Faculté des Sciences Appliquées
Département des Sciences et de Technologie

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Durée : 1h :30 min

Solution détaillée de l'examen De Math 04

Exercice 01 : Soit $z = x + iy$ où $x, y \in \mathbb{R}$.

1) $f(z) = e^{iz} = e^{i(x+iy)} = e^{-y}e^{ix} = e^{-y}(\cos x + i\sin x) = e^{-y}\cos x + ie^{-y}\sin x$ **0.5**

$\Longrightarrow$ $\boxed{\text{P(x,y)}=\text{e}^{-y}\cos x}$ et $\boxed{\text{Q(x,y)}=\text{e}^{-y}\sin x}$.

2) Montrons que f est holomorphe sur $\mathbb{C}$:

Méthode 01 : $f'(z) = ie^{iz}\ \forall z \in \mathbb{C}$. **0.5**

Méthode 02 : Par les conditions de Cauchy-Riemann : **0.5** + **0.5**

$$(1)\begin{cases} \dfrac{\partial P}{\partial x} = -e^{-y}\sin x \\ \dfrac{\partial Q}{\partial y} = -e^{-y}\sin x \end{cases} \Longrightarrow \frac{\partial P}{\partial x} = \frac{\partial Q}{\partial y} \text{ et } (2)\begin{cases} \dfrac{\partial P}{\partial y} = -e^{-y}\cos x \\ \dfrac{\partial Q}{\partial x} = e^{-y}\cos x \end{cases} \Longrightarrow \frac{\partial P}{\partial y} = -\frac{\partial Q}{\partial x}.$$

3-1) Le module de f :

$$|f(z)| = \sqrt{P^2 + Q^2} = \sqrt{e^{-2y}\cos^2 x + e^{-2y}\sin^2 x} = \sqrt{e^{-2y}(\cos^2 x + \sin^2 x)} = \boxed{e^{-y}}.$$ **0.75**

3-2) L'argument de f :

$$\begin{cases} \cos\theta = \dfrac{Re(f)}{|f|} = \dfrac{e^{-y}\cos x}{e^{-y}} = \cos x \\ \sin\theta = \dfrac{Im(f)}{|f|} = \dfrac{e^{-y}\sin x}{e^{-y}} = \sin x \end{cases} \Longrightarrow \boxed{\arg(f)=\theta = x + 2k\pi};\ k \in \mathbb{Z}.$$ **0.75**

4) Soit l'équation : **0.25** **0.25**

$$2\cos z + e^{-iz} = 2 \Longrightarrow 2\left(\frac{\overbrace{e^{iz} + e^{-iz}}}{2}\right) + e^{-iz} = 2 \Longrightarrow e^{iz} + 2e^{-iz} = 2 \overset{\times e^{iz}}{\Longrightarrow} \overbrace{e^{2iz} - 2e^{iz} + 2 = 0}.$$

Posons $e^{iz} = M$; on obtient : $M^2 - 2M + 2 = 0 \Longrightarrow \Delta = 4 - 8 = -4 = (2i)^2$.

$$\begin{cases} M_1 = \dfrac{2+2i}{2} = 1 + i \Longrightarrow e^{iz} = 1 + i \Longrightarrow iz = \log(1+i) = \ln|1+i| + i(\dfrac{\pi}{4} + 2k\pi), \\ M_2 = \dfrac{2-2i}{2} = 1 - i \Longrightarrow e^{iz} = 1 - i \Longrightarrow iz = \log(1-i) = \ln|1-i| + i(\dfrac{7\pi}{4} + 2k\pi). \end{cases}$$ **0.25** + **0.75** / **0.25** + **0.75**

Donc : $\boxed{z=\dfrac{\pi}{4} + 2k\pi - i\ln\sqrt{2}}$ ou $\boxed{z=\dfrac{7\pi}{4} + 2k\pi - i\ln\sqrt{2}}$; $k \in \mathbb{Z}$. **0.5** + **0.5**

Exercice 02 : Soit $P(x,y) = x^3 - 3xy^2 + x^2 - y^2 - y$.

1) Montrons que P est harmonique sur $\mathbb{R}^2$:

$$\begin{cases} \dfrac{\partial P}{\partial x} = 3x^2 - 3y^2 + 2x \\ \dfrac{\partial^2 P}{\partial x^2} = 6x + 2. \end{cases} \boxed{01} \text{ et } \begin{cases} \dfrac{\partial P}{\partial y} = -6xy - 2y - 1 \\ \dfrac{\partial^2 P}{\partial y^2} = -6x - 2. \end{cases} \boxed{01} \Longrightarrow \frac{\partial^2 P}{\partial x^2} + \frac{\partial^2 P}{\partial y^2} = 0.$$

2) Puisque f est holomorphe sur $\mathbb{C}$ alors le couple (P,Q) vérifie les conditions de Cauchy-Riemann, c'est à dire :

$$\begin{cases} \dfrac{\partial P}{\partial x} = \dfrac{\partial Q}{\partial y} \cdots\cdots (1) \\ \dfrac{\partial P}{\partial y} = -\dfrac{\partial Q}{\partial x} \cdots\cdots (2) \end{cases} \qquad \boxed{0.5}$$

De l'équation (1) on tire $\dfrac{\partial Q}{\partial y} = 3x^2 - 3y^2 + 2x$, d'où

$$\begin{aligned} Q(x,y) &= \int (3x^2 - 3y^2 + 2x)dy \\ &= 3x^2y - y^3 + 2xy + C(x). \qquad \boxed{01} \end{aligned}$$

d'une part d'autre part on a :

$$(2) \Longrightarrow -6xy - 2y - 1 = -\overbrace{(6xy + 2y + C'(x))}^{0.75} \Longleftrightarrow \overbrace{C'(x) = 1}^{0.25} \Longrightarrow \overbrace{C(x) = x + c}^{0.5}\ ;\ c \in \mathbb{R}.$$

Finalement :

$$\boxed{Q(x,y) = 3x^2y - y^3 + 2xy + x + c}\ ;\ c \in \mathbb{R}.$$

3) On a $f(z) = f(x,y) = P(x,y) + iQ(x,y) \cdots\cdot (3)$ telle que : $f(0,0) = 0$

$\Longrightarrow f(0,0) = P(0,0) + iQ(0,0) \Longrightarrow ic = 0 \Longleftrightarrow \boxed{c=0}$. $\boxed{0.5}$

En substituant ceci dans (3), on obtient :

$$\begin{aligned} f(z) &= x^3 - 3xy^2 + x^2 - y^2 - y + i(3x^2y - y^3 + 2xy + x) \\ &= \boxed{z^3 + z^2 + iz}. \qquad \boxed{01} \end{aligned}$$

4) La dérivée de f :

<u>Méthode 01(directe)</u> : $\boxed{f'(z) = 3z^2 + 2z + i}$, $\forall z \in \mathbb{C}$. $\boxed{0.5}$

<u>Méthode 02</u> : Puisque f est holomorphe sur $\mathbb{C}$ donc :

$$\begin{aligned} f'(z) &= \frac{\partial P}{\partial x} + i\frac{\partial Q}{\partial x} \qquad \boxed{0.5} \\ &= 3x^2 - 3y^2 + 2x + i(6xy + 2y + 1) \\ &= \boxed{3z^2 + 2z + i}. \qquad \boxed{0.5} \end{aligned}$$

Exercice 03 : Pour calculer l'intégrale il faut factoriser $2z^2 - 3iz - 1$.

$$2z^2 - 3iz - 1 = 0 \Longrightarrow \Delta = (-3i)^2 + 8 = -9 + 8 = -1 = i^2 \Longrightarrow \begin{cases} z_0 = \dfrac{3i - i}{4} = \dfrac{i}{2}, & \boxed{0.5} \\ z_1 = \dfrac{3i + i}{4} = i. & \boxed{0.5} \end{cases}$$

$$\Longrightarrow 2z^2 - 3iz - 1 = 2(z - z_0)(z - z_1) = 2(z - \tfrac{i}{2})(z - i). \qquad \boxed{0.75}$$

$$i\ ,\ \frac{i}{2} \in int(C)\ ;\ C = |z - 1 + \frac{i}{2}| = \frac{11}{6}\ ?$$

$$\begin{cases} |i - 1 + \dfrac{i}{2}| = |-1 + \dfrac{3i}{2}| = \sqrt{1 + \dfrac{9}{4}} = \dfrac{\sqrt{13}}{2} < \dfrac{11}{6} \Longrightarrow \boxed{i \in int(C)}. & \boxed{0.5} \\ |\dfrac{i}{2} - 1 + \dfrac{i}{2}| = |-1 + i| = \sqrt{1+1} = \sqrt{2} < \dfrac{11}{6} \Longrightarrow \boxed{i/2 \in int(C)}. & \boxed{0.5} \end{cases}$$

Donc il faut décomposer $\dfrac{1}{(z - \frac{i}{2})(z - i)}$ en éléments simples.

$$\frac{1}{(z - \frac{i}{2})(z - i)} = \frac{a}{z - \frac{i}{2}} + \frac{b}{z - i} = \frac{(a+b)z - i(a + \frac{b}{2})}{(z - \frac{i}{2})(z - i)} \Longrightarrow \begin{cases} a + b = 0, \\ -i(a + \frac{b}{2}) = 1. \end{cases} \Longrightarrow \begin{cases} a = 2i, & \boxed{0.25} \\ b = -2i. & \boxed{0.25} \end{cases}$$

$$\Longrightarrow I = \int_C \frac{e^{\pi z}}{2(z - \frac{i}{2})(z - i)} dz = \int_C e^{\pi z} \frac{1}{2} \left(\frac{2i}{z - \frac{i}{2}} - \frac{2i}{z - i} \right) dz = i \left(\int_C \frac{e^{\pi z}}{z - \frac{i}{2}} dz - \int_C \frac{e^{\pi z}}{z - i} dz \right). \boxed{0.25}$$

On sait que $f(z) = e^{\pi z}$ est holomorphe car $\boxed{f'(z) = \pi e^{\pi z}}$ dans l'intérieur de C $\boxed{0.5}$ qui est un chemin fermé (cercle). Donc :

$$\begin{aligned} I &= i \left(\int_C \frac{e^{\pi z}}{z - \frac{i}{2}} dz - \int_C \frac{e^{\pi z}}{z - i} dz \right) \\ &= i \Big[2\pi i \big(e^{\pi \frac{i}{2}} - e^{\pi i} \big) \Big] \qquad \boxed{0.5} \\ &= i \Big[2\pi i \big(i - (-1) \big) \Big] \\ &= \boxed{-2\pi - 2\pi i}. \qquad \boxed{0.5} \end{aligned}$$

FIN