PHYS (2) :

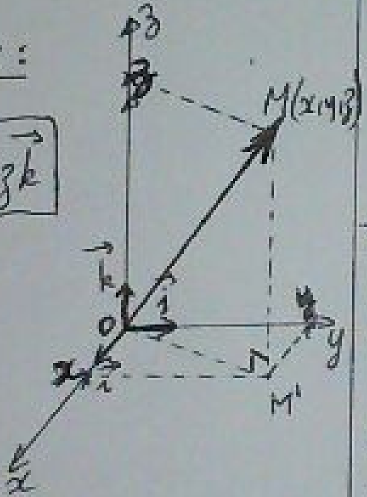
A.U. : 14/15

Éléments de réponses TD N° 1 : Compléments Mathématiques

• Exercice N° 1 :

1/- • Coor. cartésiennes :

$$\boxed{\overrightarrow{OM} = \vec{r} = x\vec{i} + y\vec{j} + z\vec{k}}$$

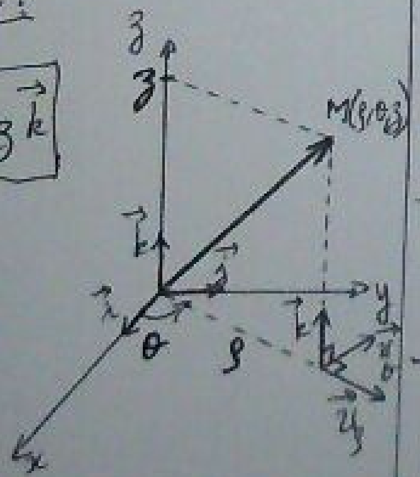


• Coor. cylindriques :

$$\boxed{\overrightarrow{OM} = \vec{r} = \rho\,\vec{u_\rho} + z\vec{k}}$$

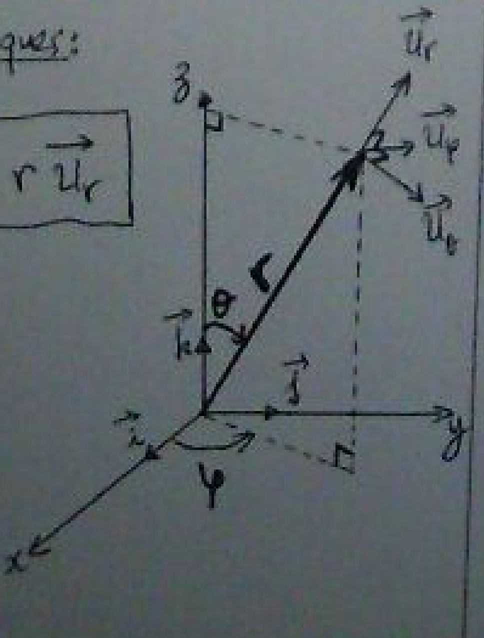
Coor. polaires :

$$\boxed{\overrightarrow{OM} = \rho\,\vec{u_\rho}}$$



• Coor. sphériques :

$$\boxed{\overrightarrow{OM} = \vec{r} = r\,\vec{u_r}}$$



2/-

• Coor. cartésiennes :

→ Déplacement élémentaire : $M\begin{pmatrix}x\\y\\z\end{pmatrix}$; $M'\begin{pmatrix}x+dx\\y+dy\\z+dz\end{pmatrix}$

$$\boxed{d\overrightarrow{OM} = d\vec{l} = \overrightarrow{MM'} = dx\,\vec{i} + dy\,\vec{j} + dz\,\vec{k}}$$

→ Surfaces élémentaires :

$$\boxed{\begin{cases} x = cte \rightarrow dS_x = dy\,dz \\ y = cte \rightarrow dS_y = dx\,dz \\ z = cte \rightarrow dS_z = dx\,dy \end{cases}}$$

→ Volume élémentaire :

$$\boxed{dV = dx\,dy\,dz}$$

• Coor. cylindriques :

→ Déplacement élémentaire : $M\begin{pmatrix}\rho\\\theta\\z\end{pmatrix}$; $M'\begin{pmatrix}\rho+d\rho\\\theta+d\theta\\z+dz\end{pmatrix}$

$$\boxed{d\overrightarrow{OM} = d\vec{l} = \overrightarrow{MM'} = d\rho\,\vec{u_\rho} + \rho\,d\theta\,\vec{u_\theta} + dz\,\vec{k}}$$

→ Surfaces élémentaires :

$$\boxed{\begin{cases} \rho = cte \rightarrow dS_\rho = \rho\,d\theta\,dz \\ \theta = cte \rightarrow dS_\theta = d\rho\,dz \\ z = cte \rightarrow dS_z = \rho\,d\rho\,d\theta \end{cases}}$$

→ Volume élémentaire :

$$\boxed{dV = \rho\,d\rho\,d\theta\,dz}$$

• Coor. sphériques :

→ Déplacement élémentaires : $M\begin{pmatrix}r\\\theta\\\varphi\end{pmatrix}$; $M'\begin{pmatrix}r+dr\\\theta+d\theta\\\varphi+d\varphi\end{pmatrix}$

$$\boxed{d\overrightarrow{OM} = d\vec{l} = \overrightarrow{MM'} = dr\,\vec{u_r} + r\,d\theta\,\vec{u_\theta} + r\sin\theta\,d\varphi\,\vec{u_\varphi}}$$

→ Surfaces élémentaires :

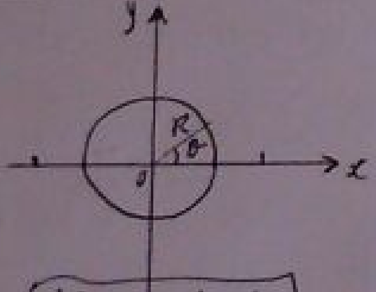
$$\begin{cases} r = cste \longrightarrow dS_r = r^2 \sin\theta \, d\theta \, d\varphi \\ \theta = cste \longrightarrow dS_\theta = r \sin\theta \, dr \, d\varphi \\ \varphi = cste \longrightarrow dS_\varphi = r \, dr \, d\theta \end{cases}$$

→ Volume élémentaire :

$$\boxed{dV = r^2 \sin\theta \, dr \, d\theta \, d\varphi}$$

3/- a)- Surface d'un disque : → Coor. polaires

- $\rho = R = cste$
- la surface élémentaire à considérer :



$$\boxed{dS_z = \rho \, d\rho \, d\theta} \rightarrow \boxed{dS_\rho = \rho \, d\rho \, d\theta}$$

- La surface du disque tout entier :

$$S = \iint_{disque} dS_\rho = \int_0^R \rho \, d\rho \int_0^{2\pi} d\theta$$

Soit : $\boxed{S = \pi R^2}$

b)- Surface d'un cylindre : → Coor. cylindri...

- $\rho = R$; hauteur $= h$
- la surface élémentaire à choisir :

$$\boxed{dS_\rho = \rho \, d\theta \, dz}$$

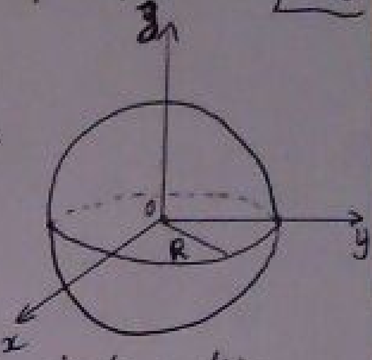
- la surface du cylindre tout entier :

$$S = \iint_{cylindre} dS_\rho = R \int_0^{2\pi} d\theta \int_0^h dz$$

Soit : $\boxed{S = 2\pi R h}$

c)- Sphère : → Coor. sphériques

- $r = R = cste$
- la surface élémentaire à considérer :



$$\boxed{dS_r = r^2 \sin\theta \, d\theta \, d\varphi}$$

- la surface de la sphère toute entière :

$$S = \iint_{sphère} dS_r = R^2 \int_0^{\pi} \sin\theta \, d\theta \int_0^{2\pi} d\varphi$$

Soit : $\boxed{S = 4\pi R^2}$

4/- a)- Volume d'un cylindre :

- Le volume élémentaire s'écrit donc :

$$\boxed{dV = \rho \, d\rho \, d\theta \, dz}$$

- le volume du cylindre tout entier :

$$V = \iiint_{cylindre} dV = \int_0^R \rho \, d\rho \int_0^{2\pi} d\theta \int_0^h dz$$

Soit : $\boxed{V = \pi R^2 h}$

b)- Volume d'une sphère :

- le volume élémentaire :

$$\boxed{dV = r^2 \sin\theta \, dr \, d\theta \, d\varphi}$$

- le volume de la sphère toute entière :

$$V = \iiint_{sphère} dV = \int_0^R r^2 dr \int_0^{\pi} \sin\theta \, d\theta \int_0^{2\pi} d\varphi$$

Soit : $\boxed{V = \frac{4}{3}\pi R^3}$

• Exercice N° 2 :

1/- Définition :

$$\boxed{\overrightarrow{grad}(f) = \vec{\nabla}\cdot f = \frac{\partial f}{\partial x}\vec{i} + \frac{\partial f}{\partial y}\vec{j} + \frac{\partial f}{\partial z}\vec{k}}$$

$\vec{\nabla}$ ≡ opérateur Nabla.

2/-

$$\boxed{df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz}$$

↳ Différentielle totale de $f(x,y,z)$

et : $d\vec{r} = dx\,\vec{i} + dy\,\vec{j} + dz\,\vec{k}$

donc :

$$\overrightarrow{grad}(f)\cdot d\vec{r} = \left(\frac{\partial f}{\partial x}\vec{i} + \frac{\partial f}{\partial y}\vec{j} + \frac{\partial f}{\partial z}\vec{k}\right)\cdot\left(dx\vec{i} + dy\vec{j} + dz\vec{k}\right)$$

$$= \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz$$

$$= df(x,y,z)$$

3/- • Coordonnées cylindriques :

$$M(\rho,\theta,z) \longrightarrow df = \frac{\partial f}{\partial \rho}d\rho + \frac{\partial f}{\partial \theta}d\theta + \frac{\partial f}{\partial z}dz$$

et :

$$df = \overrightarrow{grad}(f)\cdot d\vec{r}$$

$$df = \left(grad(f)_\rho\,\vec{u}_\rho + grad(f)_\theta\,\vec{u}_\theta + grad(f)_z\,\vec{k}\right)\cdot$$

$$\left(d\rho\,\vec{u}_\rho + \rho\,d\theta\,\vec{u}_\theta + dz\,\vec{k}\right)$$

$$df = grad(f)_\rho\,d\rho + \rho\,grad(f)_\theta\,d\theta + grad(f)_z\,dz$$

En identifiant les 2 expressions de df on aura :

$$\boxed{grad(f)_\rho = \frac{\partial f}{\partial \rho};\quad grad(f)_\theta = \frac{1}{\rho}\frac{\partial f}{\partial \theta};\quad grad(f)_z = \frac{\partial f}{\partial z}}$$

• Coordonnées sphériques :

Un raisonnement analogue au précédent conduit à :

$$\boxed{\overrightarrow{grad}(f) = \frac{\partial f}{\partial r}\vec{u}_r + \frac{1}{r}\frac{\partial f}{\partial \theta}\vec{u}_\theta + \frac{1}{r\sin\theta}\frac{\partial f}{\partial \varphi}\vec{u}_\varphi}$$

4/- $\boxed{\overrightarrow{grad}(r) = \vec{u}_r}$; $\boxed{\overrightarrow{grad}\left(\frac{1}{r}\right) = -\frac{1}{r^2}\vec{u}_r}$

• Exercice N° 3 :

$$\vec{V}(x,y,z) = (2x-y)\,\vec{i} + (2y-x)\,\vec{j} - 4z\,\vec{k}$$

1/- $div(\vec{V}) = \vec{\nabla}\cdot\vec{V} = \frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z}$

$$\Rightarrow \boxed{div(\vec{V}) = 0}$$

2/- $\boxed{\overrightarrow{rot}(\vec{V}) = \vec{\nabla}\wedge\vec{V} = \vec{0}}$

3/- $\overrightarrow{rot}(\vec{V})$ est nul ⇒ Il existe une fct scalaire $\varphi(x,y,z)$ telle que : $\vec{V} = \overrightarrow{grad}(\varphi)$

donc :

$$\begin{cases} V_x = \frac{\partial \varphi}{\partial x} = 2x - y \text{ --- } 1 \\ V_y = \frac{\partial \varphi}{\partial y} = 2y - x \text{ --- } 2 \\ V_z = \frac{\partial \varphi}{\partial z} = -4z \text{ --- } 3 \end{cases}$$

De 1 : $\boxed{\varphi(x,y,z) = x^2 - yx + h(y,z)}$ ---(*)

$h(y,z)$ fct quelconque de y et z.

En remplaçant (*) dans 2 on tire $h(y,z)$:

$\boxed{h(y,z) = y^2 + g(z)}$ ---(**)

$g(z)$: fct quelconque de z.

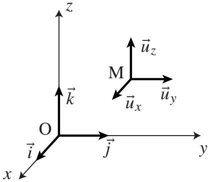
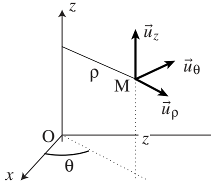
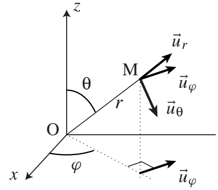
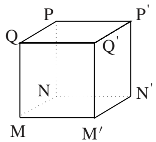
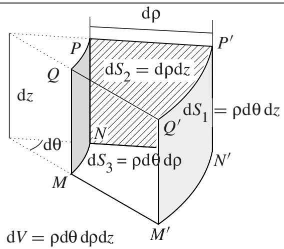
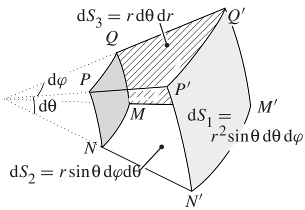
En remplaçant (**) dans 3 on tire $g(z)$:

$\boxed{g(z) = -2z^2 + C}$ (C : constante arbitraire)

Soit :

$$\boxed{\varphi(x,y,z) = x^2 - yx + y^2 - 2z^2 + C}$$

FIN

COORDONNÉES CARTÉSIENNES	COORDONNÉES CYLINDRIQUES	COORDONNÉES SPHÉRIQUES
		
x y z	$= \rho\cos\theta$ $= \rho\sin\theta$ $= z$	$= r\sin\theta\cos\varphi$ $= r\sin\theta\sin\varphi$ $= r\cos\theta$
	$0 \leq \theta \leq 2\pi$	$0 \leq \theta \leq \pi,\ 0 \leq \varphi \leq 2\pi$
$\vec{u}_x = \vec{i}$ $\vec{u}_y = \vec{j}$ $\vec{u}_z = \vec{k}$	$\vec{u}_\rho = \vec{i}\cos\theta + \vec{j}\sin\theta$ $\vec{u}_\theta = -\vec{i}\sin\theta + \vec{j}\cos\theta$ $\vec{u}_z = \vec{k}$	$\vec{u}_r = \vec{i}\sin\theta\cos\varphi + \vec{j}\sin\theta\sin\varphi + \vec{k}\sin\theta$ $\vec{u}_q = \vec{i}\cos\theta\cos\varphi + \vec{j}\cos\theta\sin\varphi - \vec{k}\sin\theta$ $\vec{u}_\varphi = -\vec{i}\sin\varphi + \vec{j}\cos\varphi$
$\mathrm{d}\overrightarrow{\mathrm{OM}} = \mathrm{d}l \begin{bmatrix} \mathrm{d}x \\ \mathrm{d}y \\ \mathrm{d}z \end{bmatrix}$ $\mathrm{d}l^2 = \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2$	$\mathrm{d}\overrightarrow{\mathrm{OM}} = \mathrm{d}l \begin{bmatrix} \mathrm{d}\rho \\ \rho\mathrm{d}\theta \\ \mathrm{d}z \end{bmatrix}$ $\mathrm{d}l^2 = \mathrm{d}\rho^2 + \rho^2\,\mathrm{d}\theta^2 + \mathrm{d}z^2$	$\mathrm{d}\overrightarrow{OM} = \mathrm{d}l \begin{bmatrix} \mathrm{d}r \\ r\mathrm{d}\theta \\ r\sin q\,\mathrm{d}\varphi \end{bmatrix}$ $\mathrm{d}l^2 = \mathrm{d}r^2 + r^2\,\mathrm{d}\theta^2 + r^2\sin^2\theta\,\mathrm{d}\varphi^2$
$\mathrm{d}S = \mathrm{d}x\mathrm{d}y = \mathrm{d}x\mathrm{d}z = \mathrm{d}y\mathrm{d}z$ $\mathrm{d}V = \mathrm{d}x\mathrm{d}y\mathrm{d}z$	$\mathrm{d}V = \rho\mathrm{d}\theta\,\mathrm{d}\rho\mathrm{d}z$	$\mathrm{d}V = r^2\sin\theta\,\mathrm{d}\theta\,\mathrm{d}r\mathrm{d}\varphi$
$\vec{\nabla}\Phi = \frac{\partial\Phi}{\partial x}\vec{\imath} + \frac{\partial\Phi}{\partial y}\vec{\jmath} + \frac{\partial\Phi}{\partial z}\vec{k}$	$\vec{\nabla}\Phi = \frac{\partial\Phi}{\partial\rho}\vec{u}_\rho + \frac{1}{\rho}\frac{\partial\Phi}{\partial\theta}\vec{u}_\theta + \frac{\partial\Phi}{\partial z}\vec{u}_z$	$\vec{\nabla}\Phi = \frac{\partial\Phi}{\partial r}\vec{u}_r + \frac{1}{r}\frac{\partial\Phi}{\partial\theta}\vec{u}q + \frac{1}{r\sin\theta}\frac{\partial\Phi}{\partial\varphi}\vec{u}_\varphi$
$\vec{\nabla}.\vec{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$	$\vec{\nabla}.\vec{A} = \frac{1}{\rho}\frac{\partial(\rho A_\rho)}{\partial\rho} + \frac{1}{\rho}\frac{\partial A_\theta}{\partial\theta} + \frac{\partial A_z}{\partial z}$	$\vec{\nabla}.\vec{A} = \frac{1}{r^2}\frac{\partial(r^2 A_r)}{\partial r} + \frac{1}{r\sin\theta}\left(\frac{\partial(A_\theta\sin\theta)}{\partial\theta} + \frac{\partial A_\varphi}{\partial\varphi}\right)$
$\vec{\nabla}\wedge\vec{A} = \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix}$	$\vec{\nabla}\wedge\vec{A} = \frac{1}{\rho}\begin{vmatrix} \vec{u}_\rho & \rho\vec{u}_\theta & \vec{u}_z \\ \frac{\partial}{\partial\rho} & \frac{\partial}{\partial\theta} & \frac{\partial}{\partial z} \\ A_\rho & \rho A_\theta & A_z \end{vmatrix}$	$\vec{\nabla}\wedge\vec{A} = \frac{1}{r^2\sin\theta}\begin{vmatrix} \vec{u}_r & r\vec{u}_\theta & r\sin\theta\vec{u}_\varphi \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\theta} & \frac{\partial}{\partial\varphi} \\ A_r & rA_\theta & r\sin\theta A_\varphi \end{vmatrix}$
$\Delta\Phi = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\Phi$	$\Delta\Phi = \left(\frac{1}{\rho}\frac{\partial}{\partial}\left(\rho\frac{\partial}{\partial}\right) + \frac{1}{\rho^2}\frac{\partial^2}{\partial\theta^2} + \frac{\partial^2}{\partial z^2}\right)\Phi$	$\Delta\Phi = \left(\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\right)\Phi$