with a changed sign.

Once the ***h*** unknowns are determined, the structure to be solved becomes isostatic. The determination of internal forces ***M,N,V*** in the complete structure can then be done in two ways:

- Either by proceeding, as with any isostatic structure, through cuts and free-body diagrams;

- Or by superimposing the diagrams of the (***h+1***) reference isostatic structures, replacing unit forces with the values of hyperstatic unknowns.

2.4 NUMERICAL DETERMINATION OF F_{IJ} AND F_{IP} COEFFICIENTS

For beams primarily subjected to bending, it is common to neglect deformations due to normal forces and shear forces compared to those caused by bending moments. In this case, the expressions for the coefficients f_{ij} and f_{ip} can be simplified to:

$$f_{ij} = \int_0^{str} \frac{M_i M_j}{EI} dx \text{ and : } f_{ip} = \int_0^{str} \frac{M_i M_p}{EI} dx \quad \text{.......................................(4.5)}$$

When dealing with cables or truss members, these elements are, solely or predominantly, subjected to normal forces. The expressions for f_{ij} and f_{ip} are then reduced to:

$$f_{ij} = \int_0^{str} \frac{N_i N_j}{EA} dx \text{ and } f_{ip} = \int_0^{str} \frac{N_i N_p}{EA} dx \quad \text{...(4.6)}$$

Furthermore, it is noteworthy that any integral involved in the calculations of f_{ij} and f_{ip}, coefficients involves two forces that can have different signs. Thus, f_{ij} and f_{ip} coefficients may be negative for ***i*** and ***j***.

In numerical applications, the moment of inertia of a beam section or the area of a cable section is often constant along the length of the element. Additionally, given the usually considered loads, the moments vary linearly or parabolically, and the normal forces are often constant or can be considered as such. It has therefore been interesting to calculate a series of integrals of the form $\int_0^L M.m.dx$ based on the characteristic quantities of rectangular, triangular, trapezoidal, and parabolic diagrams, independently of ***EI*** or ***EA*** (the functions ***M*** and ***m*** must be considered with their relative sign).

2.5 APPLICATION :

Consider the hyperstatic straight beam represented below:

1- Calculate the degree of hyperstaticity.
2- Determine the reactions at the supports.

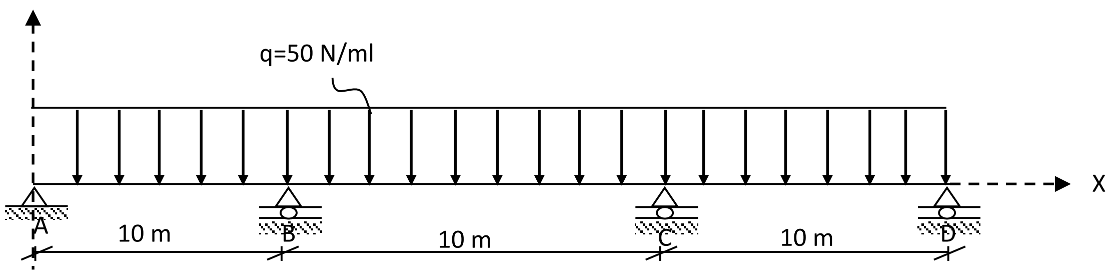


Figure 4.7 One-ended Fixed Console

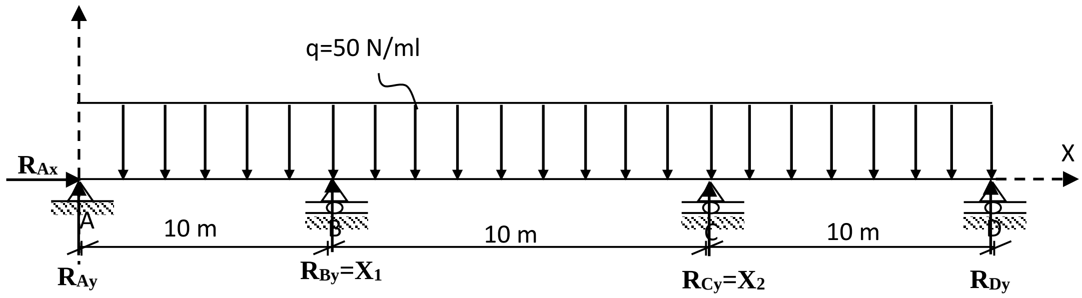
2.5.1 Solution

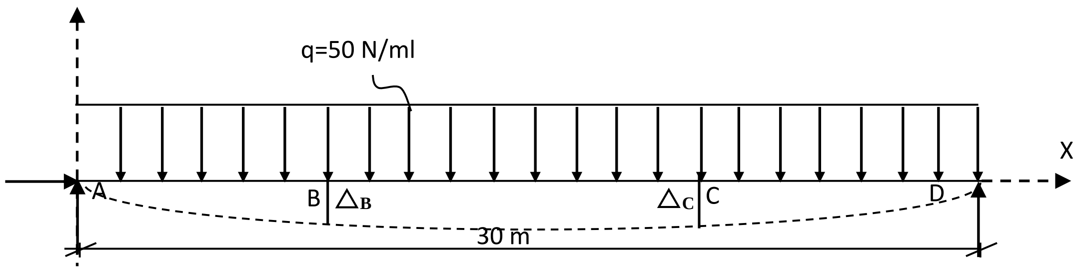
Degree of hyperstaticity:

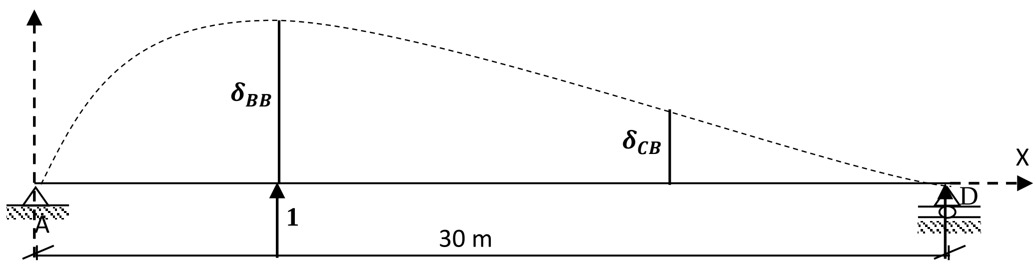
$h = r - s$ with ***: r =5 :*** number of unknowns***; S=3 :*** number of equations

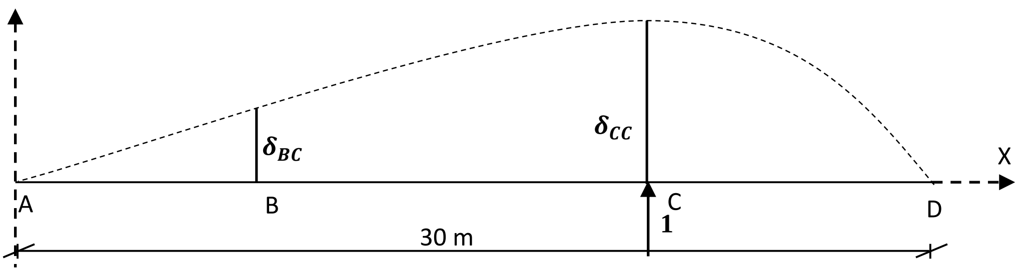
Therefore:: $h = 5 - 3 = 2$

Determination of support reactions:





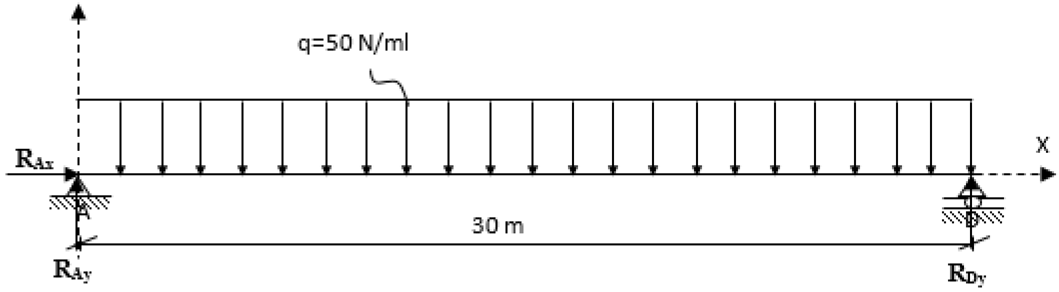




$$\begin{cases} \Delta_B = R_{By}.\delta_{BB} + R_{Cy}.\delta_{BC} \\ \Delta_C = R_{By}.\delta_{CB} + R_{Cy}.\delta_{CC} \end{cases}$$

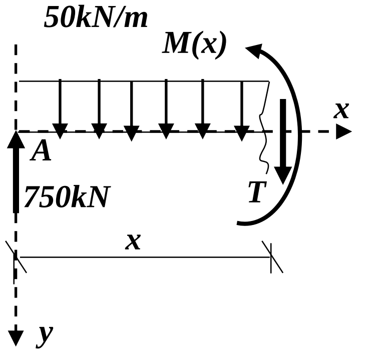
To solve this equation, we start by calculating $\triangle_B$

$\triangle_B$= ?



In static equilibrium:

$\sum \vec{F}_x = \sum \vec{F}_z = \sum \vec{M}_y = \vec{0}$; therefore:



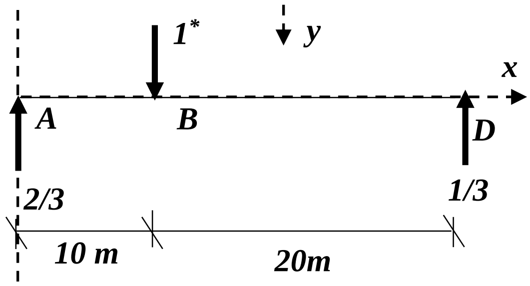
$$\sum \mathrm{F_x} = 0 \Rightarrow \mathrm{R_{AX}} = 0$$
$$\sum \mathrm{F_y} = 0 \Rightarrow \mathrm{R_{AY}} + \mathrm{R_{DY}} - 1500 = 0 \Rightarrow \mathrm{R_{AY}} = 1500 - \mathrm{R_{DY}}$$
$$\sum M_z / A = 0 \Rightarrow \mathrm{R_{DY}}.30 - 1500.15 = 0 \Rightarrow \mathrm{R_{DY}} = 750N$$
$$\Rightarrow \mathrm{R_{AY}} = \mathrm{R_{DY}} = 750N$$

$$\Delta_B = \frac{1}{EI}\int_0^{30} M(x).m^*(x).dx$$

$M(x) = ?$

1^{st} section : $0 \le x \le 30$

$$M(x) = 750.x - 50.\frac{x^2}{2} = 750.x - 25.x^2$$



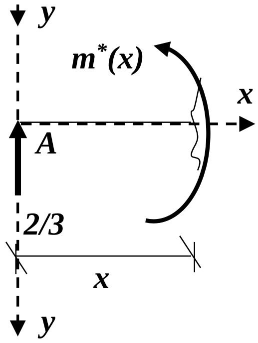
$m*(x) = ?$

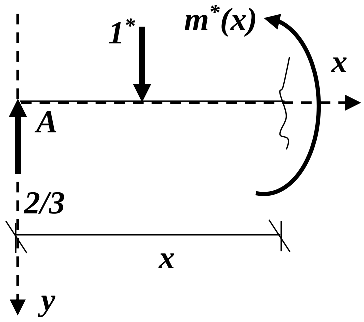
1^{st} section : $0 \le x \le 10$

$$m*(x) = \frac{2}{3}.x$$

2^{nd} section : $10 \le x \le 30$

$$m*(x) = \frac{2}{3}.x - 1.(x-10) = \frac{-x}{3} + 10$$

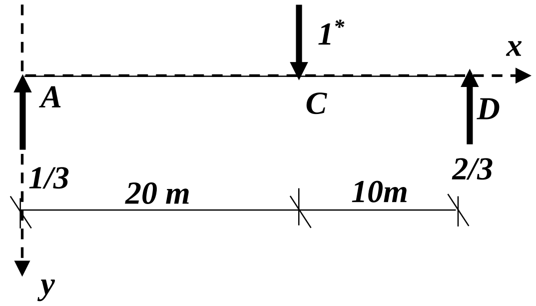




$$\Delta_B = \frac{1}{EI}\int_0^{30} M(x).m^*(x).dx = \frac{1}{EI}\left[\int_0^{10}(750.x - 25.x^2).(\frac{2}{3}x).dx + \int_{10}^{30}(750.x - 25.x^2).(10 - \frac{x}{3}).dx\right]$$

$$\Delta_B = \frac{1375000}{3EI}$$

$\triangle_C$= ?



$$\Delta_C = \frac{1}{EI}\int_0^{30} M(x).m^*(x).dx$$

$$M(x) = 750.x - 50.\frac{x^2}{2} = 750.x - 25.x^2$$

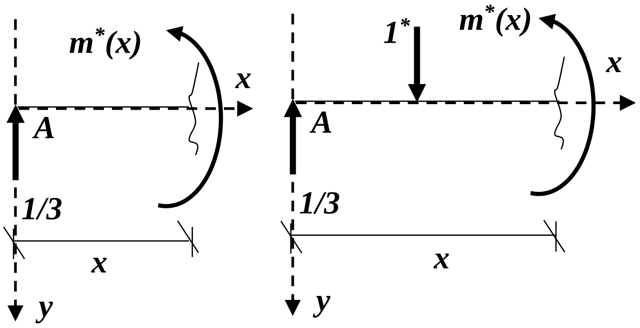
$m*(x) = ?$

1^{st} section : $0 \le x \le 20$

$$m*(x) = \frac{x}{3}$$

2^{nd} section : $20 \le x \le 30$

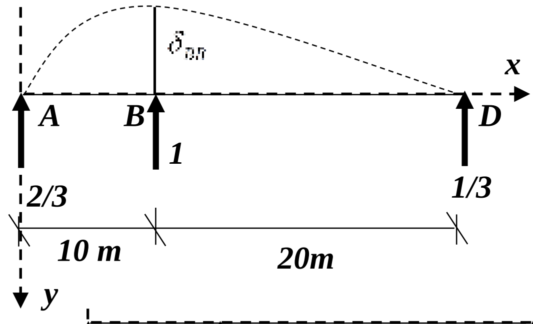
$$m*(x) = \frac{x}{3} - 1.(x-20) = \frac{-2.x}{3} + 20$$



$$\Delta_C = \frac{1}{EI}\int_0^{30} M(x).m^*(x).dx = \frac{1}{EI}\left[\int_0^{20}(750.x - 25.x^2).(\frac{x}{3}).dx + \int_{20}^{30}(750.x - 25.x^2).(20 - \frac{2x}{3}).dx\right]$$

$$\Delta_C = \frac{1375000}{3EI}$$

$\delta_{BB} = ?$

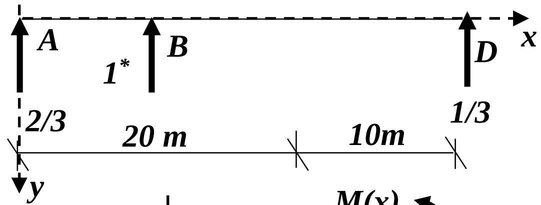


$$\delta_{BB} = \frac{1}{EI}\int_0^{30} M(x).m^*(x).dx$$

$M(x) = ?$

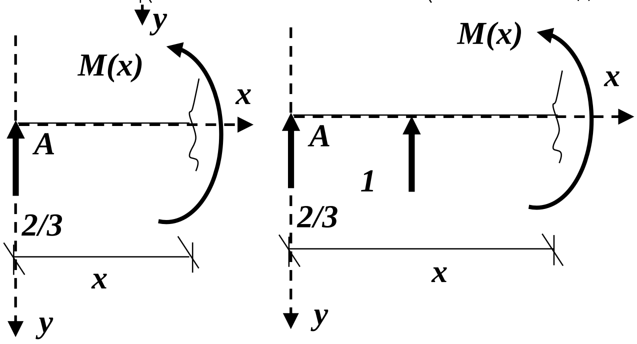
1^{st} section : $0 \le x \le 10$

$$M(x) = \frac{-2x}{3}$$



2^{nd} section : $10 \le x \le 30$

$$M(x) = \frac{-2x}{3} + x - 10 = \frac{x}{3} - 10$$



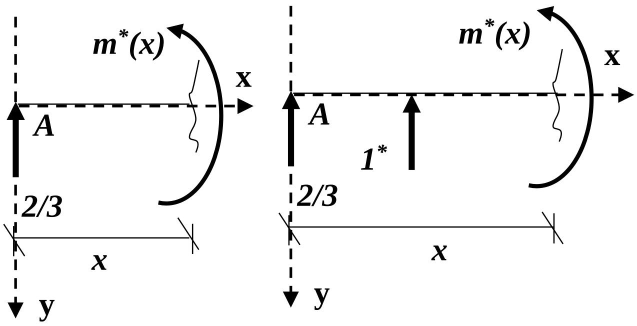
$m^*(x) = ?$

1^{st} section : $0 \le x \le 10$

$$m^*(x) = \frac{-2x}{3}$$

2^{nd} section : $10 \le x \le 30$

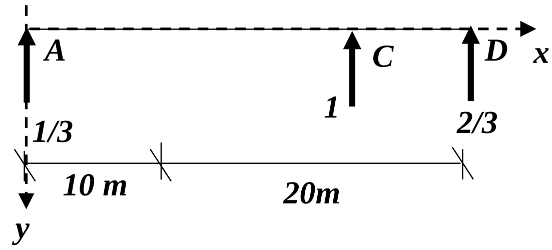
$$m^*(x) = \frac{-2x}{3} + 1.(x-10) = \frac{x}{3} - 10$$



$$\delta_{BB} = \frac{1}{EI}\int_0^{30} M(x).m^*(x).dx$$

$$\delta_{BB} = \frac{1}{EI}\left[\int_0^{10}(\frac{-2.x}{3}).(\frac{-2.x}{3}).dx + \int_{10}^{30}(\frac{x}{3}-10).(\frac{x}{3}-10).dx\right]$$

$$\delta_{BB} = \frac{12000}{27EI}$$



$$\delta_{BC} = ?$$

$$\delta_{BC} = \frac{1}{EI}\int_0^{30} M(x).m^*(x).dx$$

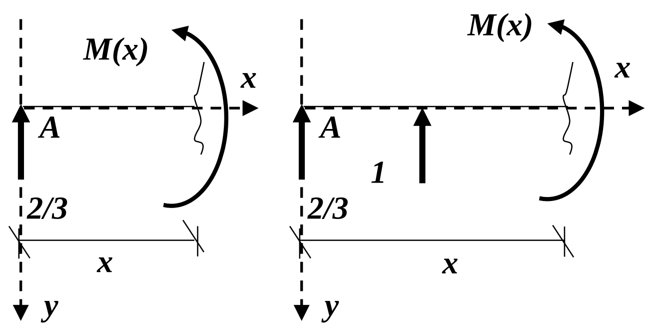
$M(x) = ?$

1^{st} section: $0 \le x \le 20$

$$M(x) = \frac{-x}{3}$$

2^{nd} section: $20 \le x \le 30$

$$M(x) = \frac{-x}{3} + 1.(x-20) = \frac{2.x}{3} - 20$$



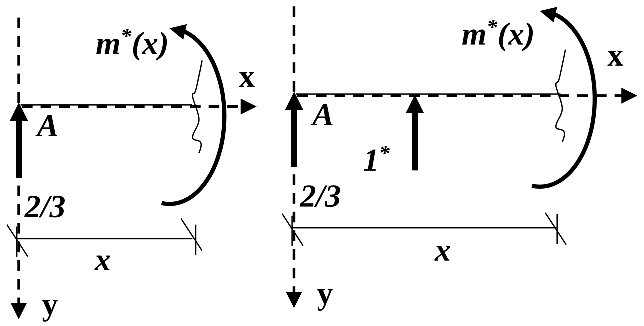
$m^*(x) = ?$

1^{st} section: $0 \le x \le 10$

$$m^*(x) = \frac{-2x}{3}$$

2^{nd} section: $10 \le x \le 30$

$$m^*(x) = \frac{-2x}{3} + 1.(x-10) = \frac{x}{3} - 10$$

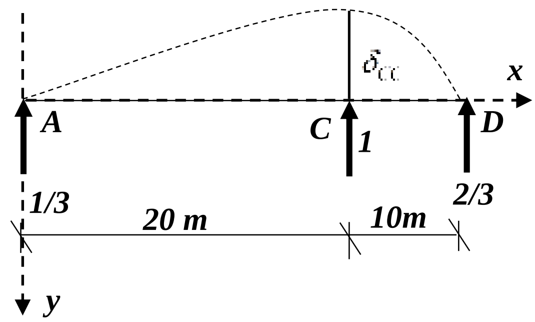


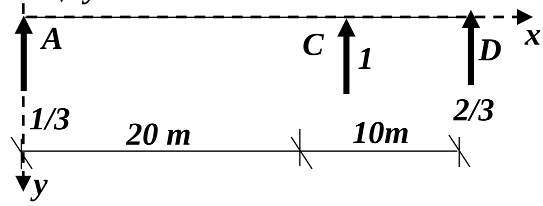
$$\delta_{BC} = \frac{1}{EI}\int_0^{30} M(x).m^*(x).dx$$

$$= \frac{1}{EI}\left[\int_0^{10}(\frac{-x}{3}).(\frac{-2.x}{3}).dx + \int_{10}^{20}(\frac{2.x}{3}-20).(\frac{x}{3}-10).dx\right] +$$

$$\int_{20}^{30}(\frac{2.x}{3}-20).(\frac{x}{3}-10).dx$$

$$\delta_{BC} = \frac{10500}{27EI}$$





$$\delta_{CC} = ?$$

$$\delta_{CC} = \frac{1}{EI}\int_0^{30} M(x).m^*(x).dx$$

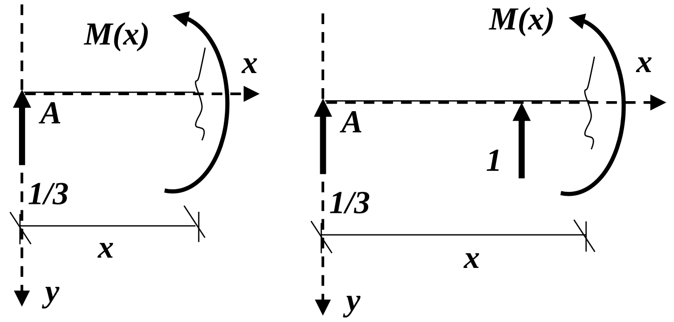
$M(x) = ?$

1^{st} section: $0 \le x \le 20$

$$M(x) = \frac{-x}{3}$$

2^{nd} section: $20 \le x \le 30$

$$M(x) = \frac{-x}{3} + x - 20 = \frac{2.x}{3} - 20$$



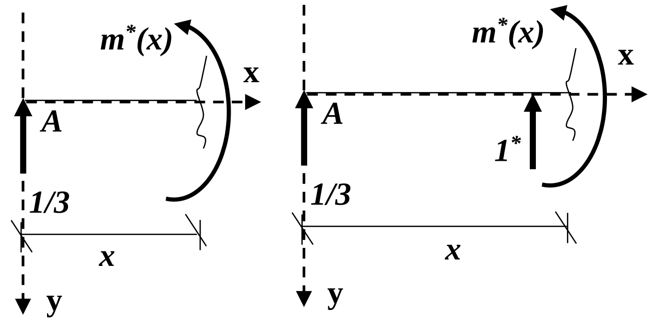
$m^*(x) = ?$

1^{st} section: $0 \le x \le 20$

$$m^*(x) = \frac{-x}{3}$$

2^{nd} section: $20 \le x \le 30$

$$m^*(x) = \frac{-x}{3} + 1.(x - 20) = \frac{2.x}{3} - 20$$

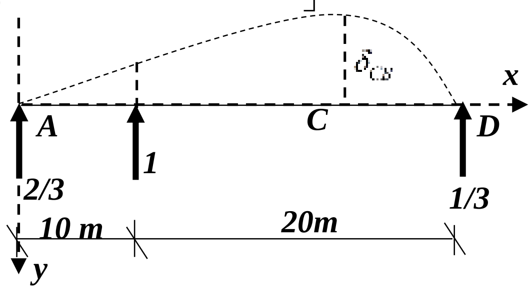


$$\delta_{CC} = \frac{1}{EI}\int_0^{30} M(x).m^*(x).dx = \frac{1}{EI}\left[\int_0^{20} (\frac{-x}{3}).(\frac{-x}{3}).dx + \int_{20}^{30} (\frac{2.x}{3} - 20).(\frac{2.x}{3} - 20).dx\right]$$

$$\delta_{CC} = \frac{12000}{27EI}$$

$\delta_{CB} = ?$

$$\delta_{CB} = \frac{1}{EI}\int_0^{30} M(x).m^*(x).dx$$



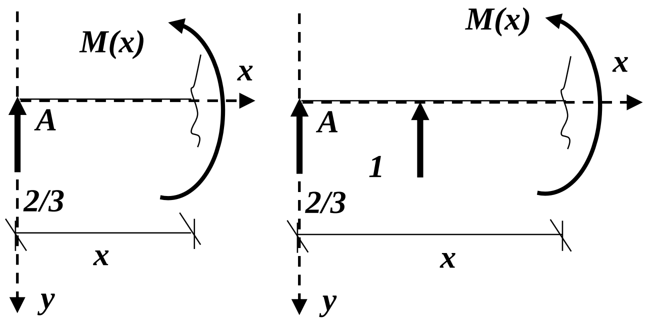
$M(x) = ?$

1^{st} section: $0 \le x \le 10$

$$M(x) = \frac{-2.x}{3}$$

2^{nd} section: $20 \le x \le 30$

$$M(x) = \frac{-2.x}{3} + 1.(x - 10) = \frac{x}{3} - 10$$



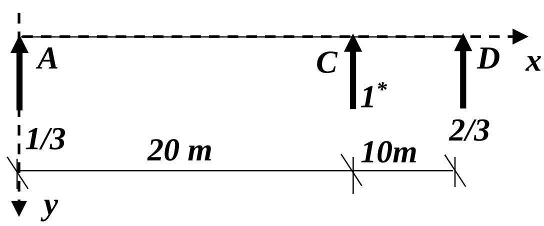
$m^*(x) = ?$

1^{st} section: $0 \le x \le 20$

$$m^*(x) = \frac{-x}{3}$$

2^{nd} section: $20 \le x \le 30$

$$m^*(x) = \frac{-x}{3} + 1.(x - 20) = \frac{2.x}{3} - 20$$

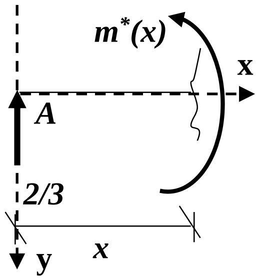


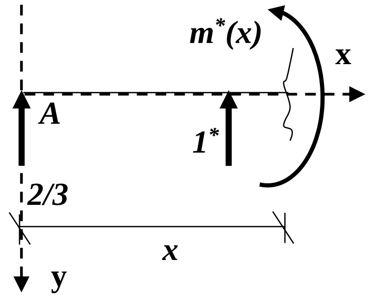
$$\delta_{CB} = \frac{1}{EI}\int_0^{30} M(x).m^*(x).dx$$

$$= \frac{1}{EI}\left[\int_0^{10}(\frac{-x}{3}).(\frac{-2.x}{3}).dx + \int_{10}^{20}(\frac{-x}{3}).(\frac{x}{3}-10).dx\right]$$

$$+\int_{20}^{30}(\frac{2.x}{3}-20).(\frac{x}{3}-10).dx$$

$$\delta_{CB} = \frac{10500}{27EI}$$

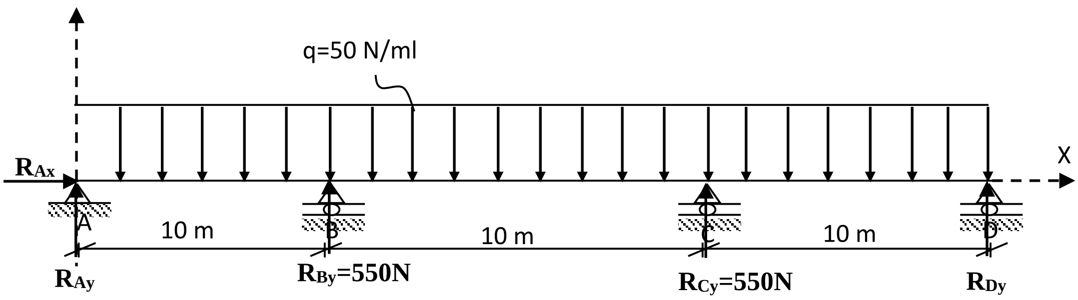




So, $\Delta_B = \Delta_C = \dfrac{1375000}{3EI}$; $\qquad \delta_{BB} = \delta_{CC} = \dfrac{12000}{27EI}$; $\qquad \delta_{CB} = \delta_{BC} = \dfrac{10500}{27EI}$

We have : $\begin{cases} \Delta_B = R_{By}.\delta_{BB} + R_{Cy}.\delta_{BC} \\ \Delta_C = R_{By}.\delta_{CB} + R_{Cy}.\delta_{CC} \end{cases} \Rightarrow \begin{cases} \dfrac{1375000}{3EI} = B_y.\dfrac{12000}{27EI} + C_y.\dfrac{10500}{27EI} \\ \dfrac{1375000}{3EI} = B_y.\dfrac{10500}{27EI} + C_y.\dfrac{12000}{27EI} \end{cases}$

We find: $\begin{cases} R_{By} = 550N \\ R_{Cy} = 550N \end{cases}$



$$\sum F_x = 0 \Rightarrow R_{AX} = 0$$

$$\sum F_y = 0 \Rightarrow R_{AY} + 550 + 550 + R_{DY} - 1500 = 0 \Rightarrow R_{AY} = 1500 - 100 - R_{DY}$$

$$\sum M_z / A = 0 \Rightarrow R_{DY}.30 + 550.20 + 550.10 - 1500.15 = 0 \Rightarrow R_{DY} = \frac{1500.15 - 550.20 - 550.10}{30}$$

$$\Rightarrow R_{DY} = 200N \quad ; \; R_{AY} = 200N \quad ; \; R_{AX} = 0N$$

CONCLUSION

This set of lecture notes and exercises is a continuation of the Mechanics of Materials course taught in the fourth semester. We will cover compound stresses, energy methods, and hyperstatic systems. The purpose of this set of notes is to serve as a guide following the outline of the third-year Bachelor's degree in mechanical engineering for mechanical engineering students and individuals seeking an overview of materials resistance.

Bibliographical references are provided below to allow the reader to delve deeper into each topic covered.

REFERENCES

[01] **S. Timoshenk**. « Résistance des matériaux – Tome 1: Théorie élémentaire et problèmes ». 3ème Ed. Dunod, Paris, 1968, 420p1405.

[02] **A. Bazergui «** Résistance des matériaux **»,** 3ème Ed, Ecole Polytechnique de Montréal, 2002.

[03] **M. Khelifa.** « Résistance des matériaux Cours et exercices corriges », Ed Casteilla, - 9, rue Michael Faraday, 78180 Montigny-le-Bretonneux.2010

[04] **K. Hadjazi.** « Résistance des matériaux ». Cours _Université d'Oran 2013

[05] **Ir. Jacques Dehard. «** Calcul des structures hyperstatiques; La méthode des forces » Méthode des forces - Deh 2007

[06] **S. Laroze «** Mécanique des structures – Tome 2 : Poutres ». Cépaduès-Editions. France, 2005

[07] **A.Delaplace, F. Gatuingt, F. Ragueneau**. « Mécanique des structures –Résistance des matériaux ». Dunod, Paris, 2008

[08] **R. El Fatmi** « Cours de résistance des matériaux : mécanique des poutres ». Ecole Polytechnique de Tunisie.

[09] **P. Agati - N. Mattera. «** Manique appliques, Résistance des matériaux, Mécanique des fluides et Thermodynamique ».DUNOD, Paris, 1996

[10] **Jean-Claude Doubrère.** « Résistance des matériaux Cours et exercices corrigés » 11éme édition. Paris 2010.

[11] **R. Montagner.** « Cours de Résistance des matériaux» 2éme édition. Paris 1988.